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三维涡度方程单向周期初边值问题的拟谱-有限差分方法*

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摘 要

本文对三维涡度方程单向周期初边值问题建立了一种Fourier拟谱-有限差分格式，分析了其广义稳定性和收敛性．数值结果显示了这种方法的优点．

关键词 三维涡度方程 单向周期边界条件 拟谱-差分格式

一、引 言

在研究边界层和一些其他问题中，需考虑单向周期流体流动问题，例如见[1～4]．解决这种问题的一种自然的方式是对周期方向采用Fourier逼近，非周期方向采用其它方式来近似，例如：Fourier 谱-差分方法(见[5～6])．但是这种方法难以处理非线性项且耗费很多时间来计算Fourier 系数．因此采用Fourier拟谱-差分方法更合适（见[7]）．本文致力于研究三维涡度方程的数值解．

记 $x=(x_1,x_2,x_3)^*$，$\Omega=Q\times I$，

这里 $Q=\{(x_1,x_2)\,|\,0<x_1,x_2<1\}$，$I=\{x_3\,|\,0<x_3<2\pi\}$．

设$\xi(x,t)$，$\psi(x,t)$ 和 ν分别为涡度向量，流函数向量和运动粘性系数，$\xi_0(x)$，$f_l(x,t)$ 为已知函数，它们的分量分别记为$\xi^{(p)}(x,t)$，$\psi^{(p)}(x,t)$，$\xi_0^{(p)}(x,t)$和$f_l^{(p)}(x,t)$，$(p=1,2,3;$ $l=1,2)$．涡度方程具有下述形式．

$$\left.\begin{aligned}&\frac{\partial\xi}{\partial t}+[(\nabla\times\psi)\cdot\nabla]\xi-(\xi\cdot\nabla)(\nabla\times\psi)-\nu\nabla^2\xi=f_1 &&(\Omega\times(0,T])\\&-\nabla^2\psi=\xi+f_2&&(\Omega\times(0,T)\\&\xi(x,0)=\xi_0(x)&&(\bar{\Omega})\end{aligned}\right\}\tag{1.1}$$

假设所有的函数关于变量 x_3 以 2π 为周期．郭本瑜[5]对(1.1)提出了一种谱-差分格式．本文构造(1.1)的拟谱-差分方法．针对于拟谱方法产生的混迭效应，我们采用了限制算子的方法来弥补这方面的不足，克服了计算中的非线性不稳定性．本文对各种边值问题给出了严格的误差估计，数值结果证实了这种新的混合格式的优点．

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二、格 式

设h为变量$x_p(p=1,2)$的网格步长，$Mh=1$，且

$$\bar{Q}_h=\{(x_1,x_2)=(j_1h,j_2h)\,|\,0\leqslant j_1,j_2\leqslant M\} \qquad (Q_h=\bar{Q}_h\cap Q),$$

$$\bar{\Omega}_h=\bar{Q}_h\times I \qquad (\Omega_h=\bar{\Omega}_h\cap\Omega),$$

$$\Gamma_h=\{x\in\bar{\Omega}_h\,|\,x_p=0 \text{ 或 } 1,\ p=1,2\},$$

$$\Omega_h^*=\{x\in\Omega_h\,|\,x_p=h \text{ 或 } 1-h,\ p=1,2\}.$$

设τ为时间t的步长，

$$S_\tau=\{t=k\tau\,|\,k=0,1,2,\cdots\}.$$

设$u(x,t)$，$v(x,t)$为三维向量函数且关于变量x_3以2π为周期．记$e_1=(1,0,0)^*$，$e_2=(0,1,0)^*$．定义

$$u_{x_p}(x,t)=\frac{1}{h}(u(x+he_p,t)-u(x,t)),\quad u_{\bar{x}_p}(x,t)=u_{x_p}(x-he_p,t),$$

$$u_{\hat{x}_p}(x,t)=\frac{1}{2}(u_{x_p}(x,t)+u_{\bar{x}_p}(x,t)) \qquad (p=1,2)$$

$$u_n(x,t)=\begin{cases}-u_{x_p}(x,t) & (x_p=0,\ p=1,2)\\ u_{\bar{x}_p}(x,t) & (x_p=1,\ p=1,2)\end{cases}$$

$$\Delta u(x,t)=u_{x_1\bar{x}_1}(x,t)+u_{x_2\bar{x}_2}(x,t)+\frac{\partial^2u}{\partial x_3^2}(x,t),$$

$$u_t(x,t)=\frac{1}{\tau}(u(x,t+\tau)-u(x,t))$$

半离散内积和范数定义如下：

$$(u(x_1,x_2,t),\ v(x_1,x_2,t))_I=\frac{1}{2\pi}\int_0^{2\pi}u(x,t)\bar{v}(x,t)dx_3,$$

$$\|u(x_1,x_2,t)\|_I^2=(u(x_1,x_2,t),\ u(x_1,x_2,t))_I,$$

$$(u(x_3,t),v(x_3,t))_{Q_h}=h^2\sum_{(x_1,x_2)\in Q_h}u(x,t)\bar{v}(x,t)$$

$$\|u(x_3,t)\|_{Q_h}^2=(u(x_3,t),u(x_3,t))_{Q_h}$$

$$(u(t),v(t))=h^2\sum_{(x_1,x_2)\in Q_h}(u(x_1,x_2,t),v(x_1,x_2,t))_I$$

$$\|u(t)\|^2=(u(t),u(t)),\quad \|u(t)\|_{l^4}^4=\|u^2(t)\|^2,$$

$$|u(t)|_1^2=\frac{1}{2}\sum_{p=1,2}(\|u_{\bar{x}_p}(t)\|^2+\|u_{x_p}(t)\|^2)+\left\|\frac{\partial u}{\partial x_3}(t)\right\|^2,$$

$$\|u(t)\|_1^2=\|u(t)\|^2+|u(t)|_1^2,$$

$$|u(t)|_2^2=\frac{1}{2}\sum_{p=1,2}(|u_{\bar{x}_p}(u)|_{1,\tilde{\Omega}_h}^2+|u_{x_p}(t)|_{1,\tilde{\Omega}_h}^2)+\left|\frac{\partial u}{\partial x_3}(t)\right|_1^2,$$

$$\|u(t)\|_2^2=\|u(t)\|_1^2+|u(t)|_2^2,$$

这里$|u(t)|_{1,\tilde{\Omega}_h}$的定义与$|u(t)|_1$类似，但仅对$x\in\tilde{\Omega}_h=\Omega\backslash\Omega_h^*$求和．我们还需下面的范数，

$$\|u(t)\|_{\Gamma_h}^2=h\sum_{j=1}^{M-1}(\|u(0,jh,t)\|_I^2+\|u(1,jh,t)\|_I^2+\|u(jh,0,t)\|_I^2+\|u(jh,1,t)\|_I^2),$$

$$\|u(t)\|^2_{\Omega^*_h}=h\sum_{j=1}^{M-1}(\|u(h,jh,t)\|^2_I+\|u(1-h,jh,t)\|^2_I+\|u(jh,h,t)\|^2_I$$
$$+\|u(jh,1-h,t)\|^2_I).$$

类似地可定义 $\|u(t)\|_{l^4(\Gamma_h)}$ 和 $\|u(t)\|_{l^4(\Omega^*_h)}$.

对任意正整数N，记

$$V_N=\text{span}\{\exp[ilx_3]\,|\,|l|\leqslant N\}.$$

设P_N 为正交投影算子，即对任意$u\in(L^2(I))^3$,

$$\int_Q(P_Nu-u)\bar{v}dx_3=0\qquad(\forall v\in[V_N]^3).$$

设P_C：$[C(I)]^3\to[V_N]^3$为插值算子，它满足

$$P_Cu\left(\frac{2\pi j}{2N+1}\right)=u\frac{2\pi j}{2N+1})\qquad(0\leqslant j\leqslant 2N)$$

定义限制算子$R=R(\gamma)$，使得对$u\in[V_N]^3$,

$$Ru\equiv\sum_{|l|\leqslant N}\left(1-\left|\frac{l}{N}\right|^\gamma\right)u_l\exp[ilx_3].$$

构造格式的关键在于模拟下面的守恒律

$$\|\xi(t)\|^2_{L^2(\Omega)}+\int_0^t L(y)dy=\|\xi_0\|^2_{L^2(\Omega)}+2\int_0^t(f_1(y),\xi(y))_{L^2(\Omega)}dy\tag{2.1}$$

这里

$$L(y)=2\nu|\xi(y)|^2_{H^1(\Omega)}-2\nu\int_\Gamma\xi(x,y)\frac{\partial\xi}{\partial n}(x,y)ds$$
$$-\int_{\Gamma_1^-}\xi^2(x,y)\left(\frac{\partial\psi^{(3)}}{\partial x_2}(x,y)-\frac{\partial\psi^{(2)}}{\partial x_3}(x,y)\right)ds$$
$$+\int_{\Gamma_1^+}\xi^2(x,y)\left(\frac{\partial\psi^{(3)}}{\partial x_2}(x,y)-\frac{\partial\psi^{(2)}}{\partial x_3}(x,y)\right)ds$$
$$+\int_{\Gamma_2^-}\xi^2(x,y)\left(\frac{\partial\psi^{(3)}}{\partial x_1}(x,y)-\frac{\partial\psi^{(1)}}{\partial x_3}(x,y)\right)ds$$
$$+\int_{\Gamma_2^+}\xi^2(x,y)\left(\frac{\partial\psi^{(3)}}{\partial x_1}(x,y)-\frac{\partial\psi^{(1)}}{\partial x_3}(x,y)\right)ds$$
$$-2((\xi(y)\cdot\nabla)(\nabla\times\psi(y)),\ \xi(y_1))_{L^2(\Omega)},$$
$$\Gamma_p^-=\{x\in\bar{\Omega}\,|x_p=0\},\ \Gamma_p^+=\{x\in\bar{\Omega}\,|\,x_p=1\}\quad(p=1,\ 2).$$

为此目的，把对流项进行反对称分解．今定义

$$z^{(1)}(w)=w^{(3)}_{\hat{x}_2}-\frac{\partial w^{(2)}}{\partial x_3},\quad z^{(2)}(w)=\frac{\partial w^{(1)}}{\partial x_3}-w^{(3)}_{\hat{x}_1},\quad z^{(3)}(w)=w^{(2)}_{\hat{x}_1}-w^{(1)}_{\hat{x}_2},$$
$$z(w)=(z^{(1)}(w),\ z^{(2)}(w),\ z^{(3)}(w))^*,$$
$$J_1(u,w)=P_C\left(\sum_{p=1}^2 z^{(p)}(w)u_{\hat{x}_p}+z^{(3)}(w)\frac{\partial u}{\partial x_3}\right),$$
$$J_2(u,w)=\sum_{p=1}^2(P_C(z^{(p)}(w)u))_{\hat{x}_p}+\frac{\partial}{\partial x_3}P_C(z^{(3)}(w)u)),$$

且由下式来逼近$((\nabla\times\psi)\cdot\nabla)\xi$,

$$J(u,w)=\frac{1}{2}J_1(u,w)+\frac{1}{2}J_2(u,w),$$

又定义

$$H(u,w)=P_{\sigma}\Big(\sum_{p=1}^{2} u^{(p)}(z(w))_{\hat{x}_p}+u^{(3)}\frac{\partial z}{\partial x_3}(w)\Big).$$

令

$$\eta^{(N)}(x,t)=\sum_{|l|\leqslant N}\eta_l^{(N)}(x_1,x_2,t)\exp[ilx_3],$$

$$\varphi^{(N)}(x,t)=\sum_{|l|\leqslant N}\varphi_l^{(N)}(x_1,x_2,t)\exp[ilx_3].$$

则(1.1)的Fourier拟谱-差分格式为

$$\left.\begin{aligned}&\eta_t^{(N)}(x,t)+RJ(R(\eta^{(N)}(x,t)+\delta\tau\eta_t^{(N)}(x,t)),\ R\varphi^{(N)}(x,t))\\&\quad-RH(R\eta^{(N)}(x,t)),R\varphi^{(N)}(x,t))-\nu\Delta(\eta^{(N)}(x,t)+\sigma\tau\eta_t^{(N)}(x,t))\\&\quad=P_{\sigma}f_1(x,t)\\&-\Delta\varphi^{(N)}(x,t)=\eta^{(N)}(x,t)+P_{\sigma}f_2(x,t)\\&\eta^{(N)}(x,0)=\eta_0^{(N)}(x)=P_{\sigma}\xi_0(x)\end{aligned}\right\}\tag{2.2}$$

其中 $0\leqslant\delta,\ \sigma\leqslant 1$.

现在我们来验证守恒律，因为（见[8]）

$$(P_{\sigma}(uv)(t),\ w(t))=(u(t),\ P_{\sigma}(vw)(t))\qquad(\forall u,v,w\in[V_N]^3)\tag{2.3}$$

故

$$\begin{aligned}&(u(t),J(v(t),w(t)))+(v(t),J(u(t),w(t)))\\&\quad=\frac{1}{2}A(u(t),v(t),w(t))+\frac{1}{2}A(v(t),u(t),w(t))\end{aligned}\tag{2.4}$$

这里

$$\begin{aligned}A(u(t),v(t),w(t))=\frac{h}{2}\sum_{j=1}^{M-1}[&(u(1,jh,t),P_{\sigma}(z^{(1)}(w(1-h,jh,t))v(1-h,jh,t)))_I\\&+(u(1-h,jh,t),P_{\sigma}(z^{(1)}(w(1,jh,t))v(1,jh,t)))_I\\&+(u(jh,1,t),P_{\sigma}(z^{(2)}(w(jh,1-h,t))v(jh,1-h,t)))_I\\&+(u(jh,1-h,t),P_{\sigma}(z^{(2)}(w(jh,1,t))v(jh,1,t)))_I\\&-(u(h,jh,t),P_{\sigma}(z^{(1)}(w,0,jh,t))v(0,jh,t)))_I\\&-(u(0,jh,t),P_{\sigma}(z^{(1)}(w(h,jh,t))v(h,jh,t)))_I\\&-(u(jh,h,t),P_{\sigma}(z^{(2)}(w(jh,0,t))v(jh,0,t)))_I\\&-(u(jh,0,t),P_{\sigma}(z^{(2)}(w(jh,h,t))v(jh,h,t)))_I].\end{aligned}$$

特别，

$$(u(t),J(u(t),w(t)))=\frac{1}{2}A(u(t),u(t),w(t))\tag{2.5}$$

另一方面，可证明

$$\begin{aligned}&(u(t),\Delta v(t))+\frac{1}{2}\sum_{p=1}^{2}[(u_{x_p}(t),v_{x_p}(t))+(u_{\bar{x}_p}(t),v_{\bar{x}_p}(t)]+\Big(\frac{\partial u}{\partial x_3}(t),\ \frac{\partial v}{\partial x_3}(t)\Big)\\&\quad=B(u(t),v(t))\end{aligned}\tag{2.6}$$

这里

$$B(u(t),v(t))=\frac{h}{2}\sum_{j=1}^{M-1}[(u(1,jh,t)+u(1-h,jh,t),v_n(1,jh,t))_I+$$

$$+(u(jh,1,t)+u(jh,1-h,t),v_n(jh,1,t))_I$$
$$+(u(0,jh,t)+u(h,jh,t),v_n(0,jh,t))_I$$
$$+(u(jh,0,t)+u(jh,h,t),v_n(jh,0,t))_I].$$

特别，

$$(u(t),\Delta u(t))+|u(t)|_1^2=B(u(t),u(t)) \tag{2.7}$$

此外，若 $u(x,t)|_{\Gamma_h}=0$，则

$$(u(t),\Delta u(t))+|u(t)|_1^2+S(u(t))=0 \tag{2.8}$$

这里

$$S(u(t))=\frac{1}{2}\sum_{j=1}^{M-1}[\|u(h,jh,t)\|_I^2+\|u(1-h,jh,t)\|_I^2+\|u(jh,h,t)\|_I^2$$
$$+\|u(jh,1-h,t)\|_I^2] \tag{2.9}$$

令 $\delta=\sigma=\frac{1}{2}$，(2.2)的第一式与 $\eta^{(N)}(x,t)+\eta^{(N)}(x,t+\tau)$ 作内积，然后对 $t\in S_\tau$ 求和。由(2.5)，(2.7)得

$$\|\eta^{(N)}(t)\|^2+\tau\sum_{\substack{y\in S_\tau\\ y\leq t-\tau}}\Big[\frac{\nu}{2}|\eta^{(N)}(y)+\eta^{(N)}(y+\tau)|_1^2$$
$$-\frac{\nu}{2}B(\eta^{(N)}(y)+\eta^{(N)}(y+\tau),\eta^{(N)}(y)+\eta^{(N)}(y+\tau))$$
$$+\frac{1}{4}A(R(\eta^{(N)}(y)+\eta^{(N)}(y+\tau)),R(\eta^{(N)}(y)+\eta^{(N)}(y+\tau)),R\varphi^{(N)}(y))$$
$$-(H(R\eta^{(N)}(y),R\varphi^{(N)}(y)),\ R(\eta^{(N)}(y)+\eta^{(N)}(y+\tau)))\Big]$$
$$=\|\eta^{(N)}(0)\|^2+\tau\sum_{\substack{y\in S_\tau\\ y\leq t-\tau}}(P_cf_1(y),\eta^{(N)}(y)+\eta^{(N)}(y+\tau)).$$

显然，上式为(2.1)的合理模拟，因此(2.2)能提供好的数值结果。

三、数 值 结 果

本节中我们对$x_l\in\Gamma_{h,l}(l=1,2)$上的Dirichlet 边界条件问题给出(2.2)的数值结果。取试验函数为

$$\xi^{(p)}(x_1,x_2,x_3,t)=A_p\exp(B_p\sin(C_px_1+D_px_2+x_3)+E_pt),$$
$$\psi^{(p)}(x_1,x_2,x_3,t)=A_p\sin C_px_1\sin D_px_2\sin x_3\exp(E_pt)\qquad(p=1,2,3).$$

为了比较数值结果，我们还需考虑[9]中的纯差分格式。令 $\bar h=\frac{2\pi}{2N+1}$为 x_3 的网格步长，$v_{x_3},v_{\bar x_3}$，$v_{\hat x_3}$的定义与$v_{x_l},v_{\bar x_l}$和$v_{\hat x_l}$ $(l=1,2)$类似。定义

$$\bar z^{(1)}(w)=w^{(3)}_{\hat x_2}-w^{(2)}_{\hat x_3},\ \bar z^{(2)}(w)=w^{(1)}_{\hat x_3}-w^{(3)}_{\hat x_1},\ \bar z^{(3)}(w)=w^{(2)}_{\hat x_1}-w^{(1)}_{\hat x_2},$$

$$\bar J(v,w)=\frac{1}{2}\sum_{p=1}^{3}(\bar z^{(p)}(w)v_{\hat x_p}+(\bar z^{(p)}(w)v)_{\hat x_p}),\quad \bar H(v,w)=\sum_{p=1}^{3}v^{(p)}\bar z^{(p)}_{\hat x_p}(w),$$

$$\bar\Delta v(x,t)=\sum_{p=1}^{3}v_{x_p\bar x_p}(x,t).$$

(1.1)的纯差分格式为(见[9])

$$\left.\begin{array}{l}\eta_t^{\bar{k}}(x,t)+J(\eta^{\bar{k}}(x,t)+\delta\tau\eta_t^{\bar{k}}(x,t),\varphi^{\bar{k}}(x,t))-\tilde{H}(\eta^{\bar{k}}(x,t),\varphi^{\bar{k}}(x,t))\\ \quad-\nu\tilde{\Delta}(\eta^{\bar{k}}(x,t)+\sigma\tau\eta^{\bar{k}}(x,t))=f_1(x,t)\\ -\tilde{\Delta}\varphi^{\bar{k}}(x,t)=\eta^{\bar{k}}(x,t)+f_2(x,t)\end{array}\right\} \tag{3.1}$$

为度量计算误差，我们引进下面的相对范数，

$$\tilde{E}(z,t)=\frac{\left(\sum_{p=1}^{3}\sum_{j_1=1}^{M-1}\sum_{j_2=1}^{M-1}\sum_{j_3=0}^{2N}(z^{(p)}(j_1h,j_2h,j_3\bar{h},t)-\xi^{p}(j,h,j_2h,j_3\bar{h},t))^2\right)^{1/2}}{\left(\sum_{p=1}^{3}\sum_{j_1=1}^{M-1}\sum_{j_2=1}^{M-1}\sum_{j_3=0}^{2N}(\xi^{(p)}(j_1h,j_2h,j_3\bar{h},t))^2\right)^{1/2}},$$

$z=\eta^{(N)}$或$\eta^{\bar{k}}$.

计算时参数选为$\nu=10^{-3}$, $A_l=B_l=E_l=0.1$, $C_1=0.3$, $C_2=0.2$, $C_3=0.1$, $D_1=0.1$, $D_2=0.2$, $D_3=0.1$, $M=8$, $N=4$, $\delta=\sigma=0$. 表1，表2给出了格式(2.2)和(3.1) 的数值结果，且显示限制算子改进了稳定性并提高了精度.

表 1 相对误差 ($\tau=0.005$)

t	格式 (2.2)				格式 (3.1)
	$\gamma=1$	$\gamma=2$	$\gamma=5$	$\gamma=\infty$	
1.0	$0.1866E-1$	$0.2139E-1$	$0.2447E-1$	$0.2660E-1$	$0.3721E-1$
2.0	$0.3309E-1$	$0.3862E-1$	$0.4436E-1$	$0.4815E-1$	$0.7166E-1$
3.0	$0.4511E-1$	$0.5352E-1$	$0.6167E-1$	$0.6684E-1$	$0.1044E+0$
4.0	$0.5585E-1$	$0.6723E-1$	$0.7765E-1$	$0.8403E-1$	$0.1362E+0$
5.0	$0.6604E-1$	$0.8046E-1$	$0.9309E-1$	$0.1066E+0$	$0.1679E+0$

表 2 相对误差 ($\tau=0.002$)

t	格 式 (2.2)				格 式 (3.1)
	$\gamma=1$	$\gamma=2$	$\gamma=5$	$\gamma=\infty$	
0.2	$0.4196E-2$	$0.4732E-2$	$0.5391E-2$	$0.5870E-2$	$0.7718E-2$
1.0	$0.1867E-1$	$0.2139E-1$	$0.2447E-1$	$0.2660E-1$	$0.3765E-1$
2.0	$0.3309E-1$	$0.3862E-1$	$0.4436E-1$	$0.1815E-1$	$0.7165E-1$

四、一 些 引 理

为了分析广义稳定性和收敛性，需给出一些引理. 本文假设C为与N, h和任何函数无关的常数，但在不同的式子中C的值可以不一样. 设$H^\beta(I)$为通常的Sobolev空间，相应的范数和半范数分别为$\|\cdot\|_{H^\beta(I)}$和$|\cdot|_{H^\beta(I)}$. 记

$$H_p^\beta(I)=\{v(x_3)\in H^\beta(I)\,|\,v\text{关于}x_3\text{以}2\pi\text{为周期}\}.$$

引理1 若$u\in[H_p^\beta(I)]^3$, $v\in[V_N]^3$, 则

$$\|P_N u-u\|_{H^\alpha(I)}\leqslant CN^{\alpha-\beta}|u|_{H^\beta(I)}\qquad(0\leqslant\alpha\leqslant\beta)$$

$$\|P_Qu-u\|_{H^\alpha(I)}\leqslant CN^{\alpha-\beta}|u|_{H^\beta(I)}\qquad\left(0\leqslant\alpha\leqslant\beta,\ \beta>\frac{1}{2}\right)$$

$$\|R(\gamma)v-v\|_{H^\alpha(I)}\leqslant CN^{\alpha-\beta}|u|_{H^\beta(I)}\qquad(0\leqslant\alpha\leqslant\beta,\ \gamma\geqslant\beta-\alpha)$$

引理2 对任意函数$u(x,t)$，有

$$2(u(w_1,x_2,t),u_t(x_1,x_2,t))_{\Gamma_h}=(\|u(x_1,x_2,t)\|_I^2)_{\Gamma_h}-\tau\|u_t(x_1,x_2,t)\|_{\Gamma_h}^2,$$

$$2(u(t),u_t(t))=(\|u(t)\|^2)_t-\tau\|u_t(t)\|^2.$$

引理3 若对$(x_1,x_2)\in Q_h$和$t\in S_\tau$，$u(x,t)\in[V_N]^3$，则

$$2(u(t),\ \Delta u_t(t))+(|u(t)|_1^2)_t-\tau|u_t(t)|_1^2=2B(u(t),\ u_t(t)),$$

$$2(u_t(t),\Delta u(t))+(|u(t)|_1^2)_t-\tau|u_t(t)|_1^2=2B(u_t(t),u(t)).$$

引理4 若对$(x_1,x_2)\in Q_h$和$t\in S_\tau$，$u(x,t)\in[V_N]^3$，则

$$|u|_1^2\leqslant\left(N^2+\frac{8}{h^2}\right)\|u\|^2+\frac{1}{2}\min\left(h\|u_n\|_{\Gamma_h}^2,\ \frac{2}{h}\|u\|_{\Gamma_h}^2\right).$$

引理5 若对$(x_1,x_2)\in Q_h$，$u(x,t)\in[V_N]^3$且$u(x)|_{\Gamma_h}=0$，则

$$\|u\|^2\leqslant C(|u|_1^2+S(u(t))).$$

引理6 若对$(x_1,x_2)\in Q_h$，$u(x,t)\in[V_N]^3$且$u(x)|_{\Gamma_h}=0$，则

$$|u|_2^2\leqslant\|\Delta u\|^2.$$

引理7 若对$(x_1,x_2)\in Q_h$，$u(x)$，$v(x)\in[V_N]^3$，则

$$\|u(x_1,x_2)v(x_1,x_2)\|_I^2\leqslant(2N+1)\|u(x_1,x_2)\|_I^2\|v(x_1,x_2)\|_I^2,$$

$$\|u(x_3)v(x_3)\|_{Q_h}^2\leqslant\frac{1}{h^2}\|u(x_3)\|_{Q_h}^2\|v(x_3)\|_{Q_h}^2,$$

$$\|uv\|^2\leqslant\frac{2N+1}{h^2}\|u\|^2\|v\|^2,$$

$$\|uv\|_{\Omega_h^*}^2\leqslant\frac{2N+1}{h}\|u\|_{\Omega_h^*}^2\|v\|_{\Omega_h^*}^2,$$

$$\|uv\|_{\Gamma_h}^2\leqslant\frac{2N+1}{h}\|u\|_{\Gamma_h}^2\|v\|_{\Gamma_h}^2.$$

引理8 若对$(x_1,x_2)\in Q_h$，$u(x)$，$v(x)\in[V_N]^3$，则

$$\|uv\|^2\leqslant CN(\|u\|^2|v|_1^2+|u|_1^2\|v\|^2+\|u\|_{\Gamma_h}^2\|v\|_1^2+\|u\|_1^2\|v\|_{\Gamma_h}^2$$
$$+\|u\|_{\Gamma_h}^2\|v\|_{\Gamma_h}^2+h^2|u|_1^2|v|_1^2).$$

证明 假设

$$u(0,x_2,x_3)=g_1(x_2,x_3),\quad v(x_1,0,x_3)=g_2(x_1,x_3),$$

显然，

$$\max_{1\leqslant j_1\leqslant M-1}u^2(j_1h,x_2,x_3)=\max_{1\leqslant j_1\leqslant M-1}\left|g_1^2(x_2,x_3)+\sum_{1\leqslant j_1'\leqslant j_1}h(u^2(j_1'h,x_2,x_3))_{\bar{x}_1}\right|$$

$$\leqslant g_1^2(x_2,x_3)+h\sum_{1\leqslant j_1'\leqslant j_1}|u(j_1'h,x_2,x_3)+u(j_1'h-h,x_2,x_3)||u_{\bar{x}_1}(j_1'h,x_2,x_3)|$$

$$\leqslant\frac{3}{2}g_1^2(x_2,x_3)+2h\sqrt{\sum_{1\leqslant j_1\leqslant M-1}u^2(j_1h,x_2,x_3)}\sqrt{\sum_{1\leqslant j_1\leqslant M-1}u_{\bar{x}_1}^2(j_1h,x_2,x_3)}$$

$$+\frac{h^2}{2}u_{\bar{x}_1}^2(h,x_2,x_3).$$

类似地,

$$\max_{1\le j_2\le M-1} v^2(x_1,j_2h,x_3)\le\frac{3}{2}g_2^2(x_1,x_3)$$

$$+2h\sqrt{\sum_{1\le j_2\le M-1}v^2(x_1,jh,x_3)}\sqrt{\sum_{1\le j_2\le M-1}v_{\bar{x}_2}^2(x_1,j_2h,x_3)}$$

$$+\frac{h^2}{2}v_{\bar{x}_2}^2(x_1,h,x_3).$$

令

$$A=h\sum_{1\le j_2\le M-1}g_1^2(j_2h,x_3),\quad B=h\sum_{1\le j_1\le M-1}g_2^2(j_1h,x_3),$$

$$|u|_{1,Q_h}^2=\frac{1}{2}\sum_{p=1,2}(\|u_{x_p}\|_{Q_h}^2+\|u_{\bar{x}_p}\|_{Q_h}^2).$$

则有

$$\|uv\|_{Q_h}^2\le\Big(\sum_{1\le j_2\le M-1}h\max_{1\le j_1\le M-1}u^2(j_1h,j_2h,x_3)\Big)$$

$$\cdot\Big(\sum_{1\le j_1\le M-1}h\max_{1\le j_2\le M-1}v^2(j_1h,j_2h,x_3)\Big)$$

$$\le4\|u\|_{Q_h}|u|_{1,Q_h}\|v\|_{Q_h}|v|_{1,Q_h}+h|u|_{1,Q_h}^2\|v\|_{Q_h}|v|_{1,Q_h}$$

$$+h\|u\|_{Q_h}|u|_{1,Q_h}|v|_{1,Q_h}^2+\frac{1}{4}h^2|u|_{1,Q_h}^2|v|_{1,Q_h}^2+\frac{3h}{4}A|v|_{1,Q_h}^2$$

$$+\frac{3h}{4}B|u|_{1,Q_h}^2+3A\|v\|_{Q_h}|v|_{1,Q_h}+3B\|u\|_{Q_h}|u|_{1,Q_h}+\frac{9}{4}AB.$$

进一步，我们有

$$4\|u\|_{Q_h}|u|_{1,Q_h}\|v\|_{Q_h}|v|_{1,Q_h}\le2\|u\|_{Q_h}^2\|v\|_1^2\|^2{}_{1,\Omega_h}+2|u|_{1,Q_h}^2\|v\|_{Q_h}^2,$$

$$h|u|_{1,Q_h}^2\|v\|_{Q_h}|v|_{1,Q_h}\le\frac{1}{2}|u|_{1,Q_h}^2\|v\|_{Q_h}^2+\frac{h^2}{2}|u|_{1,Q_h}^2,$$

$$h\|u\|_{Q_h}|u|_{1,Q_h}|v|_{1,Q_h}^2\le\frac{1}{2}\|u\|_{Q_h}^2|v|_{1,Q_h}^2+\frac{h^2}{2}|u|_{1,Q_h}^2|v|_{1,Q_h}^2,$$

$$3A\|v\|_{Q_h}|v|_{1,Q_h}\le\frac{3A}{2}\|v\|_{Q_h}^2+\frac{3A}{2}|v|_{1,Q_h}^2$$

以及

$$3B\|u\|_{Q_h}|u|_{1,Q_h}\le\frac{3B}{2}\|u\|_{Q_h}^2+\frac{3B}{2}|u|_{1,Q_h}^2.$$

再应用引理7即得本引理。

引理9 若对$(x_1,x_2)\in Q_h$, $u(x)\in[V_N]^3$, 则

$$\|u\|_{l_4}^4\le\frac{C}{h^2}\|u\|^3\|u\|_1.$$

引理10 若$h<2\varepsilon$且对$(x_1,x_2)\in Q_h$, $u(x)\in[V_N]^4$, 则

$$\|u\|_{\Gamma_h}^2\le\varepsilon|u|_1^2+C(e)\|u\|^2,$$

$$\|u\|^2_{\Omega_h^*}\leqslant\varepsilon|u|_1^2+C(\varepsilon)\|u\|^2.$$

引理11 假设

(i) $E(t)$为S_τ上的非负函数;

(ii) a,b,ρ,M_l为非负常数;

(iii) 函数$F(E)$满足对所有$E\leqslant M_3$，成立$F(E)\leqslant 0$;

(iv) 对所有$t\in S_\tau$,

$$E(t)\leqslant\rho+\tau\sum_{\substack{y\in S_\tau\\ y\leqslant t-\tau}}[M_1E(y)+M_2N^ah^{-b}E^2(y)+F(E(y))];$$

(V) $E(0)\leqslant\rho$且$\rho\exp[(M_1+M_2)T]\leqslant\min\left(M_3,\dfrac{h^b}{N^a}\right)$.

则对所有$t\in S_\tau$，$t\leqslant T$,

$$E(t)\leqslant\rho\exp[(M_1+M_2)t].$$

特别，若$M_2=0$，且对任意E有$F(E)\leqslant 0$，则对任意ρ和t有

$$E(t)\leqslant\rho\exp[M_1t].$$

引理1来源于文[10,11]，引理2～引理7以及引理9～引理11可在[5]中找到.

五、Dirichlet 边值条件问题的误差估计

考虑下面的边界条件

$$\eta^{(N)}(x,t)=P_c\xi(x,t)=P_cg(x,t),\ \ \varphi^{(N)}(x,t)=P_c\psi(x,t)=P_c\chi(x,t),\ \ (x,t)\in\Gamma_h\times S_\tau.$$

假定 $h=O\left(\dfrac{1}{N}\right)$, $\tau=O\left(\dfrac{1}{N^2}\right)$, $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{\xi}_0$, $\tilde{g}$分别表示f_1, f_2, ξ_0, g的误差，由此导致$\eta^{(N)}$和$\varphi^{(N)}$的误差，记为$\tilde{\eta}^{(N)}$和$\tilde{\varphi}^{(N)}$. 为简化起见，假设在Γ_h上$\tilde{\varphi}(x,t)=0$. 误差满足下述方程:

$$\left.\begin{aligned}&\tilde{\eta}_t^{(N)}(x,t)+RJ(R(\tilde{\eta}^{(N)}(x,t)+\delta\tau\tilde{\eta}_t^{(N)}(x,t)),\ R(\tilde{\varphi}^{(N)}(x,t)\\&\qquad+\varphi^{(N)}(x,t)))+RJ(R(\eta^{(N)}(x,t)+\delta\tau\eta_t^{(N)}(x,t),R\tilde{\varphi}^{(N)}(x,t))\\&\qquad-RH(R\tilde{\eta}^{(N)}(x,t),R(\tilde{\varphi}^{(N)}(x,t)+\varphi^{(N)}(x,t))\\&\qquad-RH(R\eta^{(N)}(x,t),R\tilde{\varphi}^{(N)}(x,t))-\nu\Delta(\tilde{\eta}^{(N)}(x,t)+\sigma\tau\tilde{\eta}_t^{(N)}(x,t))\\&\quad=P_c\tilde{f}_1(x,t),\\&-\Delta\tilde{\varphi}^{(N)}(x,t)=\tilde{\eta}^{(N)}(x,t)+P_c\tilde{f}_2(x,t),\\&\tilde{\eta}^{(N)}(x,0)=P_c\tilde{\xi}_0(x).\end{aligned}\right\}\tag{5.1}$$

此外，在$\Gamma_h\times S_\tau$上$\tilde{\eta}^{(N)}(x,t)=P_c\tilde{g}(x,t)$. 令

$$\|u(t)\|_{q,\infty}=\max_{r_0+r_1+r_2+r_3+r_4\leqslant q}\ \max_{x\in\Omega_h}\left|\left(\frac{\partial^{r_0}u}{\partial x_3^{r_0}}\right)_{\underbrace{x_1\cdots x_1}_{r_1}\underbrace{\bar{x}_1\cdots\bar{x}_1}_{r_2}\underbrace{x_2\cdots x_2}_{r_3}\underbrace{\bar{x}_2\cdots\bar{x}_2}_{r_4}}\right|,$$

$$\|u\|_{q,\infty}=\max_{t\in S_\tau}\|u(t)\|_{q,\infty}.$$

把(5.1)的第一式与$2\tilde{\eta}^{(N)}(x,t)$作内积. 由(2.1), (2.6), 引理2和引理3得

$$\|\tilde\eta^{(N)}(t)\|_t^2-\tau\|\tilde\eta_t^{(N)}(t)\|^2+2\nu|\tilde\eta^{(N)}(t)|_1^2+\nu\sigma\tau(|\tilde\eta^{(N)}|_1^2)_t-\nu\sigma\tau^2|\tilde\eta_t^{(N)}(t)|_1^2$$

$$-2\delta\tau(R\tilde\eta_t^{(N)}(t),\ J(R\tilde\eta^{(N)}(t),\ R\tilde\varphi^{(N)}(t)))$$

$$+2(R\tilde\eta^{(N)}(t),\ J(R(\eta^{(N)}(t)+\delta\tau\eta_t^{(N)}(t)),\ R\tilde\varphi^{(N)}(t))$$

$$+J(R(\tilde\eta^{(N)}(t)+\delta\tau\tilde\eta_t^{(N)}(t)),\ R\varphi^{(N)}(t))$$

$$-2(R\tilde\eta^{(N)}(t),\ H(R\tilde\eta^{(N)}(t),\ R(\varphi^{(N)}(t)+\tilde\varphi^{(N)}(t)))$$

$$+H(R\eta^{(N)}(t),\ R\tilde\varphi^{(N)}(t)))+D_1(t)+D_2(t)+B_1(t)+B_2(t)$$

$$=2(\tilde\eta^{(N)}(t),\ P_c\tilde f_1(t)),\tag{5.2}$$

这里

$$D_1(t)=A(R\tilde\eta^{(N)}(t),\ R\tilde\eta^{(N)}(t)R\tilde\varphi^{(N)}(t)),$$

$$D_2(t)=\delta\tau A(R\tilde\eta^{(N)}(t),\ R\tilde\eta_t^{(N)}(t),\ R\tilde\varphi^{(N)}(t))$$

$$+\delta\tau A(R\tilde\eta_t^{(N)}(t),\ R\tilde\eta^{(N)}(t),\ R\tilde\varphi^{(N)}(t)),$$

$$B_1(t)=-2\nu B(\tilde\eta^{(N)}(t),\ \tilde\eta^{(N)}(t)),\qquad B_2(t)=-2\nu\sigma\tau B(\tilde\eta^{(N)}(t),\ \tilde\eta_t^{(N)}(t)).$$

设m为待定正常数．把(5.1)的第一式与$m\tau\tilde\eta_t^{(N)}(t)$作内积后得到

$$m\tau\|\tilde\eta_t^{(N)}(t)\|^2+\frac{1}{2}m\nu\tau(|\tilde\eta^{(N)}(t)|_1^2)_t+m\nu\tau^2\left(\sigma-\frac{1}{2}\right)|\tilde\eta_t^{(N)}|_1^2$$

$$+m\tau(\tilde\eta_t^{(N)}(t),\ RJ(R\tilde\eta^{(N)}(t),\ R\tilde\varphi^{(N)}(t)))$$

$$+m\tau(\tilde\eta_t^{(N)}(t),\ RJ(R(\tilde\eta^{(N)}(t)+\delta\tau\tilde\eta_t^{(N)}(t)),\ R\varphi^{(N)}(t))$$

$$+RJ(R(\eta^{(N)}(t)+\delta\tau\eta_t^{(N)}(t)),\ R\tilde\varphi^{(N)}(t)))$$

$$+D_3(t)+B_3(t)+B_4(t)=m\tau(\tilde\eta_t^{(N)}(t),\ P_c\tilde f_1(t))\tag{5.3}$$

其中

$$D_3(t)=\frac{1}{2}m\delta\tau^2A(R\tilde\eta_t^{(N)}(t),\ R\tilde\eta_t^{(N)}(t),\ R\tilde\varphi^{(N)}(t)),$$

$$B_3(t)=-m\nu\tau B(\tilde\eta_t^{(N)}(t),\ \tilde\eta^{(N)}(t)),$$

$$B_4(t)=-m\nu\sigma\tau^2B(\tilde\eta_t^{(N)}(t),\ \tilde\eta_t^{(N)}(t)).$$

设$\varepsilon>0$是适当小常数．把(5.2)和(5.3)相加后得到

$$\|\tilde\eta^{(N)}(t)\|_t^2+\tau(m-1-\varepsilon)\|\tilde\eta_t^{(N)}(t)\|^2+2\nu|\tilde\eta^{(N)}(t)|_1^2$$

$$+\nu\tau\left(\sigma+\frac{m}{2}\right)(|\tilde\eta^{(N)}(t)|_1^2)_t$$

$$+\nu\tau^2\left(m\sigma-\sigma-\frac{m}{2}\right)|\tilde\eta_t^{(N)}(t)|_1^2+\sum_{\alpha=1}^{5}G_\alpha(t)+\sum_{\alpha=1}^{3}D_\alpha(t)$$

$$+\sum_{\alpha=1}^{4}B_\alpha(t)\leqslant\|\tilde\eta^{(N)}(t)\|^2+\left(1+\frac{m^2\tau}{4\varepsilon}\right)\|\tilde f_1(t)\|^2\tag{5.4}$$

其中

$$G_1(t)=(R(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t)),$$
$$J(R(\eta^{(N)}(t)+\delta\tau\eta_t^{(N)}(t),\ R\tilde{\varphi}^{(N)}(t))),$$
$$G_2(t)=(R(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t)),\ J(R(\tilde{\eta}^{(N)}(t)+\delta\tau\tilde{\eta}_t^{(N)}(t)),\ R\varphi^{(N)}(t))),$$
$$G_3(t)=\tau(m-2\delta)(R\tilde{\eta}_t^{(N)}(t),\ J(R\tilde{\eta}^{(N)}(t),\ R\tilde{\varphi}^{(N)}(t))),$$
$$G_4(t)=-(R(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t)),\ H(R\tilde{\eta}^{(N)}(t),\ R\tilde{\varphi}^{(N)}(t))),$$
$$G_5(t)=-(R(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t)),\ H(R\eta^{(N)}(t),\ R\tilde{\varphi}^{(N)}(t))$$
$$+H(R\tilde{\eta}^{(N)}(t),\ R\varphi^{(N)}(t))).$$

另一方面，把(5.1)的第二式与$R^2\tilde{\varphi}^{(N)}(t)$作内积。由引理1，引理5和(2.8)得

$$|R\tilde{\varphi}^{(N)}(t)|_1^2+S(R\tilde{\varphi}^{(N)}(t))\leqslant C(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2) \tag{5.5}$$

进一步，由引理6得

$$|\tilde{\varphi}^{(N)}(t)|_2^2\leqslant C(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2) \tag{5.6}$$

现在我们来估计$|G_a(t)|$，由引理1和(5.5)式得

$$|G_1(t)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+C\left(1+\left(1+\frac{m^2\tau}{\varepsilon}\right)|||R\eta^{(N)}|||_{1,\infty}^2\right)(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2) \tag{5.7}$$

$$|G_2(t)|\leqslant\left(\varepsilon\tau+\frac{C\tau^2}{\varepsilon}|||R\varphi^{(N)}|||_{2,\infty}^2\right)\|\tilde{\eta}_t^{(N)}(t)\|^2+\varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2$$
$$+\frac{C}{\varepsilon}|||R\varphi^{(N)}|||_{2,\infty}^2\|\tilde{\eta}^{(N)}(t)\|^2$$
$$+C(1+|||R\varphi^{(M)}|||_{2,\infty}^2)\left(\frac{\tau}{h}\|\tilde{g}(t)\|_{\Gamma_h}^2+\tau h\|\tilde{g}_t(t)\|_{\Gamma_h}^2\right). \tag{5.8}$$

由引理7和(5.5)得

$$|G_3(t)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\frac{C\tau N}{\varepsilon h^2}|\tilde{\eta}^{(N)}(t)|_1^2(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2) \tag{5.9}$$

由引理4，引理7和(5.6)得

$$|G_4(t)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\frac{C\tau N}{\varepsilon h^2}\|\tilde{\eta}^{(N)}(t)\|^4$$
$$+CN\|\tilde{\eta}^{(N)}(t)\|^2|\tilde{\eta}^{(N)}(t)|_1^2+C\left(N\|\tilde{g}(t)\|_{\Gamma_h}^2\right.$$
$$\left.+\frac{\tau N}{\varepsilon h^2}\|\tilde{f}_2(t)\|^2+1\right)\|\tilde{\eta}^{(N)}(t)\|^2+CN\|\tilde{g}(t)\|_{\Gamma_h}^2|\tilde{\eta}^{(N)}(t)|_1^2$$
$$+CN\|\tilde{g}(t)\|_{\Gamma_h}^2+C\|\tilde{f}(t)\|^2. \tag{5.10}$$

类似地，

$$|G_5(t)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+C\left(1+\frac{\tau}{\varepsilon}\right)(|||R\eta^{(N)}|||_{0,\infty}^2$$
$$+|||R\varphi^{(N)}|||_{2,\infty}^2)(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2) \tag{5.11}$$

下面我们再来估计$|D_a(t)|$，由引理7，引理9和(5.6)得

$$|D_1(t)|\leqslant \varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\frac{CN}{\varepsilon\nu}\|\tilde{g}(t)\|^2_{\Gamma_h}\sum_{p=1}^{2}\|z^{(p)}(R\tilde{\varphi}^{(N)}(t))\|^2_{\Omega^*_h}$$

$$\leqslant \varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\frac{CN}{\varepsilon\nu}\|\tilde{g}(t)\|^2_{\Gamma_h}(|\tilde{\varphi}^{(N)}(t)|^2_1+|\tilde{\varphi}^{(N)}(t)|^2_2)$$

$$\leqslant \varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\frac{CN}{\varepsilon\nu}\|\tilde{g}(t)\|^2_{\Gamma_h}(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2) \tag{5.12}$$

同理有

$$|D_2(t)|\leqslant \varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t))$$

$$+\frac{CN}{\varepsilon\nu}(\|\tilde{g}(t)\|^2_{\Gamma_h}+\tau^2\|\tilde{g}_t(t)\|^2_{\Gamma_h})(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2) \tag{5.13}$$

$$|D_3(t)|\leqslant \varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t))+\frac{CN\tau^2}{\varepsilon\nu}\|\tilde{g}_t(t)\|^2_{\Gamma_h}(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2) \tag{5.14}$$

仿[5]中类似证明得到

$$B_1(t)\geqslant 2\nu S(\tilde{\eta}^{(N)}(t))-\frac{C}{\varepsilon h}\|\tilde{g}(t)\|^2_{\Gamma_h}, \tag{5.15}$$

$$B_2(t)+B_3(t)\geqslant \nu\tau\left(\sigma+\frac{m}{2}\right)[S(\tilde{\eta}^{(N)}(t))]_t-\nu^2\left(\sigma+\frac{m}{2}\right)S(\tilde{\eta}_t^{(N)}(t))$$

$$-\varepsilon\nu S(\tilde{\eta}^{(N)}(t))-\varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t))-\frac{C}{\varepsilon h}(\|\tilde{g}(t)\|^2_{\Gamma_h}+\tau h^2\|\tilde{g}_t(t)\|^2_{\Gamma_h}), \tag{5.16}$$

$$B_4(t)\geqslant m\nu\sigma\tau^2 S(\eta_t^{(N)}(t))-\frac{C\tau h}{\varepsilon}\|\tilde{g}_t(t)\|^2_{\Gamma_h}. \tag{5.17}$$

将(5.7)～(5.17)代入(5.4)后得到

$$\|\tilde{\eta}^{(N)}(t)\|^2_t+\tau\left(m-1-6\varepsilon-\frac{C\tau}{\varepsilon}\||R\varphi^{(N)}\||^2_{2,\infty}\right)\|\tilde{\eta}_t^{(N)}(t)\|^2+\nu|\tilde{\eta}^{(N)}(t)|^2_1$$

$$+\nu\tau\left(\sigma+\frac{m}{2}\right)(|\tilde{\eta}^{(N)}(t)|^2_1)_t+\nu\tau^2\left(m\sigma-\sigma-\frac{m}{2}\right)|\tilde{\eta}_t^{(N)}(t)|^2_1$$

$$+\nu(2-3\varepsilon)S(\tilde{\eta}^{(N)}(t))+\nu\tau\left(\sigma+\frac{m}{2}\right)[S(\tilde{\eta}^{(N)}(t))]_t$$

$$+\nu\tau^2\left(m\sigma-\sigma-\frac{m}{2}-3\varepsilon\right)S(\tilde{\eta}_t^{(N)}(t))$$

$$\leqslant F_0(t)\|\tilde{\eta}^{(N)}(t)\|^2+F_1(t)\|\tilde{\eta}^{(N)}(t)\|^4+F_2(t)|\tilde{\eta}^{(N)}(t)|^2_1+\tilde{R}(t), \tag{5.18}$$

这里

$$F_0(t)=C+C\left(1+\frac{\tau}{\varepsilon}\right)(\||R\eta^{(N)}\||^2_{1,\infty}+\||R\varphi^{(N)}\||^2_{2,\infty})$$

$$+CN\left(1+\frac{1}{\varepsilon\nu}\right)\|\tilde{g}(t)\|^2_{\Gamma_h}+\frac{C\tau^2N}{\varepsilon\nu}\|\tilde{g}_t(t)\|^2_{\Gamma_h}+\frac{C\tau N}{\varepsilon h^2}\|\tilde{f}_2(t)\|^2,$$

$$F_1(t)=\frac{C\tau N}{\varepsilon h^2},$$

$$F_2(t)=-\nu+\varepsilon\nu+CN\left(1+\frac{\tau}{\varepsilon h^2}\right)\|\tilde{\eta}^{(N)}(t)\|^2+\frac{C\tau N}{\varepsilon h^2}\|\tilde{f}_2(t)\|^2+CN\|\tilde{g}(t)\|^2_{V_h},$$

$$\tilde{R}(t)=C\left(1+\frac{\tau}{\varepsilon}\right)\|\tilde{f}_1(t)\|^2+C\left[1+\left(1+\frac{\tau}{\varepsilon}\right)(\|R\eta^{(N)}\|^2_{1,\infty}+\|R\varphi^{(N)}\|^2_{2,\infty})\right.$$

$$\left.+\frac{N}{\varepsilon\nu}(\|\tilde{g}(t)\|^2_{V_h}+\tau^2\|\tilde{g}_t(t)\|^2_{V_h})\right]\|\tilde{f}_2(t)\|^2$$

$$+\frac{C}{\varepsilon h}(1+\|R\varphi^{(N)}\|^2_{2,\infty})(\|\tilde{g}(t)\|^2_{V_h}$$

$$+\tau^2\|\tilde{g}_t(t)\|^2_{V_h})+CN\|\tilde{g}(t)\|^4_{V_h}).$$

假定 $\tau<\dfrac{\varepsilon^2}{C\|R\varphi^{(N)}\|^2_{2,\infty}}$. 按如下方式选择 m,

情形 I $\sigma>\frac{1}{2}$. 取

$$m>m_1=\max\left(\frac{2\sigma+\sigma\varepsilon}{2\sigma-1},\ 1+p_0+7\varepsilon\right),\quad p_0\geqslant 0.$$

则由(5.18)得

$$\|\tilde{\eta}^{(N)}(t)\|^2_t+p_0\|\tilde{\eta}_t^{(N)}(t)\|^2+\nu[\,|\tilde{\eta}^{(N)}(t)|^2_1+S(\tilde{\eta}^{(N)}(t))]$$

$$+\nu\tau\left(\sigma+\frac{m}{2}\right)[\,|\tilde{\eta}^{(N)}(t)|^2_1+S(\tilde{\eta}^{(N)}(t))]_t$$

$$\leqslant F_0(t)\|\tilde{\eta}^{(N)}(t)\|^2+F_1(t)\|\tilde{\eta}^{(N)}(t)\|^4+F_2(t)\,|\tilde{\eta}^{(N)}(t)|^2_1+\tilde{R}(t) \tag{5.19}$$

情形 II $\sigma=\frac{1}{2}$. 由引理4和$S(u)$的定义得

$$|\tilde{\eta}_t^{(N)}(t)|^2_1\leqslant\left(N^2+\frac{8}{h^2}\right)\|\tilde{\eta}_t^{(N)}(t)\|^2+\frac{1}{h}\|\tilde{g}_t(t)\|^2_{V_h},$$

$$S(\tilde{\eta}_t^{(N)}(t))\leqslant\frac{1}{2h^2}\|\tilde{\eta}_t^{(N)}(t)\|^2.$$

因此

$$\tau(m-1-7\varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2-\nu\tau^2\left(\frac{1}{2}+3\varepsilon\right)(|\tilde{\eta}_t^{(N)}(t)|^2_1+S(\tilde{\eta}_t^{(N)}(t)))$$

$$\geqslant\tau\left(m-1-7\varepsilon-\frac{\nu\tau}{4h^2}(1+6\varepsilon)(2N^2h^2+17)\right)\|\tilde{\eta}_t^{(N)}(t)\|^2$$

$$-\frac{C\tau h}{\varepsilon}\|\tilde{g}_t(t)\|^2_{V_h}.$$

此时取

$$m>m_2=1+p_0+7\varepsilon+\frac{\nu\tau(1+6\varepsilon)(2N^2h^2+17)}{4h^2},$$

则(5.19)仍然成立。

情形 III $\sigma<\frac{1}{2}$, $\tau<\dfrac{4h^2}{\nu(1-2\sigma)(17+2N^2h^2)}$. 取

$$m>m_3=\left(4+4p_0+28\varepsilon+2\nu\tau\ (\sigma+3\varepsilon)\left(2N^2+\frac{17}{h^2}\right)\right.$$

$$\left.\left(4-\nu\tau(1-2\sigma)\left(2N^2+\frac{17}{h^2}\right)\right)\right)^{-1},$$

则(5.19)亦成立。

令

$$E_1^{(N)}(t)=\|\tilde{\eta}_t^{(N)}(t)\|^2+\nu\tau(|\tilde{\eta}^{(N)}(t)|_1^2+S(\tilde{\eta}^{(N)}(t))$$

$$+\tau\sum_{\substack{y\in S_\tau\\ y\leqslant t-\tau}}[p_0\tau\|\tilde{\eta}_t^{(N)}(y)\|^2+\nu|\tilde{\eta}^{(N)}(y)|_1^2+\nu S(\tilde{\eta}^{(N)}(y))],$$

$$\rho_1^{(N)}(t)=\|\tilde{\eta}^{(N)}(0)\|^2+\tau\sum_{\substack{y\in S_\tau\\ y\leqslant t-\tau}}\tilde{R}(y).$$

将(5.19)对所有$t\in S_\tau$求和后得到

$$E_1^{(N)}(t)\leqslant\rho_1^{(N)}(t)+\tau\sum_{\substack{y\in S_\tau\\ y\leqslant t-\tau}}(F_0(y)E_1^{(N)}(y)+F_1(y)(E_1^{(N)}(y))^2$$

$$+F_2(y)|\tilde{\eta}^{(N)}(y)|_1^2). \tag{5.20}$$

最后应用引理11得下面的结论。

定理1 假设

(i) $h=O\left(\frac{1}{N}\right)$ 且$\tau=O\left(\frac{1}{N^2}\right)$;

(ii) $\sigma\geqslant\frac{1}{2}$ 或$\tau<\frac{4h^2}{\nu(1-2\sigma)(17+2N^2h^2)}$;

(iii) 对所有$t\leqslant T$, $\|\tilde{f}_2(t)\|^2\leqslant\frac{b_1}{N}$, $\|\tilde{g}(t)\|_{F_h}^2\leqslant\frac{b_2}{N}$, $\rho_1^{(N)}(t)\leqslant\frac{b_3}{N}$.

则对所有$t\leqslant T$,

$$E_1^{(N)}(t)\leqslant b_4\exp[b_5t]\rho_1^{(N)}(t),$$

这里b_a为仅依赖于$\|R\eta^{(N)}\|_{1,\infty}$, $\|R\varphi^{(N)}\|_{2,\infty}$和$\nu$的正常数。

下面我们讨论收敛性。为简便起见,用$\eta^{(N)}$记$\eta^{(N)}(x,t)$等。令

$$\xi^{(N)}=P_N\xi,\ \psi^{(N)}=P_N\psi,\ \tilde{\xi}^{(N)}=\eta^{(N)}-\xi^{(N)},\ \tilde{\psi}^{(N)}=\varphi^{(N)}-\psi^{(N)}.$$

则

$$\left.\begin{aligned}&\xi_t^{(N)}+RJ(R(\xi^{(N)}+\delta\tau\xi_t^{(N)}),\ R\psi^{(N)})-RH(R\xi^{(N)},\ R\psi^{(N)})\\&\quad-\nu\Delta(\xi^{(N)}+\sigma\tau\,\xi_t^{(N)})=P_Nf_1+\sum_{a=1}^{6}M_a,\\&-\Delta\psi^{(N)}=\xi^{(N)}+P_Nf_2+M_7,\\&\xi^{(N)}(0)=P_N\xi_0,\end{aligned}\right\}\tag{5.21}$$

这里

$$M_1=\xi_t^{(N)}-\frac{\partial\xi^{(N)}}{\partial t},\ M_2=\delta\tau RJ(R\xi_t^{(N)},\ R\psi^{(N)}),$$

$$M_3=RJ(R\xi^{(N)},\ R\psi^{(N)})-P_N[(\nabla\times\psi)\cdot\nabla]\xi,$$

$$M_4=P_N[(\xi\cdot\nabla)(\nabla\times\psi)]-RH(R\xi^{(N)},\ R\psi^{(N)}),$$

$$M_5=\nu\frac{\partial^2\xi^{(N)}}{\partial x_1^2}+\nu\frac{\partial^2\xi^{(N)}}{\partial x_2^2}-\nu\xi^{(N)}_{x_1\bar{x}_1}-\nu\xi^{(N)}_{x_2\bar{x}_2},\ M_6=\nu\sigma\tau\Delta\xi_t^{(N)},$$

$$M_7=\frac{\partial^2\psi^{(N)}}{\partial x_1^2}+\frac{\partial^2\psi^{(N)}}{\partial x_2^2}-\psi^{(N)}_{x_1\bar{x}_1}-\psi^{(N)}_{x_2\bar{x}_2}.$$

故有

$$\left.\begin{aligned}&\tilde{\xi}_t^{(N)}+RJ(R(\tilde{\xi}^{(N)}+\delta\tau\tilde{\xi}_t^{(N)}),\ R(\psi^{(N)}+\tilde{\psi}^{(N)}))\\&\qquad+RJ(R(\xi^{(N)}+\delta\tau\xi_t^{(N)}),\ R\tilde{\psi}^{(N)})\\&\qquad-\nu\Delta(\tilde{\xi}^{(N)}+\sigma\tau\tilde{\xi}_t^{(N)})-RH(R\tilde{\xi}^{(N)},\ R(\psi^{(N)}+\tilde{\psi}^{(N)}))\\&\qquad-RH\ (R\xi^{(N)},\ R\tilde{\psi}^{(N)})=P_cf_1-P_Nf_1-\sum_{a=1}^{6}M_a,\\&-\Delta\tilde{\psi}^{(N)}=\tilde{\xi}^{(N)}-M_7+P_cf_2-P_Nf_2,\\&\tilde{\xi}^{(N)}(0)=P_c\xi_0-P_N\xi_0.\end{aligned}\right\}\qquad(5.22)$$

此外在$\Gamma_h\times S_\tau$上，$\tilde{\xi}^{(N)}=P_cg-P_Ng$.

令 $\beta>\frac{1}{2}$，$\mu>0$. 应用引理1和Sobolev 嵌入定理得

$$\|M_1\|\leqslant C\tau\left\|\frac{\partial^2\xi}{\partial t^2}\right\|_{H^{1+\mu}(Q,L^2(I))},$$

$$\|M_2\|\leqslant C\tau\|\psi\|_{H^{\frac{5}{2}+\mu}(\Omega)}\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{2+\mu}(Q,L^2(I))}+\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{1+\mu}(Q,H^1(I))}\right),$$

$$\begin{aligned}\|M_3\|\leqslant C(N^{-\beta}+h^2)&(\|\psi\|_{H^{\frac{5}{2}+\mu}(\Omega)}(\|\xi\|_{H^{1+\mu}(Q,H^{\beta+1}(I))}\\&+\|\xi\|_{H^{2+\mu}(Q,H^\beta(I))}+\|\xi\|_{H^{4+\mu}(Q,L^2(I))})\\&+\|\xi\|_{H^{\frac{5}{2}+\mu}(\Omega)}(\|\psi\|_{H^{1+\mu}(Q,H^{\beta+1}(I))}+\|\psi\|_{H^{2+\mu}(Q,H^\beta(I))}\\&+\|\psi\|_{H^{4+\mu}(Q,L^2(I))}),\end{aligned}$$

$$\begin{aligned}\|M_4\|\leqslant C|(N^{-\beta}+h^2)&[\|\xi\|_{H^{\frac{3}{2}+\mu}(\Omega)}(\|\psi\|_{H^{1+\mu}(Q,H^{\beta+2}(I))}\\&+\|\psi\|_{H^{2+\mu}(Q,H^{\beta+1}(I))}+\|\psi\|_{H^{3+\mu}(Q,H^\beta(I))}\\&+\|\psi\|_{H^{4+\mu}(Q,H^1(I))}+\|\psi\|_{H^{5+\mu}(Q,L^2(I))})\\&+\|\psi\|_{H^{1/2+\mu}(\Omega)}\|\xi\|_{H^{1+\mu}(Q,H^\beta(I))}],\end{aligned}$$

$$\|M_5\|\leqslant Ch^2\|\xi\|_{H^{5+\mu}(Q,L^2(I))},$$

$$\|M_6\|\leqslant C\tau\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{3+\mu}(Q,L^2(I))}+\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{1+\mu}(Q,H^2(I))}\right),$$

$$\|M_7\|\leqslant ch^2\|\psi\|_{H^{5+\mu}(Q,L^2(I))}.$$

另外，

$$M_4=P_N[(\xi\cdot\nabla)(\nabla\times\psi)]-RH(R\xi^{(N)},\ R\psi^{(N)}),$$

$$M_5=\nu\frac{\partial^2\xi^{(N)}}{\partial x_1^2}+\nu\frac{\partial^2\xi^{(N)}}{\partial x_2^2}-\nu\xi^{(N)}_{x_1\bar{x}_1}-\nu\xi^{(N)}_{x_2\bar{x}_2},\ M_6=\nu\sigma\tau\Delta\xi_t^{(N)},$$

$$M_7=\frac{\partial^2\psi^{(N)}}{\partial x_1^2}+\frac{\partial^2\psi^{(N)}}{\partial x_2^2}-\psi^{(N)}_{x_1\bar{x}_1}-\psi^{(N)}_{x_2\bar{x}_2}.$$

故有

$$\left.\begin{aligned}&\tilde{\xi}_t^{(N)}+RJ(R(\tilde{\xi}^{(N)}+\delta\tau\tilde{\xi}_t^{(N)}),\ R(\psi^{(N)}+\tilde{\psi}^{(N)}))\\&\quad+RJ(R(\xi^{(N)}+\delta\tau\xi_t^{(N)}),\ R\tilde{\psi}^{(N)})\\&\quad-\nu\Delta(\tilde{\xi}^{(N)}+\sigma\tau\tilde{\xi}_t^{(N)})-RH(R\tilde{\xi}^{(N)},\ R(\psi^{(N)}+\tilde{\psi}^{(N)}))\\&\quad-RH\ (R\xi^{(N)},\ R\tilde{\psi}^{(N)})=P_cf_1-P_Nf_1-\sum_{a=1}^{6}M_a,\\&-\Delta\tilde{\psi}^{(N)}=\tilde{\xi}^{(N)}-M_7+P_cf_2-P_Nf_2,\\&\tilde{\xi}^{(N)}(0)=P_c\xi_0-P_N\xi_0.\end{aligned}\right\}\qquad(5\ 22)$$

此外在$\Gamma_h\times S_\tau$上，$\tilde{\xi}^{(N)}=P_cg-P_Ng$。

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$$\|M_1\|\leqslant C\tau\left\|\frac{\partial^2\xi}{\partial t^2}\right\|_{H^{1+\mu}(Q,L^2(I))},$$

$$\|M_2\|\leqslant C\tau\|\psi\|_{H^{\frac{5}{2}+\mu}(\Omega)}\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{2+\mu}(Q,L^2(I))}+\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{1+\mu}(Q,H^1(I))}\right),$$

$$\begin{aligned}\|M_3\|\leqslant C(N^{-\beta}+h^2)(\|\psi\|_{H^{\frac{5}{2}+\mu}(\Omega)}(\|\xi\|_{H^{1+\mu}(Q,H^{\beta+1}(I))}\\+\|\xi\|_{H^{2+\mu}(Q,H^\beta(I))}+\|\xi\|_{H^{4+\mu}(Q,L^2(I))})\\+\|\xi\|_{H^{\frac{5}{2}+\mu}(\Omega)}(\|\psi\|_{H^{1+\mu}(Q,H^{\beta+1}(I))}+\|\psi\|_{H^{2+\mu}(Q,H^\beta(I))}\\+\|\psi\|_{H^{4+\mu}(Q,L^2(I))}),\end{aligned}$$

$$\begin{aligned}\|M_4\|\leqslant C|(N^{-\beta}+h^2)[\|\xi\|_{H^{\frac{3}{2}+\mu}(\Omega)}(\|\psi\|_{H^{1+\mu}(Q,H^{\beta+2}(I))}\\+\|\psi\|_{H^{2+\mu}(Q,H^{\beta+1}(I))}+\|\psi\|_{H^{3+\mu}(Q,H^\beta(I))}\\+\|\psi\|_{H^{4+\mu}(Q,H^1(I))}+\|\psi\|_{H^{5+\mu}(Q,L^2(I))})\\+\|\psi\|_{H^{1/2+\mu}(\Omega)}\|\xi\|_{H^{1+\mu}(Q,H^\beta(I))}],\end{aligned}$$

$$\|M_5\|\leqslant Ch^2\|\xi\|_{H^{5+\mu}(Q,L^2(I))},$$

$$\|M_6\|\leqslant C\tau\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{3+\mu}(Q,L^2(I))}+\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{1+\mu}(Q,H^2(I))}\right),$$

$$\|M_7\|\leqslant ch^2\|\psi\|_{H^{5+\mu}(Q,L^2(I))}.$$

另外，