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复合材料单层板非弹性主方向的裂纹尖端应变能释放率*

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摘　　要

本文研究线弹性正交异性复合材料单层板非弹性主方向的断裂问题．推出了非弹性主方向坐标系和弹性主方向坐标系的特征根和柔度系数的变换公式．将裂纹尖端应力与位移代入应变能释放率的基本公式，得到了在斜对称载荷作用下，用弹性主方向坐标系的工程参量表示的裂纹尖端应变能释放率的计算公式．

关键词　应变能释放率　非弹性主方向　中心贯穿裂纹　斜对称载荷　正交异性复合材料

一、引　　言

在断裂力学中，应变能释放率是一个常用的重要参量．对于线弹性正交异性复合材料单层板弹性主方向的裂纹尖端应变能释放率，文[1]～[2]给出了相应的计算公式．本文研究含中心贯穿裂纹，受斜对称载荷作用的线弹性正交异性复合材料单层板，非弹性主方向的裂纹尖端应变能释放率．利用非弹性主方向坐标系和弹性主方向坐标系的特征根、柔度系数和应力强度因子之间的变换公式，推出了用弹性主方向坐标系的工程参量表示的，非弹性主方向的裂纹尖端应变能释放率的计算公式．作为特例，得到了弹性主方向的Ⅱ型裂纹尖端的应变能释放率的计算公式．对于含中心贯穿裂纹受对称载荷作用的情况，在文[3]中已有探讨．

二、应 力 和 位 移

设线弹性正交异性复合材料单层板，含长度$2a$且位于1轴上的中心贯穿裂纹，受斜对称载荷τ作用．其非弹性主方向坐标系x-y与弹性主方向坐标系1-2不重合，x轴转向1轴的角度为α，如图1所示．

仿照文[1]，[4]～[6]可以得到裂纹尖端的应力和位移如下：

$$\sigma_x=\frac{k_2}{\sqrt{2\pi r}}\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[\frac{\mu_2^2}{(\cos\theta+\mu_2\sin\theta)^{1/2}}-\frac{\mu_1^2}{(\cos\theta+\mu_1\sin\theta)^{1/2}}\right]\right\}$$
$$+\frac{k_1}{\sqrt{2\pi r}}\mathrm{Re}\left\{\frac{\mu_1\mu_2}{\mu_1-\mu_2}\left[\frac{\mu_2}{(\cos\theta+\mu_2\sin\theta)^{1/2}}-\frac{\mu_1}{(\cos\theta+\mu_1\sin\theta)^{1/2}}\right]\right\} \tag{2.1a}$$

* 汤任基推荐．

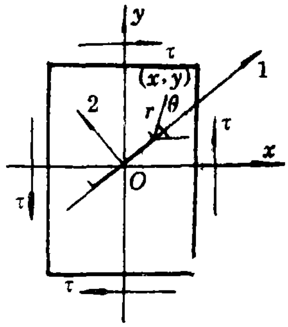

图 1

$$\sigma_y=\frac{k_2}{\sqrt{2\pi r}}\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[\frac{1}{(\cos\theta+\mu_2\sin\theta)^{1/2}}-\frac{1}{(\cos\theta+\mu_1\sin\theta)^{1/2}}\right]\right\}$$

$$+\frac{k_1}{\sqrt{2\pi r}}\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[\frac{\mu_1}{(\cos\theta+\mu_2\sin\theta)^{1/2}}-\frac{\mu_2}{(\cos\theta+\mu_1\sin\theta)^{1/2}}\right]\right\} \tag{2.1b}$$

$$\tau_{xy}=\frac{k_2}{\sqrt{2\pi r}}\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[\frac{\mu_1}{(\cos\theta+\mu_1\sin\theta)^{1/2}}-\frac{\mu_2}{(\cos\theta+\mu_2\sin\theta)^{1/2}}\right]\right\}$$

$$+\frac{k_1}{\sqrt{2\pi r}}\mathrm{Re}\left\{\frac{\mu_1\mu_2}{\mu_1-\mu_2}\left[\frac{1}{(\cos\theta+\mu_1\sin\theta)^{1/2}}-\frac{1}{(\cos\theta+\mu_2\sin\theta)^{1/2}}\right]\right\} \tag{2.1c}$$

和 $$u=k_2\sqrt{\frac{2r}{\pi}}\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[p_2(\cos\theta+\mu_2\sin\theta)^{1/2}-p_1(\cos\theta+\mu_1\sin\theta)^{1/2}\right]\right\}$$

$$+k_1\sqrt{\frac{2r}{\pi}}\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[\mu_1p_2(\cos\theta+\mu_2\sin\theta)^{1/2}-\mu_2p_1(\cos\theta+\mu_1\sin\theta)^{1/2}\right]\right\} \tag{2.2a}$$

$$v=k_2\sqrt{\frac{2r}{\pi}}\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[\frac{q_2}{\mu_2}(\cos\theta+\mu_2\sin\theta)^{1/2}-\frac{q_1}{\mu_1}(\cos\theta+\mu_1\sin\theta)^{1/2}\right]\right\}$$

$$+k_1\sqrt{\frac{2r}{\pi}}\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[\mu_1\frac{q_2}{\mu_2}(\cos\theta+\mu_2\sin\theta)^{1/2}-\mu_2\frac{q_1}{\mu_1}(\cos\theta+\mu_1\sin\theta)^{1/2}\right]\right\} \tag{2.2b}$$

其中单层板关于非弹性主方向坐标系x-y的应力强度因子k_2，k_1为

$$k_2=\tau\sqrt{\pi a}\cos^2\alpha,\quad k_1=-\tau\sqrt{\pi a}\sin\alpha\cos\alpha \tag{2.3}$$

(r,θ)是以裂纹尖端为极点的极坐标，由图1，它和(x,y)之间有关系：

$$x-a\cos\alpha=r\cos\theta,\quad y-a\sin\alpha=r\sin\theta\qquad(r\ll a) \tag{2.4}$$

Re{ }表示对{ }内的复数取实部. μ_1，μ_2是单层板关于非弹性主方向坐标系x-y的特征根，满足下列方程：

$$a_{11}\mu^4-2a_{16}\mu^3+(2a_{12}+a_{66})\mu^2-2a_{26}\mu+a_{22}=0 \tag{2.5}$$

a_{11}，a_{12}，a_{22}，a_{16}，a_{26}，a_{66}是单层板关于非弹性主方向坐标系x-y的柔度系数. 且

$$p_j=a_{11}\mu_j^2+a_{12}-a_{16}\mu_j\qquad(j=1,2) \tag{2.6a}$$

$$q_j=a_{12}\mu_j^2+a_{22}-a_{26}\mu_j\qquad(j=1,2) \tag{2.6b}$$

三、参量变换公式

单层板关于非弹性主方向坐标系x-y和弹性主方向坐标系1-2的特征根μ_1，μ_2，s_1，s_2之间，有下列变换公式[7]：

$$\mu_1=\frac{s_1\cos\alpha+\sin\alpha}{\cos\alpha-s_1\sin\alpha},\quad \mu_2=\frac{s_2\cos\alpha+\sin\alpha}{\cos\alpha-s_2\sin\alpha} \tag{3.1a,b}$$

其中s_1，s_2是单层板关于弹性主方向坐标系1-2的特征根，满足下列方程：

$$b_{11}s^4+(2b_{12}+b_{66})s^2+b_{22}=0 \tag{3.2}$$

b_{11}，b_{12}，b_{22}，b_{66}是单层板关于弹性主方向坐标系1-2的柔度系数，这时$b_{16}=b_{26}=0$。记

$$\alpha_0=\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}},\quad \beta_0=\frac{2b_{12}+b_{66}}{2b_{11}} \tag{3.3a}$$

$$\Delta=\beta_0^2-\alpha_0^2 \tag{3.3b}$$

则当$\beta_0<\alpha_0$即$\Delta<0$时，有

$$s_1=\left(\frac{\alpha_0-\beta_0}{2}\right)^{\frac{1}{2}}+\left(\frac{\alpha_0+\beta_0}{2}\right)^{\frac{1}{2}}i,\quad s_2=-\left(\frac{\alpha_0-\beta_0}{2}\right)^{\frac{1}{2}}+\left(\frac{\alpha_0+\beta_0}{2}\right)^{\frac{1}{2}}i \tag{3.4a,b}$$

当$\beta_0>\alpha_0$即$\Delta>0$时，有

$$s_1=[\beta_0-(\beta_0^2-\alpha_0^2)^{1/2}]^{1/2}i,\quad s_2=[\beta_0+(\beta_0^2-\alpha_0^2)^{1/2}]^{1/2}i \tag{3.5a,b}$$

由(3.4)，(3.5)易知

$$(s_1+s_2)i=-\sqrt{2}(\alpha_0+\beta_0)^{1/2},\quad s_1s_2=-\alpha_0 \tag{3.6a,b}$$

且
$$s_1+s_2=-(\bar{s}_1+\bar{s}_2),\quad s_1s_2=\bar{s}_1\bar{s}_2 \tag{3.7a,b}$$

由(3.1)，得到

$$\mu_1+\mu_2=\frac{s_1\cos\alpha+\sin\alpha}{\cos\alpha-s_1\sin\alpha}+\frac{s_2\cos\alpha+\sin\alpha}{\cos\alpha-s_2\sin\alpha}$$

$$=\sum_{j=1}^{2}\frac{(s_j\cos\alpha+\sin\alpha)(\cos\alpha-\bar{s}_j\sin\alpha)}{(\cos\alpha-s_j\sin\alpha)(\cos\alpha-\bar{s}_j\sin\alpha)}$$

将右端通分求和，利用(3.7)，得到

$$\text{分子}=(s_1+s_2)[\cos^4\alpha+(1-s_1s_2)\cos^2\alpha\sin^2\alpha-s_1s_2\sin^4\alpha]$$
$$+\cos\alpha\sin\alpha[2(1-s_1s_2)(\cos^2\alpha+s_1s_2\sin^2\alpha)+(s_1+s_2)^2(\cos^2\alpha-\sin^2\alpha)]$$

$$\text{分母}=\cos^4\alpha+[2s_1s_2-(s_1+s_2)^2]\cos^2\alpha\sin^2\alpha+(s_1s_2)^2\sin^4\alpha$$

由(3.6)可知s_1+s_2是纯虚数，s_1s_2是实数，从而分子的前项是纯虚数，后项是实数，且分母是实数，于是

$$\mathrm{Re}[(\mu_1+\mu_2)i]=\frac{[\cos^4\alpha+(1-s_1s_2)\cos^2\alpha\sin^2\alpha-s_1s_2\sin^4\alpha](s_1+s_2)i}{\cos^4\alpha+[2s_1s_2-(s_1+s_2)^2]\cos^2\alpha\sin^2\alpha+(s_1s_2)^2\sin^4\alpha}$$

利用$\cos^2\alpha+\sin^2\alpha=1$，上式化为

$$\mathrm{Re}[(\mu_1+\mu_2)i]=\frac{(\cos^2\alpha-s_1s_2\sin^2\alpha)(s_1+s_2)i}{\cos^4\alpha+[2s_1s_2-(s_1+s_2)^2]\cos^2\alpha\sin^2\alpha+(s_1s_2)^2\sin^4\alpha} \tag{3.8a}$$

由(3.6)和(3.3)，上式可改写为

$$\mathrm{Re}[(\mu_1+\mu_2)i]=\frac{-\sqrt{2}(\alpha_0+\beta_0)^{1/2}(\cos^2\alpha+\alpha_0\sin^2\alpha)}{\cos^4\alpha+2\beta_0\cos^2\alpha\sin^2\alpha+\alpha_0^2\sin^4\alpha} \tag{3.8b}$$

和 $$\mathrm{Re}\,[(\mu_1+\mu_2)i]=\frac{-\sqrt{2}\left[\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}+\frac{2b_{12}+b_{66}}{2b_{11}}\right]^{\frac{1}{2}}\left[\cos^2\alpha+\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}\sin^2\alpha\right]}{\cos^4\alpha+2\cdot\frac{2b_{12}+b_{66}}{2b_{11}}\cos^2\alpha\sin^2\alpha+\frac{b_{22}}{b_{11}}\sin^4\alpha}\tag{3.8c}$$

类似地，可以得到

$$\mathrm{Re}\,(\mu_1\mu_2 i)=\frac{\cos\alpha\sin\alpha(1+s_1s_2)(s_1+s_2)i}{\cos^4\alpha+[2s_1s_2-(s_1+s_2)^2]\cos^2\alpha\sin^2\alpha+(s_1s_2)^2\sin^4\alpha}\tag{3.9a}$$

$$\mathrm{Re}\,(\mu_1\mu_2 i)=\frac{-\sqrt{2}(\alpha_0+\beta_0)^{1/2}(1-\alpha_0)\cos\alpha\sin\alpha}{\cos^4\alpha+2\beta_0\cos^2\alpha\sin^2\alpha+\alpha_0^2\sin^4\alpha}\tag{3.9b}$$

和 $$\mathrm{Re}\,(\mu_1\mu_2 i)=\frac{-\sqrt{2}\left[\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}+\frac{2b_{12}+b_{66}}{2b_{11}}\right]^{\frac{1}{2}}\left[1-\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}\right]\cos\alpha\sin\alpha}{\cos^4\alpha+2\cdot\frac{2b_{12}+b_{66}}{2b_{11}}\cos^2\alpha\sin^2\alpha+\frac{b_{22}}{b_{11}}\sin^4\alpha}\tag{3.9c}$$

$$\mathrm{Re}\left(\frac{1}{\mu_1\mu_2}i\right)=\frac{-\sin\alpha\cos\alpha(1+s_1s_2)(s_1+s_2)i}{\sin^4\alpha+[2s_1s_2-(s_1+s_2)^2]\sin^2\alpha\cos^2\alpha+(s_1s_2)^2\cos^4\alpha}\tag{3.10a}$$

$$\mathrm{Re}\left(\frac{1}{\mu_1\mu_2}i\right)=\frac{\sqrt{2}(\alpha_0+\beta_0)^{1/2}(1-\alpha_0)\sin\alpha\cos\alpha}{\sin^4\alpha+2\beta_0\sin^2\alpha\cos^2\alpha+\alpha_0^2\cos^4\alpha}\tag{3.10b}$$

和 $$\mathrm{Re}\left(\frac{1}{\mu_1\mu_2}i\right)=\frac{\sqrt{2}\left[\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}+\frac{2b_{12}+b_{66}}{2b_{11}}\right]^{\frac{1}{2}}\left[1-\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}\right]\sin\alpha\cos\alpha}{\sin^4\alpha+2\cdot\frac{2b_{12}+b_{66}}{2b_{11}}\sin^2\alpha\cos^2\alpha+\frac{b_{22}}{b_{11}}\cos^4\alpha}\tag{3.10c}$$

$$\mathrm{Re}\left[\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)i\right]=\frac{-(\sin^2\alpha-s_1s_2\cos^2\alpha)(s_1+s_2)i}{\sin^4\alpha+[2s_1s_2-(s_1+s_2)^2]\sin^2\alpha\cos^2\alpha+(s_1s_2)^2\cos^4\alpha}\tag{3.11a}$$

$$\mathrm{Re}\left[\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)i\right]=\frac{\sqrt{2}(\alpha_0+\beta_0)^{1/2}(\sin^2\alpha+\alpha_0\cos^2\alpha)}{\sin^4\alpha+2\beta_0\sin^2\alpha\cos^2\alpha+\alpha_0^2\cos^4\alpha}\tag{3.11b}$$

和 $$\mathrm{Re}\left[\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)i\right]=\frac{\sqrt{2}\left[\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}+\frac{2b_{12}+b_{66}}{2b_{11}}\right]^{\frac{1}{2}}\left[\sin^2\alpha+\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}\cos^2\alpha\right]}{\sin^4\alpha+2\cdot\frac{2b_{12}+b_{66}}{2b_{11}}\sin^2\alpha\cos^2\alpha+\frac{b_{22}}{b_{11}}\cos^4\alpha}\tag{3.11c}$$

单层板关于两个坐标系x-y和1-2的柔度系数之间，有下列变换公式[7],[8]：

$$a_{11}=b_{11}\cos^4\alpha+(2b_{12}+b_{66})\cos^2\alpha\sin^2\alpha+b_{22}\sin^4\alpha\tag{3.12a}$$

$$a_{12}=b_{12}+(b_{11}+b_{22}-2b_{12}-b_{66})\sin^2\alpha\cos^2\alpha\tag{3.12b}$$

$$a_{22}=b_{11}\sin^4\alpha+(2b_{12}+b_{66})\sin^2\alpha\cos^2\alpha+b_{22}\cos^4\alpha\tag{3.12c}$$

$$a_{16}=(2b_{11}-2b_{12}-b_{66})\cos^3\alpha\sin\alpha-(2b_{22}-2b_{12}-b_{66})\cos\alpha\sin^3\alpha\tag{3.12d}$$

$$a_{26}=(2b_{11}-2b_{12}-b_{66})\sin^3\alpha\cos\alpha-(2b_{22}-2b_{12}-b_{66})\sin\alpha\cos^3\alpha\tag{3.12e}$$

$$a_{66}=b_{66}+4(b_{11}+b_{22}-2b_{12}-b_{66})\sin^2\alpha\cos^2\alpha\tag{3.12f}$$

显然，(3.12a)和(3.12c)可改写为

$$a_{11}=b_{11}\left(\cos^4\alpha+2\cdot\frac{2b_{12}+b_{66}}{2b_{11}}\cos^2\alpha\sin^2\alpha+\frac{b_{22}}{b_{11}}\sin^4\alpha\right)\tag{3.13a}$$

$$a_{11}=b_{11}(\cos^4\alpha+2\beta_0\cos^2\alpha\sin^2\alpha+\alpha_0^2\sin^4\alpha)\tag{3.13b}$$

和 $$a_{11}=b_{11}\{\cos^4\alpha+[2s_1s_2-(s_1+s_2)^2]\cos^2\alpha\sin^2\alpha+(s_1s_2)^2\sin^4\alpha\}\tag{3.13c}$$

$$a_{22}=b_{11}\left(\sin^4\alpha+2\cdot\frac{2b_{12}+b_{66}}{2b_{11}}\sin^2\alpha\cos^2\alpha+\frac{b_{22}}{b_{11}}\cos^4\alpha\right) \tag{3.14a}$$

$$a_{22}=b_{11}(\sin^4\alpha+2\beta_0\sin^2\alpha\cos^2\alpha+\alpha_0^2\cos^4\alpha) \tag{3.14b}$$

和

$$a_{22}=b_{11}\{\sin^4\alpha+[2s_1s_2-(s_1+s_2)^2]\sin^2\alpha\cos^2\alpha+(s_1s_2)^2\cos^4\alpha\} \tag{3.14c}$$

单层板关于两个坐标系x-y和1-2的应力强度因子之间，据(2.3)，有变换公式：

$$k_2=K_{\rm I}\cos^2\alpha,\quad k_1=-K_{\rm I}\sin\alpha\cos\alpha \tag{3.15}$$

其中

$$K_{\rm I}=\tau\sqrt{\pi a} \tag{3.16}$$

是单层板关于弹性主方向坐标系1-2的Ⅱ型应力力强度因子.

四、应 变 能 释 放 率

线弹性正交异性复合材料单层板非弹性主方向的裂纹尖端应变能释放率可由下列积分表示[1]：

$$G=\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta[\sigma_y(\delta-r,0)\cdot v(r,\pi)+\tau_{xy}(\delta-r,0)\cdot u(r,\pi)]dr \tag{4.1}$$

由(2.6)，有

$$\mathrm{Re}\left[\frac{1}{\mu_1-\mu_2}(p_2-p_1)i\right]=-a_{11}\mathrm{Re}[(\mu_1+\mu_2)i] \tag{4.2a}$$

$$\mathrm{Re}\left[\frac{1}{\mu_1-\mu_2}(\mu_1p_2-\mu_2p_1)i\right]=-a_{11}\mathrm{Re}(\mu_1\mu_2i) \tag{4.2b}$$

$$\mathrm{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\frac{q_2}{\mu_2}-\frac{q_1}{\mu_1}\right)i\right]=a_{22}\mathrm{Re}\left(\frac{1}{\mu_1\mu_2}i\right) \tag{4.2c}$$

$$\mathrm{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\mu_1\frac{q_2}{\mu_2}-\mu_2\frac{q_1}{\mu_1}\right)i\right]=a_{22}\mathrm{Re}\left[\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)i\right] \tag{4.2d}$$

由(2.1)、(2.2)，并利用(4.2)，得到

$$\sigma_y(\delta-r,0)=k_1/\sqrt{2\pi(\delta-r)} \tag{4.3a}$$

$$\tau_{xy}(\delta-r,0)=k_2/\sqrt{2\pi(\delta-r)} \tag{4.3b}$$

和

$$u(r,\pi)=-a_{11}\sqrt{2r/\pi}\,\{k_2\mathrm{Re}[(\mu_1+\mu_2)i]+k_1\mathrm{Re}(\mu_1\mu_2i)\} \tag{4.4a}$$

$$v(r,\pi)=a_{22}\sqrt{2r/\pi}\,\{k_2\mathrm{Re}(i/\mu_1\mu_2)+k_1\mathrm{Re}[(1/\mu_1+1/\mu_2)i]\} \tag{4.4b}$$

将(4.3)，(4.4)代入(4.1)，有

$$G=\Big\{-k_2^2a_{11}\mathrm{Re}[(\mu_1+\mu_2)i]-k_1k_2[a_{11}\mathrm{Re}(\mu_1\mu_2i)-a_{22}\mathrm{Re}(i/\mu_1\mu_2)]$$
$$+k_1^2a_{22}\mathrm{Re}\left[\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)i\right]\Big\}\cdot\lim_{\delta\to 0}\frac{1}{\pi\delta}\int_0^\delta\sqrt{\frac{r}{\delta-r}}dr$$

在积分中换元：$r=\delta\sin^2t$，$0\leqslant t\leqslant\pi/2$，则

$$\int_0^\delta\sqrt{\frac{r}{\delta-r}}\,dr=\frac{\pi\delta}{2}$$

于是得到用非弹性主方向的特征根表示的裂纹尖端应变能释放率计算公式：

$$G=-k_2^2\frac{a_{11}}{2}\mathrm{Re}[(\mu_1+\mu_2)i]-k_1k_2\left[\frac{a_{11}}{2}\mathrm{Re}(\mu_1\mu_2i)\right.$$
$$\left.-\frac{a_{22}}{2}\mathrm{Re}\left(\frac{1}{\mu_1\mu_2}i\right)\right]+k_1^2\frac{a_{22}}{2}\mathrm{Re}\left[\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)i\right] \tag{4.5}$$

因为对于复数z，有$\mathrm{Re}(zi)=-\mathrm{Im}(z)$，所以上式与文[1]中给出的公式(26a)相同．将

(3.8)～(3.11)，(3.13)～(3.15)代入(4.5)，得到依次用弹性主方向的特征根、物理常数和柔度系数表示的裂纹尖端应变能释放率计算公式：

$$G=-K_{\mathrm{I}}^{2}\frac{b_{11}}{2}\cos^2\alpha(\cos^2 2\alpha-s_1 s_2\sin^2 2\alpha)(s_1+s_2)i \tag{4.6a}$$

$$G=K_{\mathrm{I}}^{2}\frac{b_{11}}{\sqrt{2}}(\alpha_0+\beta_0)^{1/2}(\cos^2 2\alpha+\alpha_0\sin^2 2\alpha)\cos^2\alpha \tag{4.6b}$$

和

$$G=K_{\mathrm{I}}^{2}\frac{b_{11}}{\sqrt{2}}\left[\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}+\frac{2b_{12}+b_{66}}{2b_{11}}\right]^{\frac{1}{2}}\left[\cos^2 2\alpha+\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}\sin^2 2\alpha\right]\cos^2\alpha \tag{4.6c}$$

注意到

$$b_{11}=\frac{1}{E_1},\ b_{12}=-\frac{\nu_{12}}{E_1}=-\frac{\nu_{21}}{E_2},\ b_{22}=\frac{1}{E_2},\ b_{66}=\frac{1}{G_{12}} \tag{4.7}$$

其中E_1，E_2，ν_{12}，ν_{21}，G_{12}是单层板关于弹性主方向坐标系1-2的工程常数．将(4.7)代入(4.6c)，得到用工程常数表示的裂纹尖端应变能释放率计算公式：

$$G=\frac{K_{\mathrm{I}}^{2}}{E_1}\frac{1}{\sqrt{2}}\left[\left(\frac{E_1}{E_2}\right)^{\frac{1}{2}}+\frac{E_1}{2G_{12}}-\nu_{12}\right]^{\frac{1}{2}}\left[\cos^2 2\alpha+\left(\frac{E_1}{E_2}\right)^{\frac{1}{2}}\sin^2 2\alpha\right]\cos^2\alpha \tag{4.6d}$$

最后我们指出，弹性主方向的Ⅰ型裂纹，可以看作上述情况当$\alpha=0$时的特例．在(4.6)中令$\alpha=0$我们依次得到计算公式：

$$G_{\mathrm{I}}=-K_{\mathrm{I}}^{2}\frac{b_{11}}{2}(s_1+s_2)i,\quad G_{\mathrm{I}}=K_{\mathrm{I}}^{2}\frac{b_{11}}{\sqrt{2}}(\alpha_0+\beta_0)^{1/2} \tag{4.8a,b}$$

$$G_{\mathrm{I}}=K_{\mathrm{I}}^{2}\frac{b_{11}}{\sqrt{2}}\left[\left(\frac{b_{22}}{b_{11}}\right)^{\frac{1}{2}}+\frac{2b_{12}+b_{66}}{2b_{11}}\right]^{\frac{1}{2}},\quad G_{\mathrm{I}}=\frac{K_{\mathrm{I}}^{2}}{E_1}\frac{1}{\sqrt{2}}\left[\left(\frac{E_1}{E_2}\right)^{\frac{1}{2}}+\frac{E_1}{2G_{12}}-\nu_{12}\right]^{\frac{1}{2}} \tag{4.8c,d}$$

其中(4.8c)正是文[1],[2]中所给出的弹性主方向的Ⅰ型裂纹尖端应变能释放率的计算公式．

参 考 文 献

[1] Sih, G. C., P. C. Paris and G. R. Irwin, On cracks in rectilinearly anisotropic bodies, *Intern. J. Fracture Mechanics*, 1(3) (1965), 189—203.

[2] Corten, H. T., *Fracture Mechanics of Composites*, in *Fracture* Ⅶ, Ed. by H. Liebowitz, Academic Press, New York (1972), 710—716.

[3] 杨维阳、杜八先，复合材料单向板裂纹尖端的应变能释放率，太原重型机械学院学报，13(3) (1992), 143—151.

[4] Sih, G. C., E. P. Chen, S. L. Huang and E. J. McQuillen, Material characterization on the fracture of filament-reinforced composites, *J. Composite Materials*, (9) (1975), 167—186.

[5] Sih, G. C. and E. P. Chen, *Cracks in Composite Materials*, in *Mechanics of Fracture* Ⅵ, Ed. by G. C. Sih, Martinus Nijhoff Publishers, Hague (1981), 26—34.

[6] 杨维阳，正交异性复合材料单向板非弹性主方向的裂纹尖端应变与位移，太原重型机械学院学报，12(1) (1991), 30—41.

[7] С. Г. 列赫尼茨基，《各向异性板》，胡海昌译，科学出版社 (1963), 34—42.

[8] 刘锡礼、王秉权，《复合材料力学基础》，中国建筑工业出版社 (1984), 25—34.

On Crack-Tip Strain Energy Release Rate in Non-Principal Directions of Elasticity for Simple Layer Plate of Composite Materials

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Abstract

In this paper, the fracture problem in non-principal directions of elasticity for a simple layer plate of linear-elastic orthotropic composite materials is studied. The formulae of transformation between characteristic roots, coefficients of elastic compliances in non-principal directions of elasticity and corresponding parameters in principal directions of elasticity are derived. Then, the computing formulae of strain energy release rate under skew-symmetric loading in terms of engineering parameters for principal directions of elasticity are obtained by substituting crack-tip stresses and displacements into the basic formula of the strain energy release rate.

Key words strain energy release rate, non-principal directions of elasticity, central through crack, skew-symmetric loading, orthotropic composites