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简支夹层矩形板的非线性弯曲

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摘　要

本文应用变分法导出了具有软夹心的夹层矩形板的非线性弯曲理论的基本方程和边界条件．然后，使用摄动法研究了均布横向载荷作作用下简支夹层矩形板的非线性弯曲问题，得到了相当精确的解析解．

关键词　夹层矩形板　非线性弯曲

一、引　言

作为结构元件的夹层板在航空、宇航和航海工程中得到了广泛的应用．因此近年来，许多研究者对这种板进行了研究．但是，因为面临非线性微分方程和夹层结构复杂的巨大困难，仅有少数人研究了夹层板的非线性问题．首先，Reissner[1]建立了具有软夹心的夹层矩形板的非线性弯曲理论．此时，视表层如薄膜一样，忽略了表层的抗弯刚度．然后，刘人怀[2]进一步建立了计及表层抗弯刚度的具有软夹心的夹层圆板的更为精确的非线性弯曲理论，并且给出了忽略表层抗弯刚度的简化理论．刘人怀[2~7]、Kan[8]、Alwan[9]和Kamiya[10]等人先后讨论了夹层圆板和夹层矩形板的非线性弯曲和振动问题．然而，所得的简支夹层矩形板的非线性弯曲问题的结果尚不能令人满意．所以，进一步研究这一问题是很有意义的．

值得指出，应用文献[1]的方程来求解夹层矩形板的实际问题是不大方便的．为此，须推导这种板的新方程．在忽略表层抗弯刚度的情况下，我们使用变分法导出了均布横向载荷作用下具有软夹心的夹层矩形板的非线性弯曲的基本方程和边界条件．然后，对上述方程和边界条件进行了简化．最后，使用摄动法研究了均布横向载荷作用下的简支夹层矩形板，获得了相当精确的解析解．本文所得结果可供工程设计时参考应用．

二、基 本 方 程

考虑在任意横向载荷$q(x, y)$作用下的夹层矩形薄板，如图1所示．这里x，y和z为直

角坐标，$2a$和$2b$为边长，t为表层厚度，h_0为上下表层中面间的距离，1、2和3分别表示上表层、夹心和下表层。

在推导基本方程和边界条件时，采用Reissner[1]的假定：

（1） 材料服从于虎克定律.

（2） 夹心横向不可压缩.

（3） 夹心沿板面方向不能承受载荷.

（4） 表层处于薄膜应力状态.

（5） 夹心中面法线在变形后保持直线.

在这些假定下，夹层矩形板中任意一点的位移为：

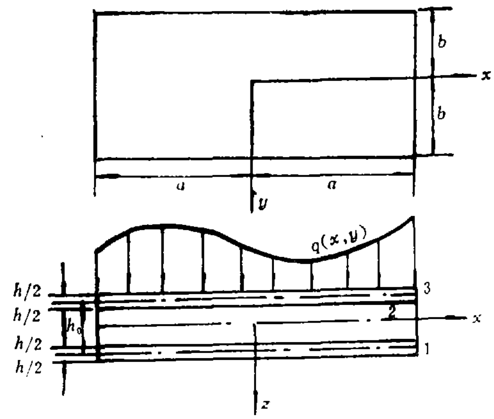


图 1 夹层矩形板的几何形状

上表层$\left[\frac{1}{2}(h_0-t)\leqslant z\leqslant\frac{1}{2}(h_0+t)\right]$

$$u_1=u+\frac{1}{2}h_0\psi_x,\quad v_1=v+\frac{1}{2}h_0\psi_y,\quad w_1=w \tag{2.1}$$

下表层$\left[-\frac{1}{2}(h_0+t)\leqslant z\leqslant-\frac{1}{2}(h_0-t)\right]$

$$u_3=u-\frac{1}{2}h_0\psi_x,\quad v_3=v-\frac{1}{2}h_0\psi_y,\quad w_3=w \tag{2.2}$$

夹心$\left[-\frac{1}{2}(h_0-t)\leqslant z\leqslant\frac{1}{2}(h_0-t)\right]$

$$u_2=u+z\psi_x,\quad v_2=v+z\psi_y,\quad w_2=w \tag{2.3}$$

其中 u_i, v_i和$w_i(i=1,\ 2,\ 3)$分别为上表层、夹心和下表层在x,y和z轴方向的位移，u,v和和w分别为板中面在x，y和z轴方向的位移，ψ_x和ψ_y分别为夹心中面法线在xz和yz平面内的转角。

设ε_{xi}，ε_{yi}，ε_{zi}，γ_{xyi}，γ_{yzi}和$\gamma_{xzi}(i=1,\ 2,\ 3)$分别为上表层、夹心和下表层的应变分量，则将式(2.1)～(2.3)分别代入下述夹层矩形板的几何方程

$$\left.\begin{aligned}
\varepsilon_{xi}&=\frac{\partial u_i}{\partial x}+\frac{1}{2}\left(\frac{\partial w_i}{\partial x}\right)^2\\
\varepsilon_{yi}&=\frac{\partial v_i}{\partial y}+\frac{1}{2}\left(\frac{\partial w_i}{\partial y}\right)^2\\
\gamma_{xyi}&=\frac{\partial v_i}{\partial x}+\frac{\partial u_i}{\partial y}+\frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial y}
\end{aligned}\right\}(i=1,\ 3) \tag{2.4}$$

$$\varepsilon_{zi}=\gamma_{xzi}=\gamma_{yzi}=0$$

$$\left.\begin{aligned}
\gamma_{xz2}&=\frac{\partial u_2}{\partial z}+\frac{\partial w_2}{\partial x}\\
\gamma_{yz2}&=\frac{\partial v_2}{\partial z}+\frac{\partial w_2}{\partial y}\\
\varepsilon_{x2}&=\varepsilon_{y2}=\varepsilon_{z2}=\gamma_{xy2}=0
\end{aligned}\right\} \tag{2.5}$$

便得：

上表层

$$\left.\begin{aligned}\varepsilon_{x1}&=\frac{\partial u}{\partial x}+\frac{h_0}{2}\frac{\partial\psi_x}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\\varepsilon_{y1}&=\frac{\partial v}{\partial y}+\frac{h_0}{2}\frac{\partial\psi_y}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\\gamma_{xy1}&=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\frac{h_0}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{aligned}\right\}\tag{2.6}$$

下表层

$$\left.\begin{aligned}\varepsilon_{x3}&=\frac{\partial u}{\partial x}-\frac{h_0}{2}\frac{\partial\psi_x}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\\varepsilon_{y3}&=\frac{\partial v}{\partial y}-\frac{h_0}{2}\frac{\partial\psi_y}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\\gamma_{xy3}&=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}-\frac{h_0}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{aligned}\right\}\tag{2.7}$$

夹心

$$\left.\begin{aligned}\gamma_{xz2}&=\psi_x+\frac{\partial w}{\partial x}\\\gamma_{yz2}&=\psi_y+\frac{\partial w}{\partial y}\end{aligned}\right\}\tag{2.8}$$

设σ_{xi}，σ_{yi}，σ_{zi}，τ_{xyi}，τ_{yzi}和τ_{xzi} $(i=1,\ 2,\ 3)$分别为上表层、夹心和下表层点的应力分量，E，ν和G分别为表层材料的弹性模量、泊松比和剪切模量，G_2为夹心的剪切模量，将式(2.6)～(2.8)分别代入下述虎克定律：

$$\left.\begin{aligned}\sigma_{xi}&=\frac{E}{1-\nu^2}(\varepsilon_{xi}+\nu\varepsilon_{yi})\\\sigma_{yi}&=\frac{E}{1-\nu^2}(\varepsilon_{yi}+\nu\varepsilon_{xi})\\\tau_{xyi}&=G\gamma_{xyi}\\\sigma_{zi}&=\tau_{xzi}=\tau_{yzi}=0\quad(i=1,\ 3)\end{aligned}\right\}\tag{2.9}$$

$$\left.\begin{aligned}\tau_{xz2}&=G_2\gamma_{xz2}\\\tau_{yz2}&=G_2\gamma_{yz2}\\\sigma_{x2}&=\sigma_{y2}=\sigma_{z2}=\tau_{yz2}=0\end{aligned}\right\}\tag{2.10}$$

便得：

上表层

$$\left.\begin{aligned}\sigma_{x1}&=\sigma_{x0}+\frac{Eh_0}{2(1-\nu^2)}\left(\frac{\partial\psi_x}{\partial x}+\nu\frac{\partial\psi_y}{\partial y}\right)\\\sigma_{y1}&=\sigma_{y0}+\frac{Eh_0}{2(1-\nu^2)}\left(\frac{\partial\psi_y}{\partial y}+\nu\frac{\partial\psi_x}{\partial x}\right)\\\tau_{xy1}&=\tau_{xy0}+\frac{Gh_0}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)\end{aligned}\right\}\tag{2.11}$$

下表层

$$\left.\begin{aligned}\sigma_{x3}&=\sigma_{x0}-\frac{Eh_0}{2(1-\nu^2)}\left(\frac{\partial\psi_x}{\partial x}+\nu\frac{\partial\psi_y}{\partial y}\right)\\ \sigma_{y3}&=\sigma_{y0}-\frac{Eh_0}{2(1-\nu^2)}\left(\frac{\partial\psi_y}{\partial y}+\nu\frac{\partial\psi_x}{\partial y}\right)\\ \tau_{xy3}&=\tau_{xy0}-\frac{Gh_0}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)\end{aligned}\right\}\tag{2.12}$$

夹心

$$\left.\begin{aligned}\tau_{xz2}&=G_2\left(\psi_x+\frac{\partial w}{\partial x}\right)\\ \tau_{yz2}&=G_2\left(\psi_y+\frac{\partial w}{\partial y}\right)\end{aligned}\right\}\tag{2.13}$$

其中 σ_{x0}，σ_{y0}和τ_{xy0}分别为夹层矩形板中面内的应力

$$\left.\begin{aligned}\sigma_{x0}&=\frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\\ \sigma_{y0}&=\frac{E}{1-\nu^2}\left[\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\\ \tau_{xy0}&=G\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\end{aligned}\right\}\tag{2.14}$$

根据弹性体应变能公式：

$$U_i=\frac{1}{2}\iiint_{V_i}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{yz}+\tau_{yz}\gamma_{yz}+\tau_{xz}\gamma_{xz})dxdydz \quad (i=1,2,3) \tag{2.15}$$

我们得到表层和夹心的应变能公式：

$$\left.\begin{aligned}U_i&=\frac{1}{2E}\iiint_{V_i}[(\sigma_{xi}+\sigma_{yi})^2+2(1+\nu)(\tau^2_{xyi}-\sigma_{xi}\sigma_{yi})]dxdydz \quad (i=1,3)\\ U_2&=\frac{1}{2G_2}\iiint_{V_2}(\tau^2_{xz2}+\tau^2_{yz2})dxdydz\end{aligned}\right\}\tag{2.16}$$

将式(2.11)～(2.13)代入式(2.16)，并对z进行积分，便得：

$$\begin{aligned}U_1=&\frac{t}{2E}\iint_{S_1}[(\sigma_{x0}+\sigma_{y0})^2+2(1+\nu)(\tau^2_{xy0}-\sigma_{x0}\sigma_{y0})dxdy\\ &+\frac{th_0}{2}\iint_{S_1}\left[\sigma_{x0}\frac{\partial\psi_x}{\partial x}+\sigma_{y0}\frac{\partial\psi_y}{\partial y}+\tau_{xy0}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)\right]dxdy\\ &+\frac{D}{4}\iint_{S_1}\left[\left(\frac{\partial\psi_x}{\partial x}\right)^2+\left(\frac{\partial\psi_y}{\partial y}\right)^2+2\nu\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_y}{\partial y}+\frac{1-\nu}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)^2\right]dxdy\end{aligned}$$

$$\left.\begin{aligned}U_3=&\frac{t}{2E}\iint\limits_{S_3}[(\sigma_{x0}+\sigma_{y0})^2+2(1+\nu)(\tau_{xy0}^2-\sigma_{x0}\sigma_{y0})]dxdy\\&-\frac{th_0}{2}\iint\limits_{S_3}\left(\sigma_{x0}\frac{\partial\psi_x}{\partial x}+\sigma_{y0}\frac{\partial\psi_y}{\partial y}+\tau_{xy0}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)\right]dxdy\\&+\frac{D}{4}\iint\limits_{S_3}\left[\left(\frac{\partial\psi_x}{\partial x}\right)^2+\left(\frac{\partial\psi_y}{\partial y}\right)^2+2\nu\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_y}{\partial y}+\frac{1-\nu}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)^2\right]dxdy\\U_2=&\frac{G_2h_0}{2}\iint\limits_{S_2}\left[\left(\psi_x+\frac{\partial w}{\partial x}\right)^2+\left(\psi_y+\frac{\partial w}{\partial y}\right)^2\right]dxdy\end{aligned}\right\}\quad(2.17)$$

其中 D为夹层矩形板的抗弯刚度

$$D=\frac{Eth_0^2}{2(1-\nu^2)}\tag{2.18}$$

任意横向载荷$q(x,\ y)$的外力功为

$$V=\iint\limits_{S}qwdxdy\tag{2.19}$$

这样，夹层矩形板的总势能为

$$\begin{aligned}U=&U_1+U_2+U_3-V\\=&\frac{t}{E}\iint\limits_{S}[(\sigma_{x0}+\sigma_{y0})^2+2(1+\nu)(\tau_{xy0}^2-\sigma_{x0}\sigma_{y0})]dxdy\\&+\frac{D}{2}\iint\limits_{S}\left[\left(\frac{\partial\psi_x}{\partial x}\right)^2+\left(\frac{\partial\psi_y}{\partial y}\right)^2+2\nu\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_y}{\partial y}+\frac{1-\nu}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)^2\right]dxdy\\&+\frac{G_2h_0}{2}\iint\limits_{S}\left[\left(\psi_x+\frac{\partial w}{\partial x}\right)^2\left(\psi_y+\frac{\partial w}{\partial y}\right)^2\right]dxdy-\iint\limits_{S}qwdxdy\end{aligned}\tag{2.20}$$

按照势能原理，并以u，v，w，ψ_x和ψ_y作为自变量，便有

$$\delta U=0\tag{2.21}$$

将式（2.20）代入此式，经部分积分后，可得在任意横向载荷$q(x,\ y)$作用下夹层矩形板大挠度理论的平衡方程和边界条件：

$$\left.\begin{aligned}&\frac{\partial\sigma_{x0}}{\partial x}+\frac{\partial\tau_{xy0}}{\partial y}=0\\&\frac{\partial\tau_{xy0}}{\partial x}+\frac{\partial\sigma_{y0}}{\partial y}=0\\&G_2h_0\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}+\nabla^2w\right)=-q-2t\left(\sigma_{x0}\frac{\partial^2w}{\partial x^2}+\sigma_{y0}\frac{\partial^2w}{\partial y^2}+2\tau_{xy0}\frac{\partial^2w}{\partial x\partial y}\right)\\&\frac{D}{G_2h_0}\left(\frac{\partial^2\psi_x}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2\psi_x}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2\psi_y}{\partial x\partial y}\right)-\left(\psi_x+\frac{\partial w}{\partial x}\right)=0\\&\frac{D}{G_2h_0}\left(\frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2}+\frac{\partial^2\psi_y}{\partial y^2}\right)-\left(\psi_y+\frac{\partial w}{\partial y}\right)=0\end{aligned}\right\}\quad(2.22\text{a}\sim\text{e})$$

$$
\left.\begin{aligned}
&\text{当}x=\pm a\text{时，}\\
&\sigma_{x0}=0,\ \text{或}\delta u=0\\
&\tau_{xy0}=0,\ \text{或}\delta v=0\\
&G_2h_0\left(\psi_x+\frac{\partial w}{\partial x}\right)+2t\left(\sigma_{x0}\frac{\partial w}{\partial x}+\tau_{xy0}\frac{\partial w}{\partial y}\right)=0,\ \text{或}\delta w=0\\
&D\left(\frac{\partial\psi_x}{\partial x}+\nu\frac{\partial\psi_y}{\partial y}\right)=0,\ \text{或}\delta\psi_x=0\\
&\frac{D(1-\nu)}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)=0;\ \text{或}\delta\psi_y=0
\end{aligned}\right\}\qquad(2.23)
$$

$$
\left.\begin{aligned}
&\text{当}y=\pm b\text{时，}\\
&\tau_{xy0}=0,\ \text{或}\delta u=0,\\
&\sigma_{y0}=0,\ \text{或}\delta v=0,\\
&G_2h_0\left(\psi_y+\frac{\partial w}{\partial y}\right)+2t\left(\frac{\partial w}{\partial x}+\sigma_{y0}\frac{\partial w}{\partial y}\right)=0,\ \text{或}\delta w=0\\
&D\left(\frac{\partial\psi_y}{\partial y}+\nu\frac{\partial\psi_x}{\partial x}\right)=0,\ \text{或}\delta\psi_y=0\\
&\frac{D(1-\nu)}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)=0,\ \text{或}\delta\psi_x=0
\end{aligned}\right\}\qquad(2.24)
$$

其中∇^2是二阶拉普拉斯算子，

$$\nabla^2(\cdots)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\cdots)\qquad(2.25)$$

至于弯矩M_x和M_y，扭矩M_{xy}，横向剪力Q_x和Q_y，则有公式：

$$
\left.\begin{aligned}
&M_x=D\left(\frac{\partial\psi_x}{\partial x}+\nu\frac{\partial\psi_y}{\partial y}\right),\ M_y=D\left(\frac{\partial\psi_y}{\partial y}+\nu\frac{\partial\psi_x}{\partial x}\right)\\
&M_{xy}=\frac{D(1-\nu)}{2}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)\\
&Q_x=G_2h_0\left(\psi_x+\frac{\partial w}{\partial x}\right)\\
&Q_y=G_2h_0\left(\psi_y+\frac{\partial w}{\partial y}\right)
\end{aligned}\right\}\qquad(2.26\text{a}\sim\text{e})
$$

由方程(2.14)消去位移u和v，便得应变协调方程：

$$
\begin{aligned}
&\frac{\partial^2\sigma_{x0}}{\partial y^2}+\frac{\partial^2\sigma_{y0}}{\partial x^2}-\nu\left(\frac{\partial^2\sigma_{x0}}{\partial x^2}+\frac{\partial^2\sigma_{y0}}{\partial y^2}\right)-2(1+\nu)\frac{\partial^2\tau_{xy0}}{\partial x\partial y}\\
&=E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]
\end{aligned}\qquad(2.27)
$$

于是，我们得到了夹层矩形板大挠度理论的基本方程和边界条件(2.22)，(2.27)，(2.23)和(2.24)。显然此问题相当繁杂，因而进一步简化是十分必要的。

应用文献[12]的处理夹层板小挠度方程的方法，我们可将ψ_x，ψ_y用两个新函数ω，f来表示：

$$\psi_x=\frac{\partial\omega}{\partial x}+\frac{\partial f}{\partial y}\ ,\ \psi_y=\frac{\partial\omega}{\partial y}-\frac{\partial f}{\partial x} \tag{2.28}$$

将式 (2.28) 代入方程(2.22d, e)，便得

$$\left.\begin{aligned}&\frac{\partial}{\partial x}\left(\frac{D}{G_2h_0}\nabla^2\omega-\omega-w\right)+\frac{\partial}{\partial y}\left[\frac{D(1-\nu)}{2G_2h_0}\nabla^2 f-f\right]=0\\&\frac{\partial}{\partial y}\left(\frac{D}{G_2h_0}\nabla^2\omega-\omega-w\right)-\frac{\partial}{\partial x}\left[\frac{D(1-\nu)}{2G_2h_0}\nabla^2 f-f\right]=0\end{aligned}\right\} \tag{2.29}$$

显而易见，若把方程(2.29)中括号里的量看作两个独立的函数，则此方程就是Cauchy-Riemann方程．因而它的解可表示为

$$\frac{D(1-\nu)}{2G_2h_0}\nabla^2 f-f+i\left(\frac{D}{G_2h_0}\nabla^2\omega-\omega-w\right)=F(x+iy) \tag{2.30}$$

这是一个关于f，ω和w的非齐次偏微分方程，其通解为方程 (2.30) 的任一特解与相应的齐次方程的通解之和．由于$F(x+iy)$ 的实部和虚部都是调和函数，所以方程 (2.30) 的特解f_1，ω_1，w_1 可取为

$$f_1+i\omega_1=-F(x+iy),\qquad w_1=0 \tag{2.31}$$

相应的齐次方程为

$$\left.\begin{aligned}&\frac{D(1-\nu)}{2G_2h_0}\nabla^2 f-f=0\\&\frac{D}{G_2h_0}\nabla^2\omega-\omega-w=0\end{aligned}\right\} \tag{2.32a, b}$$

特解(2.31)既不产生挠度，也不影响ψ_x和ψ_y的值，因此这组特解可以略去。这样，f，ω和w只要理解为满足齐次方程 (2.32) 的函数便可以了．由方程 (2.32b) 得到

$$w=\frac{D}{G_2h_0}\nabla^2\omega-\omega \tag{2.33}$$

将(2.28)和(2.33)代入方程(2.22c)，得到ω需满足的方程如下：

$$D\nabla^4\omega=-q-2t\left(\sigma_{x0}\frac{\partial^2}{\partial x^2}+\sigma_{y0}\frac{\partial^2}{\partial y^2}+2\tau_{xy0}\frac{\partial^2}{\partial x\partial y}\right)\left(\frac{D}{G_2h_0}\nabla^2\omega-\omega\right) \tag{2.34}$$

这样一来，夹层矩形板的挠度w和转角ψ_x，ψ_y便通过式 (2.28) 和 (2.33) 用两个函数ω，f来表示，而ω，f则应分别满足方程 (2.34) 和(2.32a)。

下面，我们引入如下的应力函数$\varphi(x, y)$：

$$\sigma_{x0}=\frac{\partial^2\varphi}{\partial y^2},\quad \sigma_{y0}=\frac{\partial^2\varphi}{\partial x^3},\ \tau_{xy0}=-\frac{\partial^2\varphi}{\partial x\partial y} \tag{2.35}$$

这样，方程 (2.22a, b) 已被满足，而方程 (2.34) 和 (2.27) 成为

$$D\nabla^4\omega=-q-2tL(w,\ \varphi), \tag{2.36a}$$

$$\frac{1}{E}\nabla^4\varphi=-\frac{1}{2}L(w,\ w) \tag{2.36b}$$

其中 w的表达式如式 (2.33) 所示，

$$L(w,\ \varphi)=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2\varphi}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y} \tag{2.37}$$

由于引进了函数ω，f以及应力函数φ，使得基本方程组大大简化．但是必须指出，对于大多数具体问题，在边界条件的表达式中，ω与f是耦合的，因此还必须联立求解．在个别问题的边界条件中，ω与f不同时出现，这时问题就更为简化，ω与f便可以分别独立地求解了．

今以四边简支夹层矩形板作为一个例子．由式（2.23）和（2.24）知，在这种情况下，它的边界条件为：

当$x=\pm a$时，$\sigma_{x0}=0$，$\tau_{xy0}=0$，$w=0$，$\psi_y=0$，$M_x=0$ (2.38)

当$y=\pm b$时，$\sigma_{y0}=0$，$\tau_{xy0}=0$，$w=0$，$\psi_x=0$，$M_y=0$ (2.39)

现在，我们将这些边界条件表示为ω和φ的显式．为此，考虑一块简支多边形板．于是在任一边界l上有

$$\sigma_{n0}=0,\ \tau_{nl0}=0,\ w=0,\ \psi_l=0,\ M_n=0 \tag{2.40a～e}$$

这里n为边界法向，

$$M_n=D\left(\frac{\partial\psi_n}{\partial n}+\nu\frac{\partial\psi_l}{\partial l}\right) \tag{2.41}$$

显然，式（2.40e）可简化为如下形式：

$$\frac{\partial\psi_n}{\partial n}=0 \tag{2.42}$$

由式(2.28)，有

$$\psi_n=\frac{\partial\omega}{\partial n}+\frac{\partial f}{\partial l},\qquad \psi_l=\frac{\partial\omega}{\partial l}-\frac{\partial f}{\partial n} \tag{2.43}$$

应用式(2.33)，(2.42)和(2.43)，边界条件(2.40c～e)化为

$$\frac{D}{G_2h_0}\nabla^2\omega-\omega=0,\ \frac{\partial\omega}{\partial l}-\frac{\partial f}{\partial n}=0,\qquad \frac{\partial^2\omega}{\partial n^2}+\frac{\partial^2 f}{\partial n\partial l}=0 \tag{2.44}$$

这些边界条件可进一步简化，最后化成如下形式：

$$\omega=0,\ \frac{\partial f}{\partial n}=0,\qquad \frac{\partial^2\omega}{\partial n^2}=0 \tag{2.45a～c}$$

这里，函数f还满足方程(2.32a)．

将矢量场的Ganss定理应用于平面情形，有

$$\iint\limits_S\nabla\mathbf{v}dS=\oint\limits_l\mathbf{v}\cdot\mathbf{n}dl \tag{2.46}$$

其中 $\mathbf{v}$是平面矢量，S为多边形板面区域，

$$\nabla(\cdots)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial x}\right)(\cdots) \tag{2.47}$$

取$\mathbf{v}=\nabla f$，则式（2.46）成为

$$\iint\limits_S\nabla^2 fdS=\oint_l\nabla f\cdot\mathbf{n}dl=\oint_l\frac{\partial f}{\partial n}dl \tag{2.48}$$

应用式（2.32a）和(2.45b)，我们由上式得到

$$\iint_S f dS=0 \tag{2.49}$$

由S的任意性可知

$$f\equiv 0 \tag{2.50}$$

因此，四边简支夹层矩形板的边界条件（2.38）和（2.39）化为

$$\text{当 } x=\pm a \text{ 时，} \omega=0,\ \frac{\partial^2\omega}{\partial x^2}=0,\ \frac{\partial^2\varphi}{\partial y^2}=0,\ \frac{\partial^2\varphi}{\partial x\, y}=0 \tag{2.51}$$

$$\text{当 } y=\pm b \text{ 时，} \omega=0,\ \frac{\partial^2\omega}{\partial y^2}=0,\ \frac{\partial^2\varphi}{\partial x^2}=0,\ \frac{\partial^2\varphi}{x\, y}=0 \tag{2.52}$$

为简单起见，本文仅讨论在均布载荷q_0作用下简支夹层矩形板的大挠度问题。于是，问题归结为求解非线性边值问题(2.36)，(2.51)和(2.52)。

为了计算方便，引入下列无量纲量：

$$\left.\begin{aligned}&\lambda=\frac{a}{b},\quad \xi=\frac{x}{a},\quad \eta=\frac{y}{b},\quad \Omega=\frac{2\sqrt{1-\nu^2}}{h_0}\omega\\ &\Phi=\frac{4(1-\nu^2)}{Eh_0^2}\varphi,\quad W=\frac{2\sqrt{1-\nu^2}}{h_0}w,\quad k=\frac{D}{G_2h_0a^2},\quad Q=\frac{2a^4\sqrt{1-\nu^2}}{Dh_0}q_0\end{aligned}\right\} \tag{2.53}$$

利用这些无量纲量，非线性边值问题(2.36)，(2.51) 和 (2.52) 化为下面的无量纲形式：

$$\left.\begin{aligned}&L_1^2\Omega=-Q-\lambda^2L_2(W,\ \Phi)\\ &L_1^2\Phi=-\frac{\lambda^2}{2}L_2(W,\ W)\end{aligned}\right\} \tag{2.54a, b}$$

$$\text{当 } \xi=\pm 1 \text{ 时，} \Omega=0,\ \frac{\partial^2\Omega}{\partial\xi^2}=0,\ \frac{\partial^2\Phi}{\partial\eta^2}=0,\ \frac{\partial^2\Phi}{\partial\xi\partial\eta}=0 \tag{2.55}$$

$$\text{当 } \eta=\pm 1 \text{ 时，} \Omega=0,\ \frac{\partial^2\Omega}{\partial\eta^2}=0,\ \frac{\partial^2\Phi}{\partial\xi^2}=0,\ \frac{\partial^2\Phi}{\partial\xi\partial\eta}=0 \tag{2.56}$$

其中

$$W=kL_1\Omega-\Omega \tag{2.57}$$

$$L_1(\cdots)=\left(\frac{\partial^2}{\partial\xi^2}+\lambda^2\frac{\partial^2}{\partial\eta^2}\right)(\cdots),$$

$$L_2(W,\ \Phi)=\frac{\partial^2W}{\partial\xi^2}\frac{\partial^2\Phi}{\partial\eta^2}+\frac{\partial^2W}{\partial\eta^2}\frac{\partial^2\Phi}{\partial\xi^2}-2\frac{\partial^2W}{\partial\xi\partial\eta}\frac{\partial^2\Phi}{\partial\xi\partial\eta} \tag{2.58}$$

三、非线性边值问题的解

我们用摄动法求解无量纲非线性边值问题(2.54)～(2.56)。设夹层矩形板的无量纲中心挠度$W(0,\ 0)$记为W_0，则可将Q，Ω，Φ和W展成如下形式的升幂摄动级数：

$$\left.\begin{aligned}&Q=\alpha_1W_0+\alpha_3W_0^3+\cdots\\ &\Omega=\Omega_1(\xi,\eta)W_0+\Omega_3(\xi,\eta)W_0^3+\cdots\\ &\Phi=\Phi_2(\xi,\eta)W_0^2+\Phi_4(\xi,\eta)W_0^4+\cdots\\ &W=W_1(\xi,\eta)W_0+W_3(\xi,\eta)W_0^3+\cdots\end{aligned}\right\} \tag{3.1}$$

依照定义给出

$$W_1(0,0)=1,\ W_{2i+1}(0,0)=0 \qquad (i=1,2,3,\cdots) \tag{3.2}$$

将式(3.1)代入边值问题(2.54)～(2.56)，使W_0的同次幂相等，便得到以下一系列线性边值问题：

对于一次近似，

$$L_1^2\Omega_1=-\alpha_1 \tag{3.3}$$

$$\left.\begin{aligned}&\text{当 }\xi=\pm1\text{时，}\ \Omega_1=0,\ \frac{\partial^2\Omega_1}{\partial\xi^2}=0\\&\text{当 }\eta=\pm1\text{时，}\ \Omega_1=0,\ \frac{\partial^2\Omega_1}{\partial\eta^2}=0\end{aligned}\right\} \tag{3.4}$$

对于二次近似，

$$L_1^2\Phi_2=-\frac{\lambda^2}{2}L_2(W_1,\ W_1) \tag{3.5}$$

$$\left.\begin{aligned}&\text{当 }\xi=\pm1\text{时，}\ \frac{\partial^2\Phi_2}{\partial u^2}=0,\ \frac{\partial^2\Phi_2}{\partial\xi\partial\eta}=0\\&\text{当 }\eta=\pm1\text{时，}\ \frac{\partial^2\Phi_2}{\partial\xi^2}=0,\ \frac{\partial^2\Phi_2}{\partial\xi\partial\eta}=0\end{aligned}\right\} \tag{3.6}$$

对于三次近似，

$$L_1^2\Omega_3=-\alpha_3-\lambda^2L_2(W_1,\ \Phi_2) \tag{3.7}$$

$$\left.\begin{aligned}&\text{当 }\xi=\pm1\text{时，}\ \Omega_3=0,\ \frac{\partial^2\Omega_3}{\partial\xi^2}=0\\&\text{当 }\eta=\pm1\text{时，}\ \Omega_3=0,\ \frac{\partial^2\Omega_3}{\partial\eta^2}=0\end{aligned}\right\} \tag{3.8}$$

类似地，我们还能得到更高阶近似的线性边值问题。

由式(2.57)，我们还有

$$W_i=kL_1\Omega_i-\Omega_i \qquad (i=1,\ 3,\ \cdots) \tag{3.9}$$

为了得到一阶近似的解，我们取满足边界条件(3.4)的双重Fourier级数：

$$\Omega_1=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\Omega_{ij}^{(1)}\cos\left(i-\frac{1}{2}\right)\pi\xi\cos\left(j-\frac{1}{2}\right)\pi\eta \tag{3.10}$$

将此式代入方程(3.3)，并在方程两端同乘以$\cos(i-1/2)\pi\xi\cos(j-1/2)\pi\eta$，且对$\xi$和$\eta$在各自区间内积分，得

$$\Omega_{ij}^{(1)}=\frac{(-1)^{i+j+1}4\alpha_1}{\pi^6\left(i-\frac{1}{2}\right)\left(j-\frac{1}{2}\right)\left[\left(i-\frac{1}{2}\right)^2+\lambda^2\left(j-\frac{1}{2}\right)^2\right]^2} \tag{3.11}$$

将式(3.10)代入式(3.9)，可得

$$W_1=\alpha_1\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}W_{ij}^{(1)}\cos\left(i-\frac{1}{2}\right)\pi\xi\cos\left(j-\frac{1}{2}\right)\pi\eta \tag{3.12}$$

其中

$$W_{ij}^{(1)}=(-1)^{i+j}\frac{4\pi^2\left[\left(i-\frac{1}{2}\right)^2+\lambda^2\left(j-\frac{1}{2}\right)^2\right]k+4}{\pi^6\left(i-\frac{1}{2}\right)\left(j-\frac{1}{2}\right)\left[\left(i-\frac{1}{2}\right)^2+\lambda^2\left(j-\frac{1}{2}\right)^2\right]^2} \tag{3.13}$$

应用式(3.12)和(3.2)，便有

$$\alpha_1=\beta^{-1} \tag{3.14}$$

其中

$$\beta=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}W_{ij}^{(1)} \tag{3.15}$$

于是，式(3.12)可写为如下形式：

$$W_1=\beta^{-1}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}W_{ij}^{(1)}\cos\left(i-\frac{1}{2}\right)\pi\xi\cos\left(j-\frac{1}{2}\right)\pi\eta \tag{3.16}$$

对于二阶近似，假定方程(3.5)的解为

$$\Phi_2=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_{mn}^{(2)}X_m(\xi)Y_n(\eta) \tag{3.17}$$

它满足边界条件(3.6)，而且$X_m(\xi)$和$Y_n(\eta)$是由下式给定的梁本征函数：

$$\left.\begin{aligned}X_m(\xi)&=\frac{\mathrm{ch}\lambda_m\xi}{\mathrm{ch}\lambda_m}-\frac{\cos\lambda_m\xi}{\cos\lambda_m}\\ Y_n(\eta)&=\frac{\mathrm{ch}\lambda_n\eta}{\mathrm{ch}\lambda_n}-\frac{\cos\lambda_n\eta}{\cos\lambda_n}\qquad(m,\ n=1,\ 2,\cdots)\end{aligned}\right\} \tag{3.18}$$

其中λ_m为下述超越方程的根：

$$\mathrm{th}\lambda_m+\mathrm{tg}\lambda_m=0\qquad(m=1,2,\cdots) \tag{3.19}$$

且给在表1中。

表 1 λ_m的值

m	1	2	3	4	$m>4$
λ_m	2.3650	5.4978	8.6394	11.7810	$\pi(m-0.25)$

函数$X_m(\xi)$和$Y_n(\eta)$满足下列正交关系

$$\left.\begin{aligned}\int_{-1}^{1}X_mX_nd\xi&=\begin{cases}0&(m\neq n)\\2&(m=n)\end{cases}\\ \int_{-1}^{1}Y_mY_nd\eta&=\begin{cases}0&(m\neq n)\\2&(m=n)\end{cases}\end{aligned}\right\} \tag{3.20}$$

将式(3.16)和(3.17)代入方程(3.5)，得

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[(\lambda_m^4+\lambda^4\lambda_n^4)X_mY_n+2\lambda^2\frac{d^2X_m}{d\xi^2}\frac{d^2Y_n}{d\eta^2}\right]\Phi_{mn}^{(1)}$$

$$=\pi^4\beta^{-2}\lambda^2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(i-\frac{1}{2}\right)\left(j-\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\right.$$

$$\cdot \sin\left(i-\frac{1}{2}\right)\pi\xi\sin\left(m-\frac{1}{2}\right)\pi\xi\sin\left(j-\frac{1}{2}\right)\pi\eta\sin\left(n-\frac{1}{2}\right)\pi\eta$$

$$-\left(i-\frac{1}{2}\right)^2\left(n-\frac{1}{2}\right)^2\cos\left(i-\frac{1}{2}\right)\pi\xi\cos\left(m-\frac{1}{2}\right)\pi\xi$$

$$\cdot \cos\left(j-\frac{1}{2}\right)\pi\eta\cos\left(n-\frac{1}{2}\right)\pi\eta\Big]W_{ij}^{(1)}W_{mn}^{(1)} \qquad (3.21)$$

将此方程两端乘以$X_r(\xi)Y_s(\eta)$，对ξ和η在各自区间内积分，并且利用式（3.20），我们便导出下面一组线性代数方程：

$$(\lambda_r^4+\lambda^4\lambda_s^4)\Phi_{rs}^{(2)}+2\lambda^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_m^2\lambda_n^2K^{rm}H^{sn}\Phi_{mn}^{(2)}$$

$$=\frac{\pi^4\lambda^2}{4\beta^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\left(i-\frac{1}{2}\right)\left(j-\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\left(n-\frac{1}{2}\right)M_4^{rim}N_4^{sjn}$$

$$-\left(i-\frac{1}{2}\right)^2\left(n-\frac{1}{2}\right)^2M_1^{rim}N_1^{sjn}\Big]W_{ij}^{(1)}W_{mn}^{(1)} \qquad (r,\ s=1,2,3,\cdots) \qquad (3.22)$$

这里

$$K^{rm}=\frac{1}{2\lambda_m^2}\int_{-1}^{1}X_r\frac{d^2X_m}{d\xi^2}d\xi=\begin{cases}-\dfrac{4\lambda_r}{\lambda_r^4-\lambda_4^4}(\lambda_r\mathrm{tg}\lambda_r-\lambda_m\mathrm{tg}\lambda_m) & (r\neq m)\\ -\mathrm{tg}^2\lambda_r-\dfrac{\mathrm{tg}\lambda_r}{\lambda_r} & (r=m)\end{cases}$$

$$H^{sn}=\frac{1}{2\lambda_n^2}\int_{-1}^{1}Y_s\frac{d^2Y_n}{d\eta^2}d\eta=\begin{cases}-\dfrac{4\lambda_s^2}{\lambda_s^4-\lambda_n^4}(\lambda_s\mathrm{tg}\lambda_s-\lambda_n\mathrm{tg}\lambda_n) & (s\neq n)\\ -\mathrm{tg}^2\lambda_s-\dfrac{\mathrm{tg}\lambda_s}{\lambda_s} & (s=n)\end{cases}$$

$$M_1^{mri}=\int_{-1}^{1}X_m\cos\left(r-\frac{1}{2}\right)\pi\xi\cos\left(i-\frac{1}{2}\right)\pi\xi d\xi=-T_{m,r+i-1}-T_{m,r-i}$$

$$N_1^{nsj}=\int_{-1}^{1}Y_n\cos\left(s-\frac{1}{2}\right)\pi\eta\cos\left(j-\frac{1}{2}\right)\pi\eta d\eta=-T_{n,s+j-1}-T_{n,s-j}$$

$$M_4^{rim}=\int_{-1}^{1}X_r\sin\left(i-\frac{1}{2}\right)\pi\xi\sin\left(m-\frac{1}{2}\right)\pi\xi d\xi=T_{r,i+m-1}-T_{r,i-m}$$

$$N_4^{sjn}=\int_{-1}^{1}Y_s\sin\left(j-\frac{1}{2}\right)\pi\eta\sin\left(n-\frac{1}{2}\right)\pi\eta d\eta=T_{s,j+n-1}-T_{s,j-n}$$

$$T_{m,n}=(-1)^n\frac{2\lambda_m^3\mathrm{tg}\lambda_m}{\lambda_m^4-n^4\pi^4} \qquad (3.23)$$

联立求解这些代数方程，得到$\Phi_{mn}^{(2)}$，由此Φ_2得以确定。

对于三阶近似，我们取双重Fourier级数：

$$\Omega_3=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Omega_{mn}^{(3)}\cos\left(m-\frac{1}{2}\right)\pi\xi\cos\left(n-\frac{1}{2}\right)\pi\eta \qquad (3.24)$$

它满足边界条件(3.8)。现在，我们将此式连同式(3.16)和 (3.17)一起代入方程 (3.7)，便得

$$\pi^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(m-\frac{1}{2}\right)^2+\lambda^2\left(n-\frac{1}{2}\right)^2\right]^2 \Omega_{mn}^{(3)}\cos\left(m-\frac{1}{2}\right)\pi\xi\cos\left(n-\frac{1}{2}\right)\pi\eta$$

$$=-\alpha_3+\frac{\pi^2\lambda^2}{\beta}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(i-\frac{1}{2}\right)^2 X_m\frac{d^2Y_n}{d\eta^2}\cos\left(i-\frac{1}{2}\right)\pi\xi\right.$$

$$\cdot\cos\left(j-\frac{1}{2}\right)\pi\eta+\left(j-\frac{1}{2}\right)^2 Y_n\frac{d^2X_m}{d\xi^2}\cos\left(i-\frac{1}{2}\right)\pi\xi$$

$$\cdot\cos\left(j-\frac{1}{2}\right)\pi\eta+2\left(i-\frac{1}{2}\right)\left(j-\frac{1}{2}\right)\frac{dX_m}{d\xi}\frac{dY_n}{d\eta}\sin\left(i-\frac{1}{2}\right)\pi\xi$$

$$\left.\cdot\sin\left(j-\frac{1}{2}\right)\pi\eta\right]W_{ij}^{(1)}\Phi_{mn}^{(2)} \tag{3.25}$$

将此方程两端同乘以 $\cos(r-1/2)\pi\xi\cos(s-1/2)\pi\eta$，并对 ξ 和 η 在各自区间内积分，即得

$$\Omega_{rs}^{(3)}=(-1)^{r+s+1}\frac{4\alpha_3}{\pi^6\left(r-\frac{1}{2}\right)\left(s-\frac{1}{2}\right)\left[\left(r-\frac{1}{2}\right)^2+\lambda^2\left(s-\frac{1}{2}\right)^2\right]^2}$$

$$+\frac{\lambda^2}{\pi^2\left[\left(r-\frac{1}{2}\right)^2+\lambda^2\left(s-\frac{1}{2}\right)^2\right]^2\beta}$$

$$\cdot\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(i-\frac{1}{2}\right)^2 M_1^{mri}N_2^{nsj}+\left(j-\frac{1}{2}\right)^2 M_2^{mri}N_1^{nsj}\right.$$

$$\left.+2\left(i-\frac{1}{2}\right)\left(j-\frac{1}{2}\right)M_3^{mir}N_3^{njs}\right]W_{ij}^{(1)}\Phi_{mn}^{(2)} \tag{3.26}$$

其中

$$\begin{aligned}
M_2^{mri}&=\int_{-1}^{1}\frac{d^2X_m}{d\xi^2}\cos\left(r-\frac{1}{2}\right)\pi\xi\cos\left(i-\frac{1}{2}\right)\pi\xi d\xi\\
&=\pi^2[(r+i-1)^2T_{m,r+i-1}+(r-i)^2T_{m,r-i}]\\
N_2^{nsj}&=\int_{-1}^{1}\frac{d^2Y_n}{d\eta^2}\cos\left(s-\frac{1}{2}\right)\pi\eta\cos\left(j-\frac{1}{2}\right)\pi\eta d\eta\\
&=\pi^2[(s+j-1)^2T_{n,s+j-1}+(s-j)^2T_{n,s-j}]\\
M_3^{mir}&=\int_{-1}^{1}\frac{dX_m}{d\xi}\sin\left(i-\frac{1}{2}\right)\pi\xi\cos\left(r-\frac{1}{2}\right)\pi\xi d\xi\\
&=\pi[(r+i-1)T_{m,r+i-1}-(r-i)T_{m,r-i}]\\
N_3^{njs}&=\int_{-1}^{1}\frac{dY_n}{d\eta}\sin\left(j-\frac{1}{2}\right)\pi\eta\cos\left(s-\frac{1}{2}\right)\pi\eta d\eta\\
&=\pi[(s+j-1)T_{n,s+j-1}-(s-j)T_{n,s-j}]
\end{aligned} \tag{3.27}$$

将式 (3.24) 代入式(3.9)，便得

$$W_3=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\Big\{a_3W_{rs}^{(1)}-\frac{\pi^4\lambda^2}{4\beta}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{r+s}\Big(r-\frac{1}{2}\Big)\Big(j-\frac{1}{2}\Big)$$

$$\cdot\Big[\Big(i-\frac{1}{2}\Big)^2M_1^{mri}N_2^{nsj}+\Big(j-\frac{1}{2}\Big)^2M_2^{mri}N_1^{nsj}+2\Big(i-\frac{1}{2}\Big)\Big(j-\frac{1}{2}\Big)$$

$$\cdot M_3^{mir}N_3^{njs}\Big]W_{rs}^{(1)}W_{ij}^{(1)}\Phi_{mn}^{(2)}\Big\}\cos\Big(r-\frac{1}{2}\Big)\pi\xi\cos\Big(s-\frac{1}{2}\Big)\pi\eta \qquad (3.28)$$

应用式（3.2），由上式可得

$$\alpha_3=\frac{\pi^4\lambda^2}{4\beta^2}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{r+s}\Big(r-\frac{1}{2}\Big)\Big(s-\frac{1}{2}\Big)\Big[\Big(i-\frac{1}{2}\Big)^2$$

$$\cdot M_1^{mri}N_2^{nsj}+\Big(j-\frac{1}{2}\Big)^2M_2^{mri}N_1^{nsj}+2\Big(i-\frac{1}{2}\Big)\Big(j-\frac{1}{2}\Big)M_3^{mir}N_3^{njs}\cdot\Big]$$

$$\cdot W_{rs}^{(1)}W_{ij}^{(1)}\Phi_{mn}^{(1)} \qquad (3.29)$$

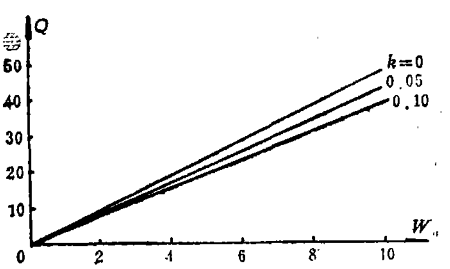
四、结果与讨论

由上一节的解(3.1)，我们得到夹层矩形板的载荷与中心挠度的特征关系式为

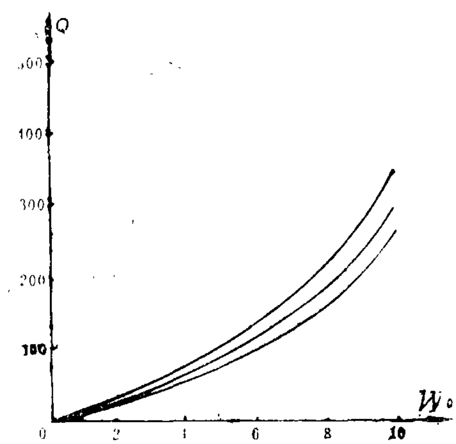
$$Q=\alpha_1W_0+\alpha_3W_0^3 \qquad (4.1)$$

应用这一公式，我们得到了下面一些有用的数值结果．对边长比λ和特征参数k的不同值，均布载荷Q对中心挠度W_0的曲线给在图2a～d中．由图看到，对于相同的Q值，具有较大k值或较小λ值的夹层矩形板将产生较大的中心挠度．

特别地，在$k=0$，$t=\dfrac{h}{2}$和$h_0=\dfrac{h}{\sqrt{3}}$的情况下，上述夹层矩形板的边值问题就转化为厚度为h的单层矩形板的边值问题．因此，前面的公式也给出了均布载荷作用下简支矩形板的大挠度问题的解．为了说明本文解的正确性，我们在图3中给出了简支正方形板（$\lambda=1$）情况下



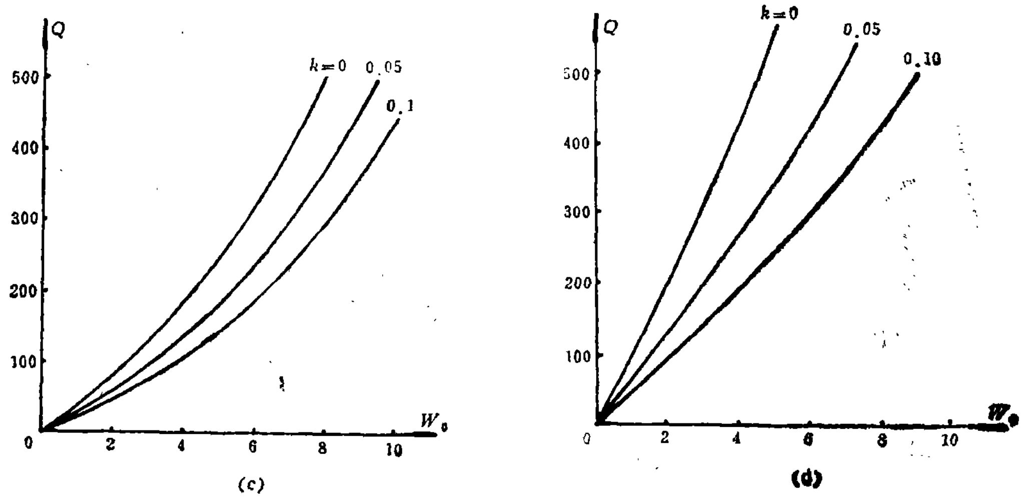
(a)



(b)

(a)$\lambda=0$，(b)$\lambda=1$，

图 2



(c)$\lambda=1.5$, (d)$\lambda=2$

图2 夹层矩形板各种k和λ值下的载荷-挠度曲线

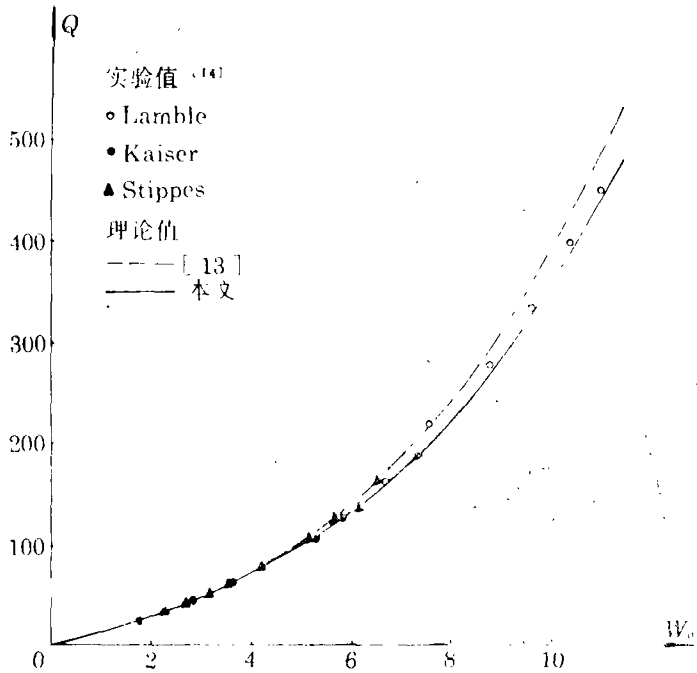


图3 简支正方形板的理论和实验结果之间的比较

的本文结果以及文献[13]和[14]的理论值与实验值。由图看到，本文结果与实验值十分吻合。由此可得出结论：本文公式的精确性是十分令人满意的。

参 考 文 献

[1] Reissner, E., Finite deflection of sandwich plates, *J. Aeron. Sci.*, 15(7)(1948), 435; 17(2)(1950), 125.

[2] 刘人怀，夹层圆板的非线性弯曲，应用数学和力学，2(2)(1981)，173.

[3] 刘人怀，在边缘力矩作用下夹层圆板的非线性轴对称弯曲问题，中国科学技术大学学报，10(2)(1980)，56.

[4] 刘人怀、施云方，夹层圆板大挠度问题的精确解，应用数学和力学，3(1)(1982)，11.

[5] Liu Ren-huai, Nonlinear bending of circular sandwich plates under the action of axisymmetric uniformly distributed line loads, *Progress in Applied Mechanics*(edited by Yeh Kai-yuan), Martinus Nijhoff Publishers, Dordrecht(1987), 293.

[6] 刘人怀、朱高秋，夹层圆板大挠度问题的进一步研究，应用数学和力学，10(12)(1989)，1041.

[7] 刘人怀、吴建成，夹层矩形板的非线性振动，中国科学，A辑，(9)(1991).

[8] Kan, H. P. and J. C. Huang, Large deflection of rectangular sandwich plates, *AIAA Journal*, 5(9)(1967), 1706.

[9] Alwan, A. M., Bending of sandwich plates with large deflection, *Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers*, 93(EM3)(1967), 83.

[10] Kamiya, N., Governing equations for large deflection of sandwich plates, *AIAA Journal*, 5(9)(1967), 1706.

[11] 中国科学院北京力学研究所固体力学研究室板壳组，《夹层板壳的弯曲、稳定和振动》，科学出版社，北京(1977).

[12] 胡海昌，各向同性夹层板反对称小挠度的若干问题，力学学报，6(1)(1963)，53.

[13] Sundara Raja Iyengar, K. T. and M. M. Naqvi, Large deflection of rectangular plates, *Int. J. Non-Linear Mech.*, 1(2)(1966), 109.

[14] Brown, J. C. and J. M. Harvey, Large deflections of rectangular plates subjected to uniform lateral pressure and compressive edge loading, *J. Mech. Eng. Sci.*, 11(3)(1969), 305.

Nonlinear Bending of Simply Supported Rectangular Sandwich Plates

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Abstract

In this paper, fundamental equations and boundary conditions of the nonlinear bending theory for a rectangular sandwich plate with a soft core are derived by means of the method of calculus of variations. Then the nonlinear bending for a simply supported rectangular sandwich plate under the uniform lateral load is investigated by use of the perturbation method and a quite accurate analytic solution is obtained.

Key words rectangular sandwich place, nonlinear bending