应用数学和力学，第15卷第4期（1994年4月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

一类非线性积分微分方程的初边值问题

崔尚斌　　屈长征

（兰州大学数学系）　　（西安 西北大学数学系）

（钱伟长推荐，1992年11月6日收到）

摘　　要

本文讨论下列初边值问题整体经典解的存在性：

$$\begin{cases} u_{tt}-u_{xx}+\int_0^t\lambda(t-s)\sigma(u,u_x)_x ds=f(x,t,u,u_t), & a<x<b,t>0 \\ u\Big|_{x=a}=0,\ u\Big|_{x=b}=0, & t\geqslant 0 \\ u\Big|_{t=0}=\varphi(x),\ u_t\Big|_{t=0}=\psi(x), & a\leqslant x\leqslant b \end{cases}$$

该问题所描述的是一类具有非线性粘弹性的粘弹性杆的非线性振动。在一定条件下，我们证明了该问题整体经典解的存在唯一性.

关键词　积分微分方程　初边值问题　整体经典解

一、引　　言

积分微分方程是近年来研究十分活跃的一个课题，其中来源于粘弹性力学的一些积分微分方程尤其为人们所重视，参见［2～11］. 在文献［9～11］中我们讨论了描述具有广义Maxwell 线性粘弹性的粘弹性杆之非线性纵振动的积分微分方程

$$u_{tt}-u_{xx}+\int_0^t\lambda(t-s)u_{xx}(x,s)ds=f(x,t,u,u_t) \tag{1.1}$$

的初边值问题整体解的存在性、稳定性以及解的爆破问题，本文我们讨论下列非线性积分微分方程

$$u_{tt}-u_{xx}+\int_0^t\lambda(t-s)\sigma(u,u_x)_x ds=f(x,t,u,u_t) \tag{1.2}$$

的初边值问题整体解的存在性问题. 熟知这一方程所描述的是具有本构关系

$$N(x,t)=u_x(x,t)-\int_0^t\lambda(t-s)\sigma(u(x,s),u_x(x,s))ds \tag{1.3}$$

（其中$u(x,t)$ 和 $N(x,t)$ 分别表示坐标为x的截面在时刻 t 的位移和所承受的内力）的均匀粘弹性杆的非线性纵振动. 我们证明了，当函数 $\sigma(s,p)$ 的两个一阶偏导数均有界时，对于一大类非线性函数 $f(x,t,s,p)$ 来说，方程(1.2)的一定初边值问题存在唯一的整体经典解.

必须指出，虽然本文是文［9］的继续且所用方法相同，但不仅本文所论方程类型更加广

泛，而且对非线性函数 $f(x,t,u,u_t)$ 的要求也要弱。另外，从下面的讨论还可看到，把积分中的二阶导数项从u_{xx}变为$\sigma(u,u_x)_x$将带来更多困难。

本文所得结果的一个意义是揭示了方程

$$u_{tt}-u_{xx}+\int_0^t\lambda(t-s)\sigma(u,u_x)_x ds=f(x,t) \tag{1.4}$$

和

$$u_{tt}-\sigma_1(u_x)_x+\int_0^t\lambda(t-s)\sigma_2(u,u_x)_x ds=f(x,t) \tag{1.5}$$

的本质区别，熟知当 $\sigma_1(p)$ 是非线性函数时，方程(1.5)的初边值问题一般只对小初值存在整体经典解[4]，而对大初值来说解的导数一般将在有限时刻发生间断从而不存在整体经典解[12,13]。但对方程(1.4)来说，本文所获结果说明只要函数$\sigma(s,p)$的一阶偏导数有界，相应的初边值问题便对任意充分光滑的初值函数都存在整体经典解。

本文沿用[9]的记号，不再一一说明。

二、主 要 结 果

我们考察方程(1.2)的下列初边值问题：

$$\begin{cases} u_{tt}-u_{xx}+\int_0^t\lambda(t-s)\sigma(u,u_x)_x ds=f(x,t,u,u_t), \quad a<x<b, t>0 & (2.1)\\ u\Big|_{x=a}=0,\ u\Big|_{x=b}=0, \quad t\geqslant 0 & (2.2)\\ u\Big|_{t=0}=\varphi(x),\ u_t\Big|_{t=0}=\psi(x), \quad a\leqslant x\leqslant b & (2.3)\end{cases}$$

其中$-\infty<a<b<+\infty$，λ，σ，f，φ，ψ都是给定的函数。

用$\{\mu_n\}_{n=1}^{\infty}$和$\{v_n(x)\}_{n=1}^{\infty}$分别表示算子$-d^2/dx^2$在区间$[a,b]$上对应于齐次Dirichlet边值条件的递增特征值序列和相应的规范特征函数序列，即

$$\mu_n=\left(\frac{n\pi}{b-a}\right)^2,\quad v_n(x)=\sqrt{\frac{2}{b-a}}\sin\frac{n\pi(x-a)}{b-a}\qquad (n=1,2,\cdots)$$

对每个自然数m，考虑下列积分微分方程组的初值问题：

$$\begin{cases} y''_{nm}(t)+\mu_n y_{nm}(t)-\int_0^t\lambda(t-s)ds\int_a^b\sigma\left(\sum_{k=1}^{m}y_{km}(s)v_k(x),\sum_{k=1}^{m}y_{km}(s)v'_k(x)\right)v'_n(x)dx \\ \qquad =\int_a^b f\left(x,t,\sum_{k=1}^{m}y_{km}(s)v_k(x),\sum_{k=1}^{m}y'_{km}(s)v_k(x)\right)v_n(x)dx, \qquad t>0 & (2.4)\\ y_{nm}(0)=A_n,\ y'_{nm}(0)=B_n & (2.5)\\ \qquad (n=1,2,\cdots,m) \end{cases}$$

其中，$A_n=\int_a^b\varphi(x)v_n(x)dx$，$B_n=\int_a^b\psi(x)v_n(x)dx \qquad (n=1,2,\cdots,m)$

应用Picard迭代技巧不难证明，当$\lambda(t)\in C[0,+\infty)$，$f(x,t,s,p)\in C([a,b]\times[0,+\infty)\times R^1\times R^1)$并关于固定的$(x,t)\in[a,b]\times[0,+\infty)$，$f(x,t,s,p)\in C^{1-0}(R^1\times R^1)$，$\sigma(s,q)\in C^{1-0}(R^1\times R^1)$时，上述问题存在唯一的局部解$\{y_{nm}(t)\}_{n=1}^{m}$。但是一般熟知上述问题不存在整体解；为保证上述问题存在整体解，函数σ和f必须满足一些更强的条件。出于力学方面的考虑，我们设函数f具有下列形式[14]：

$$f(x,t,s,p)=g(x,t)+h_1(s)+h_2(p)+h_3(s)p$$

并设函数λ，σ，$h_j(j=1,2,3)$分别满足下列条件：

(Λ) $\lambda(t)$在$[0,+\infty)$上一次可导，且$\lambda'(t)$在有界区间上有界；

(Σ) $\sigma(s,q)$在$R^1\times R^1$上有二阶偏导数，各二阶偏导数都在有界集上有界，且

$$\sigma(0,0)=0$$

$$\left|\frac{\partial\sigma}{\partial s}(s,q)\right|+\left|\frac{\partial\sigma}{\partial q}(s,q)\right|\leqslant\text{const.}\qquad\forall(s,q)\in R^1\times R^1$$

(H_1) $h_1(s)$在R^1上二次可导，$h_1''(s)$在有界区间上有界，且

$$h_1(0)=0,\ h_1(s)s\leqslant\text{const}.s^2,\qquad\forall s\in R^1$$

(H_2) $h_2(p)$在R^1上二次可导，$h_2''(p)$在有界区间上有界，且

$$h_2(0)=0,\ h_2'(p)\leqslant\text{const.},\qquad\forall p\in R^1$$

(H_3) $h_3(s)$在R^1上二次可导，$h_3''(s)$在有界区间上有界，且

$$h_3(s)\leqslant\text{const.},\qquad\forall s\in R^1$$

再设函数φ，ψ和g满足下列条件：

(C) $\varphi\in H_0^3(a,b)$，$\psi\in H_0^2(a,b)$，并对任意$t>0$有$g(x,t)\in H_0^2(a,b)$，且对任意$T>0$，$g(x,t)$，$\partial g(x,t)/\partial x$，$\partial^2 g(x,t)/\partial x^2$都在$L^2([a,b]\times[0,T])$上平方可积。

本文的主要结果是下列

定理1 在上述条件下，初边值问题(2.1)～(2.3)和初值问题(2.4)～(2.5)分别存在唯一的整体经典解$u(x,t)$和$\{y_{nm}(t)\}_{n=1}^m$，而且对任意$T>0$，函数序列

$$u_m(x,t)=\sum_{n=1}^m y_{nm}(t)v_n(x)\qquad(m=1,2,\cdots)\tag{2.6}$$

在$[a,b]\times[0,T]$上按$C^2([a,b]\times[0,T])$拓朴收敛于函数$u(x,t)$。

以下两节我们将致力于证明这一定理。

三、积分估计

以下我们总用$u_m(x,t)\,(m=1,2,\cdots)$表示由(2.6)给出的函数，并记

$$M_1(T)=\sup_{0<t<T}|\lambda(t)|,\quad M_2(T)=\sup_{0<t<T}|\lambda'(t)|$$

$$\varphi_m(x)=\sum_{n=1}^m A_n v_n(x),\quad \psi_m(x)=\sum_{n=1}^m B_n v_n(x)$$

另外，我们用C表示与m和T均无关的万用常数，而用$C(T)$表示仅与T有关而与m无关的万用常数。这里“万用”的意思是指，它们在不同的表达式中可能有不同的值，即使在同一表达式的不同位置也可能有不同的值。

显然，在定理1的条件下，(2.4)～(2.5)的解$\{y_{nm}(t)\}_{n=1}^m$在其存在区间上至少有三阶导数，进而函数$u_m(x,t)$在其存在区域上至少有三阶偏导数。

引理1 在定理1的条件下，如果对某个$T>0$问题(2.4)～(2.5)在$[0,T)$上存在解，则成立

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2\times L_T}+\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2\times L_T^\infty}\leqslant C(T) \tag{3.1}$$

证 对(2.4)乘以$y'_{nm}(t)$再关于n从1到m求和，关于t从0到$t\in(0,T)$积分得：

$$\sum_{n=1}^{m}\int_0^t y''_{nm}(\tau)\,y'_{nm}(\tau)d\tau+\sum_{n=1}^{m}\mu_n\int_0^t y_{nm}(\tau)y'_{nm}(\tau)d\tau$$

$$-\int_0^t\int_0^\tau\lambda(\tau-s)\,dsd\tau\int_a^b\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^2 u_m}{\partial x\partial\tau}(x,\tau)dx$$

$$=\int_0^t\int_a^b h_1(u_m)\frac{\partial u_m}{\partial\tau}dx\,d\tau+\int_0^t\int_a^b h_2\left(\frac{\partial u_m}{\partial\tau}\right)\frac{\partial u_m}{\partial\tau}dxd\tau$$

$$+\int_0^t\int_a^b h_3(u_m)\left(\frac{\partial u_m}{\partial\tau}\right)^2dx\,d\tau+\int_0^t\int_a^b g(x,\tau)\frac{\partial u_m}{\partial\tau}(x,\tau)dxd\tau \tag{3.2}$$

不难知道，(3.2)左端的前两项可分别写成（参考[9]引理1的证明）

$$\frac{1}{2}\left[\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2}^2-\left\|\psi_m\right\|_{L^2}^2\right],\ \frac{1}{2}\left[\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2}^2-\left\|\varphi'_m\right\|_{L^2}^2\right]$$

对于(3.2)左端的第三项，关于变元τ分部积分得

$$-\int_0^t\int_0^\tau\lambda(\tau-s)dsd\tau\int_a^b\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^2u_m}{\partial x\partial\tau}(x,\tau)dx$$

$$=-\int_0^t\lambda(t-s)ds\int_a^b\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial u_m}{\partial x}(x,t)dx$$

$$+\int_0^t\int_0^\tau\lambda'(\tau-s)dsd\tau\int_a^b\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial u_m}{\partial x}(x,\tau)dx$$

$$+\lambda(0)\int_0^t\int_a^b\sigma\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\frac{\partial u_m}{\partial x}(x,\tau)dxd\tau$$

$$\geqslant-\Big[TM_1(T)\int_0^t|\lambda(t-s)|ds\int_a^b\left|\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\right|^2dx$$

$$+\frac{1}{4TM_1(T)}\int_0^t|\lambda(t-s)|ds\int_a^b\left|\frac{\partial u_m}{\partial x}(x,t)\right|^2dx\Big]$$

$$-\frac{1}{2}\Big[\int\int_0\int_0^\tau|\lambda'(\tau-s)|dsd\tau\int_a^b\left|\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\right|^2dx$$

$$+\int_0^t\int_0^\tau|\lambda'(\tau-s)|dsd\tau\int_a^b\left|\frac{\partial u_m}{\partial x}(x,\tau)\right|^2dx\Big]$$

$$-\frac{1}{2}|\lambda(0)|\Big[\int\int_0\int_a^b\left|\sigma\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\right|^2dxd\tau$$

$$+\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial x}(x,\tau)\right|^2dxd\tau\Big]$$

$$\geqslant-M_3(T)\int_0^t\int_a^b\left|\sigma\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\right|^2dxd\tau$$

$$-M_4(T)\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial x}(x,\tau)\right|^2dxd\tau-\frac{1}{4}\int_a^b\left|\frac{\partial u_m}{\partial x}(x,t)\right|^2dx \tag{3.3}$$

其中

$$M_3(T)=TM_1(T)^2+\frac{1}{2}TM_2(T)+\frac{1}{2}|\lambda(0)|$$

$$M_4(T)=\frac{1}{2}TM_2(T)+\frac{1}{2}|\lambda(0)|$$

从条件(Σ)知$|\sigma(s,p)|\leqslant C(|s|+|p|)$, $\quad\forall(s,p)\in R^1\times R^1$

所以

$$\int_0^t\int_a^b\left|\sigma\left(u_m(x,\tau),\ \frac{\partial u_m}{\partial x}(x,\tau)\right)\right|^2dxd\tau$$

$$\leqslant C\left(\int_0^t\int_a^b|u_m(x,\tau)|^2dx\,d\tau+\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial x}(x,\tau)\right|^2dxd\tau\right)$$

$$\leqslant C\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial x}(x,\tau)\right|^2dxd\tau \tag{3.4}$$

这里用到了 Poincaré 不等式[15]

$$\int_a^b|w(x)|^2dx\leqslant\frac{1}{\mu_1}\int_a^b|w'(x)|^2dx,\qquad\forall w\in H_0^1(a,b)$$

把(3.4)代入(3.3)便得

$$-\int_0^t\int_0^\tau\lambda(\tau-s)\,dsd\tau\int_a^b\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^2u_m}{\partial x\partial\tau}(x,\tau)dx$$

$$\geqslant-C(T)\int_0^t\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2}^2dt-\frac{1}{4}\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2}^2 \tag{3.5}$$

对于(3.2)右端第一项，记

$$H_1(s)=\int_0^s h_1(\eta)d\eta$$

则由条件(H_1)知$H_1(s)\leqslant Cs^2$，$\forall s\in R^1$，所以

$$\int_0^t\int_a^b h_1(u_m)\frac{\partial u_m}{\partial\tau}\,dxd\tau=\int_0^t\frac{\partial}{\partial\tau}\left(\int_a^b H_1(u_m(x,\tau))\,dx\right)d\tau$$

$$=\int_a^b H_1(u_m(x,t))dx-\int_a^b H_1(\varphi_m(x))dx$$

$$\leqslant C\int_a^b|u_m(x,t)|^2dx-\int_a^b H_1(\varphi_m(x))dx \tag{3.6}$$

但

$$\int_a^b|u_m(x,t)|^2dx=\int_0^t\frac{\partial}{\partial\tau}\left(\int_a^b|u_m(x,\tau)|^2dx\right)d\tau+\int_a^b|u_m(x,0)|^2dx$$

$$=2\int_0^t\int_a^b u_m(x,\tau)\frac{\partial u_m}{\partial\tau}(x,\tau)dxd\tau+\int_a^b|\varphi_m(x)|^2dx$$

$$\leqslant\int_0^t\int_a^b|u_m(x,\tau)|^2dx\,d\tau+\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial\tau}(x,\tau)\right|^2dx\,d\tau+\int_a^b|\varphi_m(x)|^2dx$$

$$\leqslant\frac{1}{\mu_1}\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial x}(x,\tau)\right|^2dx\,d\tau+\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial\tau}(x,\tau)\right|^2dx\,d\tau+\int_a^b|\varphi_m(x)|^2dx$$

代入(3.6)即得

$$\int_0^t\int_a^b h_1(u_m)\frac{\partial u_m}{\partial \tau}dxd\tau$$

$$\leqslant C\left(\int_0^t\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2}^2 d\tau+\int_0^t\left\|\frac{\partial u_m}{\partial \tau}\right\|_{L^2}^2 d\tau\right)+C\left\|\varphi_m\right\|_{L^2}^2-\int_a^b H_1(\varphi_m(x))dx \tag{3.7}$$

对于(3.2)右端第二项，由条件(H_2)知$h_2(p)p\leqslant Cp^2$，$\forall p\in R^1$，所以

$$\int_0^t\int_a^b h_2\left(\frac{\partial u_m}{\partial \tau}\right)\frac{\partial u_m}{\partial \tau}dxd\tau\leqslant C\int_0^t\left\|\frac{\partial u_m}{\partial \tau}\right\|_{L^2}^2 d\tau \tag{3.8}$$

对于(3.2)右端第三项，由条件(H_3)得

$$\int_0^t\int_a^b h_3(u_m)\left(\frac{\partial u_m}{\partial \tau}\right)^2 dxd\tau\leqslant C\int_0^t\left\|\frac{\partial u_m}{\partial \tau}\right\|_{L^2}^2 d\tau \tag{3.9}$$

对(3.2)右端第四项，我们有

$$\int_0^t\int_a^b g(x,\tau)\frac{\partial u_m}{\partial \tau}(x,\tau)dxd\tau\leqslant\frac{1}{2}\left\|g\right\|_{L^2\times L_T^2}^2+\frac{1}{2}\int_0^t\left\|\frac{\partial u_m}{\partial \tau}\right\|_{L^2}^2 d\tau \tag{3.10}$$

把(3.5)，(3.7)，(3.8)，(3.9)和(3.10)都代入(3.2)便得到

$$\frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2}^2+\frac{1}{4}\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2}^2\leqslant C(T)\int_0^t\left(\left\|\frac{\partial u_m}{\partial \tau}\right\|_{L^2}^2+\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2}^2\right)d\tau$$

$$+\frac{1}{2}\left\|g\right\|_{L^2\times L_2^T}^2+\frac{1}{2}\left\|\psi_m\right\|_{L^2}^2+\frac{1}{2}\left\|\varphi_m'\right\|_{L^2}^2$$

$$+C\left\|\varphi_m\right\|_{L^2}^2-\int_a^b H_1(\varphi_m(x))dx \tag{3.11}$$

由条件(C)知当$m\to+\infty$时，$\varphi_m\xrightarrow{H^3}\varphi$，$\psi_m\xrightarrow{H^2}\psi$，进而(3.11)右端后四项之和收敛于

$$\frac{1}{2}\left\|\psi\right\|_{L^2}^2+\frac{1}{2}\left\|\varphi'\right\|_{L^2}^2+C\left\|\varphi\right\|_{L^2}^2-\int_a^b H_1(\varphi(x))dx$$

从而它们可用与m无关的常数界定，这样从(3.11)便得

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2}^2+\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2}^2\leqslant C(T)\int_0^t\left(\left\|\frac{\partial u_m}{\partial \tau}\right\|_{L^2}^2+\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2}^2\right)dt+C(T)$$

据此应用 Gronwall-Bellman 引理即得(3.1)。证毕。

应用 Sobolev 嵌入不等式，从上述引理即得

推论1 在引理1的条件下，有

$$\left\|u_m\right\|_{L^\infty\times L_T^\infty}\leqslant C(T) \tag{3.12}$$

引理2 在引理1的条件下，有

$$\left\|\frac{\partial^2 u_m}{\partial x\partial t}\right\|_{L^2\times L_T^\infty}+\left\|\frac{\partial^2 u_m}{\partial x^2}\right\|_{L^2\times L_T^\infty}\leqslant C(T) \tag{3.13}$$

证 对(2.4)乘以$\mu_n y_{nm}'(t)$再关于n从1到m求和，关于t从0到$t\in(0,T)$积分得

$$\sum_{n=1}^m \mu_n\int_0^t y_{nm}''(\tau)y_{nm}'(\tau)d\tau+\sum_{n=1}^m \mu_n^2\int_0^t y_{nm}(\tau)y_{nm}'(\tau)d\tau$$

$$+\int_0^t\int_0^\tau \lambda(\tau-s)\,dsd\tau\int_a^b\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^4 u_m}{\partial x^3\partial \tau}(x,\tau)dx$$

$$= -\int_0^t\int_a^b h_1(u_m)\frac{\partial^3 u_m}{\partial x^2\partial\tau}dxd\tau - \int_0^t\int_a^b h_2\left(\frac{\partial u_m}{\partial\tau}\right)\frac{\partial^3 u_m}{\partial x^2\partial\tau}dxd\tau$$

$$-\int_0^t\int_a^b h_3(u_m)\frac{\partial u_m}{\partial\tau}\frac{\partial^3 u_m}{\partial x^2\partial\tau}dxd\tau - \int_0^t\int_a^b g(x,\tau)\frac{\partial^3 u_m}{\partial x^2\partial\tau}dxd\tau \tag{3.14}$$

易知(3.14)左端前两项分别等于（参见[9]引理2的证明）

$$\frac{1}{2}\left[\left\|\frac{\partial^2 u_m}{\partial x\partial t}\right\|_{L^2}^2 - \left\|\psi_m'\right\|_{L^2}^2\right],\quad \frac{1}{2}\left[\left\|\frac{\partial^2 u_m}{\partial x^2}\right\|_{L^2}^2 - \left\|\varphi_m''\right\|_{L^2}^2\right]$$

对于(3.14)左端第三项，通过先关于τ分部积分，再关于x分部积分可得：

$$\int_0^t\int_0^\tau\lambda(\tau-s)\,dsd\tau\int_a^b\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^4 u_m}{\partial x^3\partial\tau}(x,\tau)dx$$

$$= -\int_0^t\lambda(t-s)ds\int_a^b\frac{\partial}{\partial x}\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\cdot\frac{\partial^2 u_m}{\partial x^2}(x,t)dx$$

$$+\int_0^t\int_0^\tau\lambda'(\tau-s)dsd\tau\int_a^b\frac{\partial}{\partial x}\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\cdot\frac{\partial^2 u_m}{\partial x^2}(x,\tau)dx$$

$$+\lambda(0)\int_0^t\int_a^b\frac{\partial}{\partial x}\sigma\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\cdot\frac{\partial^2 u_m}{\partial x^2}(x,\tau)dxd\tau$$

再类似于不等式(3.3)的推导可得

$$\int_0^t\int_0^\tau\lambda(\tau-s)\,dsd\tau\int_a^b\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^4 u_m}{\partial x^3\partial\tau}(x,\tau)dx$$

$$\geqslant -M_3(T)\int_0^t\int_a^b\left|\frac{\partial}{\partial x}\sigma\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\right|^2dxd\tau$$

$$-M_4(T)\int_0^t\int_a^b\left|\frac{\partial^2 u_m}{\partial x^2}(x,\tau)\right|^2dxd\tau - \frac{1}{4}\int_a^b\left|\frac{\partial^2 u_m}{\partial x^2}(x,t)\right|^2dx \tag{3.15}$$

根据条件(Σ)知

$$\left|\frac{\partial}{\partial x}\sigma\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\right|$$

$$= \left|\frac{\partial\sigma}{\partial s}\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\frac{\partial u_m}{\partial x}(x,\tau)\right.$$

$$\left.+\frac{\partial\sigma}{\partial p}\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\frac{\partial^2 u_m}{\partial x^2}(x,\tau)\right|$$

$$\leqslant C\left(\left|\frac{\partial u_m}{\partial x}(x,\tau)\right| + \left|\frac{\partial^2 u_m}{\partial x^2}(x,\tau)\right|\right)$$

进而

$$\int_0^t\int_a^b\left|\frac{\partial}{\partial x}\sigma\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\right|^2dxd\tau$$

$$\leqslant C\left(\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial x}(x,\tau)\right|^2dxd\tau + \int_0^t\int_a^b\left|\frac{\partial^2 u_m}{\partial x^2}(x,\tau)\right|^2dxd\tau\right)$$

$$\leqslant C(T) + C\int_0^t\int_a^b\left|\frac{\partial^2 u_m}{\partial x^2}(x,\tau)\right|^2dxd\tau \tag{3.16}$$

这里用到了引理1.把(3.16)代入(3.15)便得

$$\int_0^t\int_0^\tau\lambda(\tau-s)\,dsd\tau\int_0^b\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^4 u_m}{\partial x^3\partial\tau}(x,\tau)dx$$

$$\geqslant -C(T)-C(T)\int_0^t\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2}^2 d\tau-\frac{1}{4}\left\|\frac{\partial^2 u_m}{\partial x^2}\right\|_{L^2}^2 \tag{3.17}$$

对于(3.14)右端第一项，我们有

$$-\int_0\int_a^b h_1(u_m)\frac{\partial^3 u_m}{\partial x^2\partial\tau}\,dxd\tau=\int_0\int_a^b h_1'(u_m)\frac{\partial u_m}{\partial x}\frac{\partial^2 u_m}{\partial x\partial\tau}\,dxd\tau \tag{3.18}$$

根据推论1知当$x\in[a,b]$，$\tau\in[0,t]\subset[0,T)$时有

$$|h_1'(u_m(x,\tau))|\leqslant C(T)$$

所以从(3.18)得

$$-\int_0^t\int_a^b h_1(u_m)\frac{\partial^3 u_m}{\partial x^2\partial\tau}\,dxd\tau\leqslant C(T)\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial x}\frac{\partial^2 u_m}{\partial x\partial\tau}\right|dxd\tau$$

$$\leqslant C(T)\left[\frac{1}{2}\int_0^t\int_a\left|\frac{\partial u_m}{\partial x}\right|^2 dxd\tau+\frac{1}{2}\int_0^t\int_a^b\left|\frac{\partial^2 u_m}{\partial x\partial\tau}\right|^2 dxd\tau\right]$$

$$\leqslant C(T)+C(T)\int_0^t\left\|\frac{\partial^2 u_m}{\partial x\partial\tau}\right\|_{L^2}^2 d\tau \tag{3.19}$$

这里用到了引理1；对于(3.14)右端第二项，应用条件(H_2)得

$$-\int_0^t\int_a^b h_2\left(\frac{\partial u_m}{\partial\tau}\right)\frac{\partial^3 u_m}{\partial x^2\partial\tau}\,dxd\tau=\int_0^t\int_a^b h_2'(u_m)\left(\frac{\partial^2 u_m}{\partial x\partial\tau}\right)^2 dxd\tau$$

$$\leqslant C\int_0^t\left\|\frac{\partial^2 u_m}{\partial x\partial\tau}\right\|_{L^2}^2 d\tau \tag{3.20}$$

对于(3.14)右端第三项，我们有

$$-\int_0^t\int_a^b h_3(u_m)\frac{\partial u_m}{\partial\tau}\frac{\partial^3 u_m}{\partial x^2\partial\tau}dxd\tau$$

$$=\int_0^t\int_a^b h_3'(u_m)\frac{\partial u_m}{\partial x}\frac{\partial u_m}{\partial\tau}\frac{\partial^2 u_m}{\partial x\partial\tau}dxd\tau+\int_0^t\int_a^b h_3(u_m)\left(\frac{\partial^2 u_m}{\partial x\partial\tau}\right)^2 dxd\tau \tag{3.21}$$

根据条件(H_3)和推论1知当$x\in[a,b]$，$\tau\in[0,t]\subset[0,T)$时有

$$|h_3'(u_m(x,\tau))|\leqslant C(T),\quad |h_3(u_m(x,\tau))|\leqslant C$$

代入(3.21)得

$$-\int_0^t\int_a^b h_3(u_m)\frac{\partial u_m}{\partial\tau}\frac{\partial^3 u_m}{\partial x^2\partial\tau}dxd\tau$$

$$\leqslant C(T)\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial x}\frac{\partial u_m}{\partial\tau}\frac{\partial^2 u_m}{\partial x\partial\tau}\right|dxd\tau+C\int_0\int_a^b\left|\frac{\partial^2 u_m}{\partial x\partial\tau}\right|^2 dxd\tau$$

$$\leqslant C(T)\left[\frac{1}{4}\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial x}\right|^4 dxd\tau+\frac{1}{4}\int_0^t\int_a^b\left|\frac{\partial u_m}{\partial\tau}\right|^4 dxd\tau\right.$$

$$\left.+\frac{1}{2}\int_0^t\int_a^b\left|\frac{\partial^2 u_m}{\partial x\partial\tau}\right|^2 dxd\tau\right]+C\int_0^t\int_a^b\left|\frac{\partial^2 u_m}{\partial x\partial\tau}\right|^2 dxd\tau \tag{3.22}$$

在 Gagliardo-Nirenberg 不等式[18]

$$\left(\int_a^b\left|w^{(k)}(x)\right|^p dx\right)^{1/p}\leqslant C\left(\int_a^b\left|w^{(n)}(x)\right|^r dx\right)^{\theta/r}\left(\int_a^b\left|w(x)\right|^q dx\right)^{(1-\theta)/q}$$

(其中 $p,q,r\in[1,+\infty)$, $k/n\leqslant\theta<1$, $1/p=\theta/r+(1-\theta)/q-(n\theta-k)$) 中取 $k=0$, $n=1$, $p=4$, $q=r=2$, $\theta=1/4$, 可得下列不等式:

$$\int_a^b|w(x)|^4dx\leqslant C\left(\int_a^b|w'(x)|^2dx\right)^{\frac{1}{2}}\left(\int_a^b|w(x)|^2dx\right)^{\frac{3}{2}}$$

$$\leqslant C\left[\int_a^b|w'(x)|^2dx+\left(\int_a^b|w(x)|^2dx\right)^3\right] \tag{3.23}$$

分别对$w=\partial u_m/\partial x$和$w=\partial u_m/\partial\tau$应用这一不等式可得

$$\int_a^b\left|\frac{\partial u_m}{\partial x}\right|^4dx\leqslant C\left[\int_a^b\left|\frac{\partial^2u_m}{\partial x^2}\right|^2dx+\left(\int_a^b\left|\frac{\partial u_m}{\partial x}\right|^2dx\right)^3\right]$$

$$\int_a^b\left|\frac{\partial u_m}{\partial\tau}\right|^4dx\leqslant C\left[\int_a^b\left|\frac{\partial^2u_m}{\partial x\partial\tau}\right|^2dx+\left(\int_a^b\left|\frac{\partial u_m}{\partial\tau}\right|^2dx\right)^3\right]$$

代入(3.22), 并应用引理1便得

$$-\int_0^t\int_a^b h_3(u_m)\frac{\partial u_m}{\partial\tau}\frac{\partial^3u_m}{\partial x^2\partial\tau}dxd\tau$$

$$\leqslant C(T)\int_0^t\left\|\frac{\partial^2u_m}{\partial x\partial\tau}\right\|_{L^2}d\tau+C(T)\int_0^t\left\|\frac{\partial^2u_m}{\partial x^2}\right\|_{L^2}^2d\tau+C(T) \tag{3.24}$$

对于(3.14)右端最后一项关于x分部积分可得

$$-\int_0^t\int_a^b g(x,\tau)\frac{\partial^3u_m}{\partial x^2\partial\tau}dxd\tau\leqslant\frac{1}{2}\left\|g_x\right\|_{L^2\times L^2_T}^2+\frac{1}{2}\int_0^t\left\|\frac{\partial^2u_m}{\partial x\partial\tau}\right\|_{L^2}^2d\tau \tag{3.25}$$

把(3.17), (3.19), (3.20), (3.24)和(3.25)代入(3.14), 最后得

$$\frac{1}{2}\left\|\frac{\partial^2u_m}{\partial x\partial t}\right\|_{L^2}^2+\frac{1}{4}\left\|\frac{\partial^2u_m}{\partial x^2}\right\|_{L^2}^2\leqslant C(T)\int_0^t\left(\left\|\frac{\partial^2u_m}{\partial x\partial\tau}\right\|_{L^2}^2+\left\|\frac{\partial^2u_m}{\partial x^2}\right\|_{L^2}^2\right)d\tau$$

$$+\frac{1}{2}\left\|g_x\right\|_{L^2\times L^2_T}^2+\frac{1}{2}\left\|\psi'_m\right\|_{L^2}^2+\frac{1}{2}\left\|\varphi''_m\right\|_{L^2}^2+C(T) \tag{3.26}$$

由条件(C)知

$$\lim_{m\to\infty}\left(\frac{1}{2}\left\|\psi'_m\right\|_{L^2}^2+\frac{1}{2}\left\|\varphi''_m\right\|_{L^2}^2\right)=\frac{1}{2}\left\|\psi'\right\|_{L^2}^2+\frac{1}{2}\left\|\varphi''\right\|_{L^2}^2$$

所以从(3.26)可得

$$\left\|\frac{\partial^2u_m}{\partial x\partial t}\right\|_{L^2}^2+\left\|\frac{\partial^2u_m}{\partial x^2}\right\|_{L^2}^2\leqslant C(T)\int_0^t\left(\left\|\frac{\partial^2u_m}{\partial x\partial\tau}\right\|_{L^2}^2+\left\|\frac{\partial^2u_m}{\partial x^2}\right\|_{L^2}^2\right)d\tau+C(T)$$

据此应用 Gronwall-Bellman 引理即得(3.13). 证毕.

推论2 在引理1的条件下有

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{L^\infty\times L^\infty_T}+\left\|\frac{\partial u_m}{\partial x}\right\|_{L^\infty\times L^\infty_T}\leqslant C(T) \tag{3.27}$$

引理3 在引理1的条件下有

$$\left\|\frac{\partial^2u_m}{\partial t^2}\right\|_{L^2\times L^\infty_T}\leqslant C(T) \tag{3.28}$$

证 应用方程(2.4)我们有

$$\left\|\frac{\partial^2u_m}{\partial t^2}\right\|_{L^2}^2=\sum_{n=1}^m(y''_{nm}(t))^2$$

$$= -\sum_{n=1}^{m} \mu_n y_{nm}(t) y''_{nm}(t) + \int_0^t \lambda(t-s)ds \int_a^b \sigma\left(u_m(x,s),\right.$$

$$\left.\frac{\partial u_m}{\partial x}(x,s)\right) \frac{\partial^3 u_m}{\partial x \partial t^2}(x,t)dx$$

$$+\int_a^b f(x,t,u_m(x,t),\frac{\partial u_m}{\partial t}(x,t)) \frac{\partial^2 u_m}{\partial t^2}(x,t)dx$$

$$=\int_a^b \frac{\partial^2 u_m}{\partial x^2} \frac{\partial^2 u_m}{\partial t^2} dx - \int_0^t \lambda(t-s)ds \int_a^b \frac{\partial}{\partial x} \sigma\left(u_m(x,s),\right.$$

$$\left.\frac{\partial u_m}{\partial x}(x,s)\right) \frac{\partial^2 u_m}{\partial t^2}(x,t)dx$$

$$+\int_a^b f\left(x,t,u_m, \frac{\partial u_m}{\partial t}\right) \frac{\partial^2 u_m}{\partial t^2} dx$$

$$\leqslant \frac{1}{2}\left(3\left\|\frac{\partial^2 u_m}{\partial x^2}\right\|_{L^2}^2 + \frac{1}{3}\left\|\frac{\partial^2 u_m}{\partial t^2}\right\|_{L^2}^2\right) + \frac{1}{2}\Big[3TM_1(T)$$

$$\int_0^t |\lambda(t-s)| ds \int_a^b \left| \frac{\partial}{\partial x} \sigma\left(u_m, \frac{\partial u_m}{\partial x}\right)\right|^2 dx$$

$$+ \frac{1}{3TM_1(T)} \int_0^t |\lambda(t-s)| ds \int_a^b \left| \frac{\partial^2 u_m}{\partial t^2}(x,t)\right|^2 dx \Big]$$

$$+\frac{1}{2}\left(3\int_a^b \left| f\left(x,t,u_m, \frac{\partial u_m}{\partial t}\right)\right|^2 dx + \frac{1}{3}\left\|\frac{\partial^2 u_m}{\partial t^2}\right\|_{L^2}^2\right)$$

$$\leqslant \frac{1}{2}\left\|\frac{\partial^2 u_m}{\partial t^2}\right\|_{L^2}^2 + C(T)$$

这里用到了前面所得各结论。据此即得(3.28)。证毕。

引理4 在引理1的条件下有

$$\left\|\frac{\partial^3 u_m}{\partial x^2 \partial t}\right\|_{L^2 \times L_T^\infty} + \left\|\frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2 \times L_T^\infty} \leqslant C(T) \tag{3.29}$$

证 对(2.4)乘以$\mu_n^2 y'_{nm}(t)$再关于n从1到m求和，关于t从0到 $t\in(0,T)$积分得：

$$\sum_{n=1}^{m} \mu_n^2 \int_0^t y''_{nm}(\tau) y'_{nm}(\tau)\, d\tau + \sum_{n=1}^{m} \mu_n^3 \int_0^t y_{nm}(\tau) y'_{nm}(\tau)\, d\tau$$

$$-\int_0^t \int_0^\tau \lambda(\tau-s)\, ds d\tau \int_a^b \sigma\left(u_m(x,s), \frac{\partial u_m}{\partial x}(x,s)\right) \frac{\partial^6 u_m}{\partial x^5 \partial \tau}(x,\tau)dx$$

$$=\int_0^t \int_a^b h_1(u_m) \frac{\partial^5 u_m}{\partial x^4 \partial \tau} dxd\tau + \int_0^t \int_a^b h_2\left(\frac{\partial u_m}{\partial \tau}\right) \frac{\partial^5 u_m}{\partial x^4 \partial \tau} dxd\tau$$

$$+\int_0^t \int_a^b h_3(u_m) \frac{\partial u_m}{\partial \tau} \frac{\partial^5 u_m}{\partial x^4 \partial \tau} dxd\tau + \int_0^t \int_a^b g(x,\tau) \frac{\partial^5 u_m}{\partial x^4 \partial \tau} dxd\tau \tag{3.30}$$

易知(3.30)左端前两项分别等于

$$\frac{1}{2}\left[\left\|\frac{\partial^3 u_m}{\partial x^2 \partial t}\right\|_{L^2}^2 - \left\|\psi''_m\right\|_{L^2}^2\right], \quad \frac{1}{2}\left[\left\|\frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2}^2 - \left\|\varphi'''_m\right\|_{L^2}^2\right]$$

对于(3.30)左端第三项，通过先对τ分部积分再对x分部积分两次可得：

$$-\int_0^t\int_0^\tau \lambda(\tau-s)\,dsd\tau\int_a^b \sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^6 u_m}{\partial x^5\partial\tau}(x,\tau)dx$$

$$=-\int_0^t \lambda(t-s)ds\int_a^b \frac{\partial^2}{\partial x^2}\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^3 u_m}{\partial x^3}(x,t)dx$$

$$+\int_0^t\int_0^\tau \lambda'(\tau-s)dsd\tau\int_a^b \frac{\partial^2}{\partial x^2}\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^3 u_m}{\partial x^3}(x,\tau)dx$$

$$+\lambda(0)\int_0^t\int_a^b \frac{\partial}{\partial x}\sigma\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\frac{\partial^3 u_m}{\partial x^3}(x,\tau)dxd\tau$$

由此应用与不等式(3.3)相同的技巧估计，可得

$$-\int_0^t\int_0^\tau \lambda(\tau-s)\,dsd\tau\int_a^b \sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^6 u_m}{\partial x^5\partial\tau}(x,\tau)dx$$

$$\geqslant -M_3(T)\int_0^t\int_a^b\left|\frac{\partial^2}{\partial x^2}\sigma\left(u_m(x,\tau),\frac{\partial u_m}{\partial x}(x,\tau)\right)\right|^2 dxd\tau$$

$$-M_4(T)\int_0^t\int_a^b\left|\frac{\partial^3 u_m}{\partial x^3}(x,\tau)\right|^2 dxd\tau-\frac{1}{4}\int_a^b\left|\frac{\partial^3 u_m}{\partial x^3}(x,t)\right|^2 dx \tag{3.31}$$

易见

$$\frac{\partial^2}{\partial x^2}\sigma\left(u_m,\ \frac{\partial u_m}{\partial x}\right)=\frac{\partial\sigma}{\partial s}\left(u_m,\ \frac{\partial u_m}{\partial x}\right)\frac{\partial^2 u_m}{\partial x^2}+\frac{\partial\sigma}{\partial p}\left(u_m,\ \frac{\partial u_m}{\partial x}\right)\frac{\partial^3 u_m}{\partial x^3}$$

$$+\frac{\partial^2\sigma}{\partial s^2}\left(u_m,\ \frac{\partial u_m}{\partial x}\right)\left(\frac{\partial u_m}{\partial x}\right)^2+2\frac{\partial^2\sigma}{\partial s\partial p}\left(u_m,\ \frac{\partial u_m}{\partial x}\right)\frac{\partial u_m}{\partial x}\frac{\partial^2 u_m}{\partial x^2}$$

$$+\frac{\partial^2\sigma}{\partial p^2}\left(u_m,\ \frac{\partial u_m}{\partial x}\right)\left(\frac{\partial^2 u_m}{\partial x^2}\right)^2$$

进而根据推论1和推论2知有

$$\left|\frac{\partial^2}{\partial x^2}\sigma\left(u_m,\ \frac{\partial u_m}{\partial x}\right)\right|\leqslant C(T)+C(T)\left|\frac{\partial^2 u_m}{\partial x^2}\right|^2+C(T)\left|\frac{\partial^3 u_m}{\partial x^3}\right|$$

(这里用到$\left|\frac{\partial^2 u_m}{\partial x^2}\right|\leqslant\frac{1}{2}+\frac{1}{2}\left|\frac{\partial^2 u_m}{\partial x^2}\right|^2$)，代入(3.31)就得到

$$-\int_0^t\int_0^\tau \lambda(\tau-s)\,dsd\tau\int_a^b \sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^6 u_m}{\partial x^5\partial\tau}(x,\tau)dx$$

$$\geqslant -C(T)-C(T)\int_0^t\int_a^b\left|\frac{\partial^2 u_m}{\partial x^2}(x,\tau)\right|^4 dxd\tau-C(T)\int_0^t\int_a^b\left|\frac{\partial^3 u_m}{\partial x^3}(x,\tau)\right|^2 dxd\tau$$

$$-\frac{1}{4}\int_a^b\left|\frac{\partial^3 u_m}{\partial x^3}(x,t)\right|^2 dx \tag{3.32}$$

应用不等式(3.23)（在其中取$w=\partial^2 u_m/\partial x^2$）和引理2可得

$$\int_a^b\left|\frac{\partial^2 u_m}{\partial x^2}(x,\tau)\right|^4 dx\leqslant C\int_a^b\left|\frac{\partial^3 u_m}{\partial x^3}(x,\tau)\right|^2 dx+C(T),\qquad 0\leqslant\tau<T$$

代入(3.32)就得到

$$-\int_0^t\int_0^\tau \lambda(\tau-s)\,dsd\tau\int_a^b \sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^6 u_m}{\partial x^5\partial\tau}(x,\tau)dx$$

$$\geqslant -C(T)-C(T)\int_0^t\left\|\frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2}^2 d\tau-\frac{1}{4}\left\|\frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2}^2 \tag{3.33}$$

对于(3.30)右端各项，通过关于x分部积分两次并运用与前面所用类似的技巧估计，可知它们的和可以被形如（参见[9]引理4的证明）

$$C(T)\int_0^t\left\|\frac{\partial^3 u_m}{\partial x^2\partial\tau}\right\|_{L^2}^2 d\tau+C(T)$$

的表达式界定。这样结合(3.33)，从(3.30)我们得

$$\frac{1}{2}\left\|\frac{\partial^3 u_m}{\partial x^2\partial t}\right\|_{L^2}^2+\frac{1}{4}\left\|\frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2}^2\leqslant C(T)\int_0^t\left(\left\|\frac{\partial^3 u_m}{\partial x^2\partial\tau}\right\|_{L^2}^2+\left\|\frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2}^2\right)d\tau+C(T)$$

据此应用 Gronwall-Bellman 引理便得(3.29)。证毕。

引理5 在引理1的条件下有

$$\left\|\frac{\partial^3 u_m}{\partial t^3}\right\|_{L^2\times L_T^\infty}\leqslant C(T),\quad \left\|\frac{\partial^3 u_m}{\partial x\partial t^2}\right\|_{L^2\times L_T^\infty}\leqslant C(T) \tag{3.34}$$

证 对(2.4)求一次导数再乘以$y'''_{nm}(t)$，然后关于n从1到m求和得

$$\left\|\frac{\partial^3 u_m}{\partial t^3}\right\|_{L^2}^2=\sum_{n=1}^m\left(y'''_{nm}(t)\right)^2=\sum_{n=1}^m y'''_{nm}(t)\cdot\frac{d}{dt}y''_{nm}(t)$$

$$=-\sum_{n=1}^m y'''_{nm}(t)\cdot\mu_n y'_{nm}(t)+\int_0^t\lambda'(t-s)ds\int_a^b\sigma\left(u_m(x,s),\right.$$

$$\left.\frac{\partial u_m}{\partial x}(x,s)\right)\frac{\partial^4 u_m}{\partial x\partial t^3}(x,t)dx$$

$$+\lambda(0)\int_a^b\sigma\left(u_m(x,t),\frac{\partial u_m}{\partial x}(x,t)\right)\frac{\partial^4 u_m}{\partial x\partial t^3}(x,t)dx$$

$$+\int_a^b\frac{\partial}{\partial t}f\left(x,t,u_m(x,t),\frac{\partial u_m}{\partial t}(x,t)\right)\cdot\frac{\partial^3 u_m}{\partial t^3}(x,t)dx$$

$$=\int_a^b\frac{\partial^3 u_m}{\partial x^2\partial t}\frac{\partial^3 u_m}{\partial t^3}dx-\int_0^t\lambda'(t-s)ds\int_a^b\frac{\partial}{\partial x}\sigma\left(u_m(x,s),\right.$$

$$\left.\frac{\partial u_m}{\partial x}(x,s)\right)\cdot\frac{\partial^3 u_m}{\partial t^3}(x,t)dx$$

$$-\lambda(0)\int_a^b\frac{\partial}{\partial x}\sigma\left(u_m(x,t),\frac{\partial u_m}{\partial x}(x,t)\right)\cdot\frac{\partial^3 u_m}{\partial t^3}(x,t)dx$$

$$+\int_a^b\frac{\partial}{\partial t}f\left(x,t,u_m(x,t),\frac{\partial u_m}{\partial t}(x,t)\right)\cdot\frac{\partial^3 u_m}{\partial t^3}(x,t)dx$$

$$\leqslant\frac{1}{2}\left[4\left\|\frac{\partial^3 u_m}{\partial x^2\partial t}\right\|_{L^2}^2+\frac{1}{4}\left\|\frac{\partial^3 u_m}{\partial t^3}\right\|_{L^2}^2\right]+\frac{1}{2}\left[4TM_2(T)\int_0^t|\lambda'(t-s)|ds\right.$$

$$\cdot\int_a^b\left|\frac{\partial}{\partial x}\sigma\left(u_m(x,s),\frac{\partial u_m}{\partial x}(x,s)\right)\right|^2dx$$

$$\left.+\frac{1}{4TM_2(T)}\int_0^t|\lambda'(t-s)|ds\cdot\left\|\frac{\partial^3 u_m}{\partial t^3}\right\|_{L^2}^2\right]$$

的函数$v(x,t)$所组成的函数空间，即

$$B_T=\left\{v\in C^3([a,b]\times[0,+\infty)):\right.$$
$$\left.\|v\|_{B_T}=\max\left\{\left\|\frac{\partial^{k+l}v}{\partial x^k\partial t^l}\right\|_{L^2\times L_T^\infty},\ k+l\leqslant 3\right\}<+\infty\right\}$$

不难证明B_T按范数$\|\cdot\|_{B_T}$成为一个 Banach 空间，并且能够完全连续地嵌入到$C^2([a,b]\times[0,T])$中去。而由引理1～引理5知对每个$T>0$，$\{u_m(x,t)\}_{m=1}^\infty$是$B_T$中的有界序列，因此它必有按$C^2([a,b]\times[0,T])$拓朴收敛的子序列，记其极限函数为$u_T(x,t)$。则不难验证$u_T(x,t)$是问题(2.4)～(2.5)在$[a,b]\times[0,T]$上的解（参考[9]定理1的证明）。注意到$\{u_m(x,t)\}_{m=1}^\infty$的每个按$C^2([a,b]\times[0,T])$拓朴收敛的子序列的极限函数都可用相同的方法证明为问题(2.4)～(2.5)在$[a,b]\times[0,T]$上的解，所以由解的唯一性知整个序列$\{u_m(x,t)\}_{m=1}^\infty$都按$C^2([a,b]\times[0,T])$拓朴收敛于$u_T(x,t)$。另外，又由解的唯一性知对任意$T_1>0$和$T_2>0$，在$[a,b]\times[0,T]$ $(T=\min(T_1,T_2))$上$u_{T_1}(x,t)=u_{T_2}(x,t)=u_T(x,t)$，所以可定义$[a,b]\times[0,+\infty)$上的函数$u(x,t)$，使对每个$T>0$，在$[a,b]\times[0,T]$上$u(x,t)=u_T(x,t)$。显然函数$u(x,t)$就是问题(2.4)～(2.5)的整体经典解。定理证毕。

参 考 文 献

[1] Christensen, R. M., *Theory of Viscoelasticity, An Introduction*, 2nd edition, Academic Press, New York (1982).

[2] Renardy, M., W. E. Hrusa and J. A. Nohel, *Mathematical Problems in Viscoelasticity, Longman Group*, Boston-London-Melbourne (1987).

[3] MacCamy, R. C., A model for one-dimensional nonlinear viscc-elasticity, *Quart. Appl. Math.*, **35** (1977), 21—33.

[4] Dafermos, C. M. and J. A. Nohel, Energy methods for nonlinear hyperbolic Volterra integrodifferential equations, *Comm. Partial Differential Equations*, **4** (1979), 219—278.

[5] Dafermos, C. M. and J. A. Nohel, A nonlinear hyperbolic Volterra equation in viscoelasticity, *Amer. Math. J.*, Supplement (1981), 87—116.

[6] Hrusa, W. and M. Renardy, On wave propagation in linear viscc-elasticity, *Quart. Appl. Math.*, **43** (1985), 237—253.

[7] Hrusa, W. and J.A. Nohel, The Cauchy problem in one-dimensional nonlinear viscoelasticity, *J. Diff. Equa.*, **59** (1985), 388—412.

[8] Engler, H., Weak solutions of a class of quasilinear hyperbolic integro-differential equations describing viscoelastic materials, *Arch. Rational Mech. Anal.*, **113** (1991), 1—38.

[9] 崔尚斌，半线性卷积双曲型积分微分方程的初边值问题，应用数学学报，**11**(3) (1988), 271—286.

[10] 屈长征，关于两类方程的解，兰州大学学报，**25**(4) (1989), 5—14.

[11] 崔尚斌、马玉兰，一类半线性积分微分方程初边值问题的整体解和爆破解，应用数学，**6**(4) (1993), 445—451.

[12] Dafermos, C. A., Development of singularities in the motion of materials with fading memory, *Arch. Rational Mech. Anal.*, **91** (1986), 193—205.

[13] Klainerman, S. and A. Majda, Formation of singularities for wave equations

including the nonlinear vibrating string, *Comm. Pure Appl. Math.*, 33 (1980), 241—263.

[14] 陈予恕，《非线性振动》，天津科学技术出版社 (1983).

[15] 张恭庆，《临界点理论及其应用》，上海科学技术出版社 (1986).

Initial Boundary Value Problems for a Class of Nonlinear Integro-Partial Differential Equations

Cui Shang-bin

(*Dept. of Math., Lanzhou Univ., Lanzhou*)

Qu Chang-zheng

(*Dept. of Math., Northwest Univ., Xi'an*)

Abstract

This paper studies the global existence of the classical solutions of the following problem:

$$\begin{cases} u_{tt}-u_{xx}+\int_0^t \lambda(t-s)\sigma(u,\ u_x)_x ds=f(x,t,u,u_t), & a<x<b,\ t>0 \\ u|_{x=a}=0,\ u|_{x=b}=0, & t\geqslant 0 \\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x), & a\leqslant x\leqslant b \end{cases}$$

This problem describes the nonliear vibrations of finite rods with nonlinear viscoelasticity. Under certain conditions on σ and f, we obtained the unique existence of the global classical solution of this problem.

Key words integro-partial differetial equation, initial boundary value problem, global classical solution