厚环壳的渐近求解方程和作用弯矩M_0的解

赵 兴 华

（上海大学，上海市应用数学和力学研究所）

（1993年12月15日收到）

摘 要

本文从三维弹性力学基本方程出发，利用几何小参数 $\alpha=r_0/R_0$ 摄动展开，得到了任意载荷下，厚环壳的各级渐近求解方程。它可以分成两组类似平面应变问题和扭转问题的独立方程组。用此方程求得了厚环壳受弯矩M_0作用的两级渐近解。

关键词 厚环壳 摄动 渐近方程 应力分析

一、引 言

关于细（薄）环壳的基本方程及其解法，许多学者已作了大量的研究工作[1][2][3][6]，取得了很大的进展。但对任意载荷下厚环壳的应力分析，研究不是太多。本文则利用小参数$\alpha=r_0/R_0$（环壳子午方向圆弧内径与圆环半径之比），从三维弹性力学基本方程出发，通过摄动展开，得到厚环壳各级渐近求解方程。

研究结果表明：这些方程可以分为两组类似平面应变问题和扭转问题的独立求解方程，前面各级的解将作为已知量，影响下一级的体积力和应变修正项。这两组方程由边界条件可以分别独立进行求解。在同一级内，这两组未知量互不关联，但在下一级中又相互影响体积力和应变修正项。

利用平面应变问题类型的渐近方程，研究了在$\varphi=0$，$\pi/2$边界上受弯矩M_0作用的厚环壳的应力。结果表明，在$\varphi=0$截面上，沿壳体厚度σ_r，σ_φ，σ_θ呈非线性分布，与曲杆的应力分布十分相似。

二、厚环壳的渐近求解方程

1. 基本方程

对于图1所示厚环壳，取$\bar{r}$，θ，φ曲线坐标系，它与直角坐标的关系为：

$$\left.\begin{aligned}x&=(R_0+\bar{r}\cos\varphi)\cos\theta\\y&=(R_0+\bar{r}\cos\varphi)\sin\theta\\z&=\bar{r}\sin\varphi\end{aligned}\right\}\tag{2.1}$$

其中 R_0为环半径，r_0为子午线方向内弧半径．应力及位移符号如图1所示．

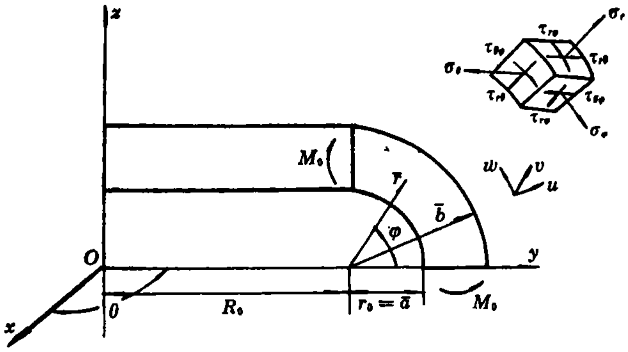

图 1

由三维弹性力学基本方程[4]，通过坐标变换，能得到r，θ，φ坐标系中的平衡方程，几何关系和弹性关系。为使上述方程变为无量纲形式，引进以下无量纲符号：

$$\left.\begin{aligned}&\frac{1}{r_0}(\bar{r},\bar{u},\bar{v},\bar{w})=r,u,v,w;\quad \frac{r_0}{R_0}=\alpha,\quad \frac{\bar{a}}{r_0}=a,\quad \frac{\bar{b}}{r_0}=b\\&\frac{1}{E}(\bar{\sigma}_r,\bar{\sigma}_\theta,\bar{\sigma}_\varphi,\bar{\tau}_{r\varphi},\bar{\tau}_{\theta\varphi},\bar{\tau}_{r\theta})=\sigma_r,\sigma_\theta,\sigma_\varphi,\tau_{r\varphi},\tau_{\theta\varphi},\tau_{r\theta}\end{aligned}\right\}\tag{2.2}$$

最后得无量纲形式的平衡方程为：

$$\left.\begin{aligned}&\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\varphi}}{\partial\varphi}+\frac{1}{r}(\sigma_r-\sigma_\varphi)+\frac{\alpha}{1+\alpha r\cos\varphi}\Big[\frac{\partial\tau_{r\theta}}{\partial\theta}\\&\quad+(\sigma_r-\sigma_\theta)\cos\varphi-\tau_{r\varphi}\sin\varphi\Big]=0\\&\frac{1}{r}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{\partial\tau_{r\varphi}}{\partial r}+\frac{2}{r}\tau_{r\varphi}+\frac{\alpha}{1+\alpha r\cos\varphi}\Big[\frac{\partial\tau_{r\theta}}{\partial\theta}\\&\quad+\tau_{r\varphi}\cos\varphi+(\sigma_\theta-\sigma_\varphi)\sin\varphi\Big]=0\\&\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi}+\frac{1}{r}\tau_{r\theta}+\frac{\alpha}{1+\alpha r\cos\varphi}\Big[\frac{\partial\sigma_\theta}{\partial\theta}\\&\quad+2\tau_{r\theta}\cos\varphi-2\tau_{\theta\varphi}\sin\varphi\Big]=0\end{aligned}\right\}\tag{2.3}$$

几何关系：

$$\left.\begin{aligned}&\varepsilon_r=\frac{\partial u}{\partial r},\quad \varepsilon_\theta=\frac{\alpha}{1+\alpha r\cos\varphi}\Big[\frac{\partial v}{\partial\theta}+u\cos\varphi-w\sin\varphi\Big]\\&\varepsilon_\varphi=\frac{1}{r}\frac{\partial w}{\partial\varphi}+\frac{u}{r},\quad \varepsilon_{r\varphi}=\frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\varphi}-\frac{w}{r}\\&\varepsilon_{r\theta}=\frac{\partial v}{\partial r}+\frac{\alpha}{1+\alpha r\cos\varphi}\Big[\frac{\partial u}{\partial\theta}-v\cos\varphi\Big]\\&\varepsilon_{\theta\varphi}=\frac{1}{r}\frac{\partial v}{\partial\varphi}+\frac{\alpha}{1+\alpha r\cos\varphi}\Big[\frac{\partial w}{\partial\theta}+v\sin\varphi\Big]\end{aligned}\right\}\tag{2.4}$$

弹性关系：

$$\left.\begin{aligned} &\varepsilon_r=\sigma_r-\mu(\sigma_\theta+\sigma_\varphi), \quad &\varepsilon_{r\varphi}=2(1+\mu)\tau_{r\varphi} \\ &\varepsilon_\varphi=\sigma_\varphi-\mu(\sigma_r+\sigma_\theta), \quad &\varepsilon_{\theta\varphi}=2(1+\mu)\tau_{\theta\varphi} \\ &\varepsilon_\theta=\sigma_\theta-\mu(\sigma_r+\sigma_\varphi), \quad &\varepsilon_{r\theta}=2(1+\mu)\tau_{r\theta} \end{aligned}\right\} \tag{2.5}$$

以上即为r，θ，φ坐标中的基本方程，求解时还必须满足相应的边界条件．这里E，μ分别为弹性模量和波桑比．

2．弯环渐近分析

对图1所示弯环，$\alpha=\frac{r_0}{R_0}\ll 1$．若设应力为$\alpha^p$量级，则位移可能是$\alpha^{p-2}$量级．将域内所有各量都展成$\alpha$的幂级数，且都从$\alpha^{p-2}$量级开始，则有

$$\left.\begin{aligned} &u=\alpha^p(u^{-2}\alpha^{-2}+u^{-1}\alpha^{-1}+u^0\alpha^0+u^1\alpha^1+\cdots) \\ &v,\ w\text{相似}; \\ &\varepsilon_r=\alpha^p(\varepsilon_r^{-2}\alpha^{-2}+\varepsilon_r^{-1}\alpha^{-1}+\varepsilon_r^0\alpha^0+\varepsilon_r^1\alpha^1+\cdots) \\ &\varepsilon_\varphi,\ \varepsilon_\theta,\ \varepsilon_{r\varphi},\ \varepsilon_{r\theta},\ \varepsilon_{\varphi\theta}\text{相似}; \\ &\sigma_r=\alpha^p(\sigma_r^{-2}\alpha^{-2}+\sigma_r^{-1}\alpha^{-1}+\sigma_r^0\alpha^0+\sigma_r^1\alpha^1+\cdots) \\ &\sigma_\varphi,\ \sigma_\theta,\ \tau_{r\varphi},\ \tau_{r\theta},\ \tau_{\varphi\theta}\text{相似}, \\ &\frac{\alpha}{1+\alpha r\cos\varphi}=\alpha[1-\alpha r\cos\varphi+\alpha^2r^2\cos^2\varphi+\cdots] \end{aligned}\right\} \tag{2.6}$$

边界应力根据边界上的已知值，也展开成从α^p级开始的幂级数．对于已知边界位移的情况，级数是从α^{p-2}级开始．将表达式(2.6)代入(2.3)～(2.5)及边界条件，随后取方程中α同次幂的系数为零，就得到如下各级渐近方程组．

（1）对α^{p-2}级渐近求解方程

平衡方程

$$\left.\begin{aligned} &\frac{\partial\sigma_r^{-2}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\varphi}^{-2}}{\partial\varphi}+\frac{1}{r}(\sigma_r^{-2}-\sigma_\varphi^{-2})=0 \\ &\frac{1}{r}\frac{\partial\sigma_\varphi^{-2}}{\partial\varphi}+\frac{\partial\tau_{r\varphi}^{-2}}{\partial r}+\frac{2}{r}\tau_{r\varphi}^{-2}=0 \\ &\frac{\partial\tau_{r\theta}^{-2}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\varphi}^{-2}}{\partial\varphi}+\frac{1}{r}\tau_{r\theta}^{-2}=0 \end{aligned}\right\} \tag{2.7}$$

几何关系

$$\left.\begin{aligned} &\varepsilon_r^{-2}=\frac{\partial u^{-2}}{\partial r}, \quad &\varepsilon_{r\varphi}^{-2}=\frac{\partial w^{-2}}{\partial r}+\frac{1}{r}\frac{\partial u^{-2}}{\partial\varphi}-\frac{w^{-2}}{r} \\ &\varepsilon_\theta^{-2}=0, \quad &\varepsilon_{r\theta}^{-2}=\frac{\partial v^{-2}}{\partial r} \\ &\varepsilon_\varphi^{-2}=\frac{1}{r}\frac{\partial w^{-2}}{\partial\varphi}+\frac{u^{-2}}{r}, \quad &\varepsilon_{\theta\varphi}^{-2}=\frac{1}{r}\frac{\partial v^{-2}}{\partial\varphi} \end{aligned}\right\} \tag{2.8}$$

弹性关系的形式与(2.5)式相同．在边界上必须满足已知位移的条件．在给定应力的边界上，α^{p-2}级的边界应力等于零．

（2）对α^{p+n}级渐近求解方程

平衡方程

$$\left.\begin{aligned}&\frac{\partial\sigma_r^n}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\varphi}^n}{\partial\varphi}+\frac{1}{r}(\sigma_r^n-\sigma_\varphi^n)+R^n(r,\theta,\varphi)=0\\&\frac{1}{r}\frac{\partial\sigma_\varphi^n}{\partial\varphi}+\frac{\partial\tau_{r\varphi}^n}{\partial r}+\frac{2}{r}\tau_{r\varphi}^n+\psi^n(r,\theta,\varphi)=0\\&\frac{\partial\tau_{r\theta}^n}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\varphi}^n}{\partial\varphi}+\frac{1}{r}\tau_{r\theta}^n+Q^n(r,\theta,\varphi)=0\end{aligned}\right\}\tag{2.9}$$

几何方程

$$\left.\begin{aligned}&\varepsilon_r^n=\frac{\partial u^n}{\partial r},\quad \varepsilon_{r\varphi}^n=\frac{\partial w^n}{\partial r}+\frac{1}{r}\frac{\partial u^n}{\partial\varphi}-\frac{w^n}{r}\\&\varepsilon_\theta^n=e_\theta^n,\quad \varepsilon_{r\theta}^n=\frac{\partial v^n}{\partial r}+e_{r\theta}^n\\&\varepsilon_\varphi^n=\frac{1}{r}\frac{\partial w^n}{\partial\varphi}+\frac{u^n}{r},\quad \varepsilon_{\theta\varphi}^n=\frac{1}{r}\frac{\partial v^n}{\partial\varphi}+e_{\theta\varphi}^n\\&\qquad(n=-1,\ 0,\ 1,\ 2,\ 3,\ \cdots)\end{aligned}\right\}\tag{2.10}$$

弹性关系的形式与(2.5)式相同．应力边界条件从α^p级开始，由给定值确定．同时还必须满足位移边界条件．式中

$$\left.\begin{aligned}R^n(r,\theta,\varphi)=&\left[\frac{\partial\tau_{r\theta}^{n-1}}{\partial\theta}+(\sigma_r^{n-1}-\sigma_\theta^{n-1})\cos\varphi-\tau_{r\varphi}^{n-1}\sin\varphi\right]\\&+\left[\frac{\partial\tau_{r\theta}^{n-2}}{\partial\theta}+(\sigma_r^{n-2}-\sigma_\theta^{n-2})\cos\varphi-\tau_{r\varphi}^{n-2}\sin\varphi\right](-r\cos\varphi)\\&+\cdots\\&+\left[\frac{\partial\tau_{r\theta}^{-2}}{\partial\theta}+(\sigma_r^{-2}-\sigma_\theta^{-2})\cos\varphi-\tau_{r\varphi}^{-2}\sin\varphi\right](-r\cos\varphi)^{n+1}\\\psi^n(r,\theta,\varphi)=&\left[\frac{\partial\tau_{\theta\varphi}^{n-1}}{\partial\theta}+\tau_{r\varphi}^{n-1}\cos\varphi+(\sigma_\theta^{n-1}-\sigma_\varphi^{n-1})\sin\varphi\right]\\&+\left[\frac{\partial\tau_{\theta\varphi}^{n-2}}{\partial\theta}+\tau_{r\varphi}^{n-2}\cos\varphi+(\sigma_\theta^{n-2}-\sigma_\varphi^{n-2})\sin\varphi\right](-r\cos\varphi)\\&+\cdots\\&+\left[\frac{\partial\tau_{\theta\varphi}^{-2}}{\partial\theta}+\tau_{r\varphi}^{-2}\cos\varphi+(\sigma_\theta^{-2}-\sigma_\varphi^{-2})\sin\varphi\right](-r\cos\varphi)^{n+1}\\Q^n(r,\theta,\varphi)=&\left[\frac{\partial\sigma_\varphi^{n-1}}{\partial\theta}+2\tau_{r\theta}^{n-1}\cos\varphi-2\tau_{\theta\varphi}^{n-1}\sin\varphi\right]\\&+\left[\frac{\partial\sigma_\theta^{n-2}}{\partial\theta}+2\tau_{r\theta}^{n-2}\cos\varphi-2\tau_{\theta\varphi}^{n-2}\sin\varphi\right](-r\cos\varphi)\\&+\cdots\\&+\left[\frac{\partial\sigma_\theta^{-2}}{\partial\theta}+2\tau_{r\theta}^{-2}\cos\varphi-2\tau_{\theta\varphi}^{-2}\sin\varphi\right](-r\cos\varphi)^{n+1}\\e_\theta^n=&\left[\frac{\partial v^{n-1}}{\partial\theta}+u^{n-1}\cos\varphi-w^{n-1}\sin\varphi\right]+\left[\frac{\partial v^{n-2}}{\partial\theta}\right.\\&\left.+u^{n-2}\cos\varphi-w^{n-2}\sin\varphi\right](-r\cos\varphi)\end{aligned}\right\}\tag{2.11}$$

$$
\left.\begin{aligned}
&+\cdots+\left[\frac{\partial v^{-2}}{\partial\theta}+u^{-2}\cos\varphi-w^{-2}\sin\varphi\right](-r\cos\varphi)^{n+1}\\
e^n_{r\theta}=&\left[\frac{\partial u^{n-1}}{\partial\theta}-v^{n-1}\cos\varphi\right]+\left[\frac{\partial u^{n-2}}{\partial\theta}-v^{n-2}\cos\varphi\right](-r\cos\varphi)\\
&+\cdots+\left[\frac{\partial u^{-2}}{\partial\theta}-v^{-2}\cos\varphi\right](-r\cos\varphi)^{n+1}\\
e^n_{\theta\varphi}=&\left[\frac{\partial w^{n-1}}{\partial\theta}+v^{n-1}\sin\varphi\right]+\left[\frac{\partial w^{n-2}}{\partial\theta}+v^{n-2}\sin\varphi\right](-r\cos\varphi)\\
&+\cdots+\left[\frac{\partial w^{-2}}{\partial\theta}+v^{-2}\sin\varphi\right](-r\cos\varphi)^{n+1}\\
&\qquad\qquad(n=-1,\ 0,\ 1,\ 2,\ 3,\ \cdots)
\end{aligned}\right\}
$$

3. 弯环渐近方程的一个性质

对方程(2.9)，(2.10)，(2.5)实际上可分为两组求解方程．即

A组：

$$
\left.\begin{aligned}
&\frac{\partial\sigma^n_r}{\partial r}+\frac{1}{r}\frac{\partial\tau^n_{r\varphi}}{\partial\varphi}+\frac{1}{r}(\sigma^n_r-\sigma^n_\varphi)+R^n(r,\theta,\varphi)=0\\
&\frac{1}{r}\frac{\partial\sigma^n_\varphi}{\partial\varphi}+\frac{\partial\tau^n_{r\varphi}}{\partial r}+\frac{2}{r}\tau^n_{r\varphi}+\psi^n(r,\theta,\varphi)=0\\
&\varepsilon^n_r=\frac{\partial u^n}{\partial r},\qquad \varepsilon^n_\varphi=\frac{1}{r}\frac{\partial w^n}{\partial\varphi}+\frac{u^n}{r}\\
&\varepsilon^n_\theta=e^n_\theta,\qquad \varepsilon^n_{r\varphi}=\frac{\partial w^n}{\partial r}+\frac{1}{r}\frac{\partial u^n}{\partial\varphi}-\frac{w^n}{r}\\
&\varepsilon^n_r=\sigma^n_r-\mu(\sigma^n_\theta+\sigma^n_\varphi),\ \varepsilon^n_\varphi=\sigma^n_\varphi-\mu(\sigma^n_r+\sigma^n_\theta)\\
&\varepsilon^n_\theta=\sigma^n_\theta-\mu(\sigma^n_r+\sigma^n_\varphi),\ \varepsilon^n_{r\varphi}=2(1+\mu)\tau^n_{r\varphi}
\end{aligned}\right\}\tag{2.12}
$$

B组：

$$
\left.\begin{aligned}
&\frac{\partial\tau^n_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\tau^n_{\theta\varphi}}{\partial\varphi}+\frac{1}{r}\tau^n_{r\theta}+Q^n(r,\theta,\varphi)=0\\
&\varepsilon^n_{r\theta}=\frac{\partial v^n}{\partial r}+e^n_{r\theta},\qquad \varepsilon^n_{\theta\varphi}=\frac{1}{r}\frac{\partial v^n}{\partial\varphi}+e^n_{\theta\varphi}\\
&\varepsilon^n_{r\theta}=2(1+\mu)\tau^n_{r\theta},\qquad \varepsilon^n_{\theta\varphi}=2(1+\mu)\tau^n_{\theta\varphi}
\end{aligned}\right\}\tag{2.13}
$$

A组方程与平面问题方程极相似，当ε^n_θ为零或常数时，即为平面应变问题方程．其中R^n，ψ^n相当于体积力，由前几级近似解求得．这组方程反映了弯环壳在$r\varphi$平面内的变形性质，可以独立求解．

B组方程与扭转问题的方程极相似，Q^n是对应的体积力，由前几级近似解得到．这组方程反映了弯环受扭转和切力作用在θ方向的变形性态，也可以独立求解．

由于体积力项是由上一级近似解求得的，因此这两组方程很容易按边界条件逐级求解下去．对同一级，这两组的未知量互不关联，而在下一级又相互影响体积力和应变修正项．由此可知：弯环的弯曲问题和扭转问题可看成两个独立的求解问题，其相互作用是在更高一级的近似解中才反映．

显然，上述两组方程比原方程容易求解，其计算模型的物理意义也是十分清楚的．必须

指出：上述方程对弯环各种受力情况都适用，对弯环的厚度、曲率、边界形状并无限制，因此它是求解 $\alpha=\frac{r_0}{R_0}\ll 1$ 这类问题的一般方程。

三、弯环壳受弯矩M₀作用的渐近解

弯环壳受弯矩 M_0 作用时，其边界条件为（图1）：

$$\left.\begin{aligned}&r=a,\ b:\ \sigma_r=0,\ \tau_{r\varphi}=0\\&\varphi=0:\ \tau_{r\varphi}=0,\ w=0\\&\varphi=\frac{\pi}{2}:\ \tau_{r\varphi}=0,\ \int_a^b\sigma_\varphi dr=0\\&\qquad\int_a^b\sigma_\varphi^0 rdr=M_0\qquad(\text{对}\alpha^p\text{级})\\&\qquad\int_a^b\sigma_\varphi^i rdr=0\qquad(\text{其余各级})\end{aligned}\right\}\tag{3.1}$$

由方程(2.12)和边界条件(3.1)，解得 α^{p-2}，α^{p-1} 级所有应力、位移分量均为零。下面求 α^p，α^{p+1} 级方程的解。

1. α^p 级方程的解

方程(2.12)化为：

$$\left.\begin{aligned}&\frac{\partial\sigma_r^0}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\varphi}^0}{\partial\varphi}+\frac{1}{r}(\sigma_r^0-\sigma_\varphi^0)=0\\&\frac{1}{r}\frac{\partial\sigma_\varphi^0}{\partial\varphi}+\frac{\partial\tau_{r\varphi}^0}{\partial r}+\frac{2}{r}\tau_{r\varphi}^0=0\\&\varepsilon_r^0=\frac{\partial u^0}{\partial r},\qquad\varepsilon_\theta^0=0\\&\varepsilon_\varphi^0=\frac{1}{r}\frac{\partial w^0}{\partial\varphi}+\frac{u^0}{r},\quad\varepsilon_{r\varphi}^0=\frac{1}{r}\frac{\partial u^0}{\partial\varphi}+\frac{\partial w^0}{\partial r}-\frac{w^0}{r}\end{aligned}\right\}\tag{3.2}$$

及弹性关系式。这是典型的平面应变问题方程。在边界条件(3.1)之下的解为[4]：

$$\left.\begin{aligned}&\sigma_r^0=-\frac{4M_0}{\Delta}\left[-\frac{a^2b^2}{r^2}\ln\frac{a}{b}+b^2\ln\frac{r}{b}-a^2\ln\frac{r}{a}\right]\\&\sigma_\varphi^0=-\frac{4M_0}{\Delta}\left[\frac{a^2b^2}{r^2}\ln\frac{a}{b}+b^2\ln\frac{r}{b}-a^2\ln\frac{r}{a}+b^2-a^2\right]\\&\sigma_\theta^0=-\mu\frac{4M_0}{\Delta}\left[2b^2\ln\frac{r}{b}-2a^2\ln\frac{r}{a}+b^2-a^2\right]\\&\tau_{r\varphi}^0=0\\&u^0=-\frac{4M_0}{\Delta}(1+\mu)\left\{\frac{a^2b^2}{r}\ln\frac{a}{b}+(1-2\mu)\left[b^2\ln\frac{r}{b}\right.\right.\\&\qquad\left.\left.-a^2\ln\frac{r}{a}\right]r-(1-\mu)(b^2-a^2)r\right\}-A^0\cos\varphi\\&w^0=-\frac{8M_0}{\Delta}(1-\mu^2)(b^2-a^2)r\varphi+A^0\sin\varphi\end{aligned}\right\}\tag{3.3}$$

其中　$\varDelta=(b^2-a^2)^2-4a^2b^2\left(\ln\frac{a}{b}\right)^2$

此解就是平面应变曲杆的解．因此弯环壳受纯弯作用作用其第一级近似解（α^p级）实际就是曲杆解．其中A^0为待定系数．

2．α^{p+1}级方程的渐近解

由方程(2.12)和关系式(2.11)，(3.3)得α^{p+1}级的求解方程为
平衡方程：

$$\left.\begin{aligned}&\frac{\partial\sigma_r^1}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\varphi}^1}{\partial\varphi}+\frac{1}{r}(\sigma_r^1-\sigma_\varphi^1)+R^1(r,\varphi)=0\\&\frac{1}{r}\frac{\partial\sigma_\varphi^1}{\partial\varphi}+\frac{\partial\tau_{r\varphi}^1}{\partial r}+\frac{2}{r}\tau_{r\varphi}^1+\psi^1(r,\varphi)=0\end{aligned}\right\}\tag{3.4}$$

其中

$$R^1(r,\varphi)=\frac{4M_0}{\varDelta}\left[\frac{a^2b^2}{r^2}\ln\frac{a}{b}-(1-2\mu)\left(b^2\ln\frac{r}{b}-a^2\ln\frac{r}{a}\right)+\mu(b^2-a^2)\right]\cos\varphi$$

$$\psi^1(r,\varphi)=\frac{4M_0}{\varDelta}\left[\frac{a^2b^2}{r^2}\ln\frac{a}{b}+(1-2\mu)\left(b^2\ln\frac{r}{b}-a^2\ln\frac{r}{a}\right)+(1-\mu)(b^2-a^2)\right]\sin\varphi$$

几何关系

$$\left.\begin{aligned}&\varepsilon_r^1=\frac{\partial u^1}{\partial r},\ \varepsilon_{r\varphi}^1=\frac{\partial w^1}{\partial r}+\frac{1}{r}\frac{\partial u^1}{\partial\varphi}-\frac{w^1}{r},\ \varepsilon_\varphi^1=\frac{1}{r}\frac{\partial w^1}{\partial\varphi}+\frac{u^1}{r}\\&\varepsilon_\theta^1=-\frac{4M_0}{\varDelta}(1+\mu)\left\{\frac{a^2b^2}{r}\ln\frac{a}{b}+(1-2\mu)\left(b^2\ln\frac{r}{b}-a^2\ln\frac{r}{a}\right)r\right.\\&\qquad\left.-(1-\mu)(b^2-a^2)r\right\}\cos\varphi\\&\qquad+\frac{8M_0}{\varDelta}(1-\mu^2)(b^2-a^2)r\varphi\sin\varphi-A^0\end{aligned}\right\}\tag{3.5}$$

引入满足方程(3.4)的应力函数$\varPhi^1$，则

$$\left.\begin{aligned}&\sigma_r^1=\frac{1}{r}\frac{\partial\varPhi^1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varPhi_1}{\partial\varphi^2}+\frac{4M_0}{\varDelta}(1-2\mu)\left[b^2r\ln\frac{r}{b}-a^2r\ln\frac{r}{a}\right]\cos\varphi\\&\sigma_\varphi^1=\frac{\partial^2\varPhi^1}{\partial^2r^2}+\frac{4M_0}{\varDelta}\left\{\frac{a^2b^2}{r}\ln\frac{a}{b}+(1-2\mu)\left[b^2r\ln\frac{r}{b}-a^2r\ln\frac{r}{a}\right]\right.\\&\qquad\left.+(1-\mu)(b^2-a^2)r\right\}\cos\varphi\\&\tau_{r\varphi}^1=\frac{1}{r^2}\frac{\partial\varPhi^1}{\partial\varphi}-\frac{1}{r}\frac{\partial^2\varPhi^1}{\partial r\partial\varphi}\end{aligned}\right\}\tag{3.6}$$

根据几何关系(3.5)，$\varPhi^1$还应满足协调方程

$$\nabla^2\nabla^2\varPhi^1=-\frac{16M_0}{\varDelta}(1-2\mu)(b^2-a^2)\frac{1}{r}\cos\varphi\tag{3.7}$$

其中　$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$．取(3.7)式的解为

$$\Phi^1=\left[B_1r^3+C_1\frac{1}{r}+D_1r\ln r\right]\cos\varphi+A_1r\varphi\sin\varphi$$

$$-\frac{M_0}{2\Delta}(1-2\mu)(b^2-a^2)\left[r^3\ln\frac{r}{a}+r^3\ln\frac{r}{b}-\frac{5}{2}r^3\right]\cos\varphi \tag{3.8}$$

代入(3.6)式得

$$\left.\begin{aligned}
\sigma_r^1=&\left[A_1\frac{2}{r}+2B_1r-2C_1\frac{1}{r^3}+D_1r\right]\cos\varphi\\
&-\frac{M_0}{\varDelta}\left(\frac{1}{2}-\mu\right)(b^2-a^2)\left[2r\ln\frac{r}{a}+2r\ln\frac{r}{b}-3r\right]\cos\varphi\\
&+\frac{4M_0}{\varDelta}(1-2\mu)\left[b^2r\ln\frac{r}{b}-a^2r\ln\frac{r}{a}\right]\cos\varphi\\
\sigma_\varphi^1=&\left[6B_1r+2C_1\frac{1}{r^3}+D_1\frac{1}{r}\right]\cos\varphi-\frac{M_0}{\varDelta}\left(\frac{1}{2}-\mu\right)(b^2-a^2)\\
&\left[6r\ln\frac{r}{a}+6r\ln\frac{r}{b}-5r\right]\cos\varphi\\
&+\frac{4M_0}{\varDelta}\left[\frac{a^2b^2}{r}\ln\frac{a}{b}+(1-2\mu)\left(b^2r\ln\frac{r}{b}-a^2r\ln\frac{r}{a}\right)\right.\\
&\left.+(1-\mu)(b^2-a^2)r\right]\cos\varphi\\
\tau_{r\varphi}^1=&\left[2B_1r-2C_1\frac{1}{r^3}+D_1\frac{1}{r}\right]\sin\varphi-\frac{M_0}{\varDelta}\left(\frac{1}{2}-\mu\right)(b^2-a^2)\\
&\cdot\left[2r\ln\frac{r}{a}+2r\ln\frac{r}{b}-3r\right]\sin\varphi
\end{aligned}\right\} \tag{3.9}$$

系数A_1，B_1，C_1，D_1由边界条件决定。当(3.9)式满足(3.1)式时，各系数为：

$$\left.\begin{aligned}
A_1=&-\frac{2M_0}{\varDelta}(1-2\mu)a^2b^2\ln\frac{a}{b}\\
B_1=&-\left(\frac{1}{2}-\mu\right)\frac{M_0}{\Delta H}\left[(a^4+b^4)\left(\ln\frac{a}{b}\right)^2\right.\\
&\left.+\frac{5}{2}(b^4-a^4)\ln\frac{a}{b}+2(b^2-a^2)^2\right]\\
C_1=&\frac{1}{2}\left(\frac{1}{2}-\mu\right)a^2b^2\frac{M_0}{H}\\
D_1=&\left(\frac{1}{2}-\mu\right)(b^2-a^2)\frac{M_0}{\Delta H}\left[4a^2b^2\ln\frac{a}{b}+b^4-a^4\right]
\end{aligned}\right\} \tag{3.10}$$

其中 $H=(a^2+b^2)\ln\frac{a}{b}+b^2-a^2$

将(3.10)式代入(3.9)式就得α^{p+1}级的σ_r^1，σ_φ^1，$\tau_{r\varphi}^1$之解。对于σ_θ^1根据弹性关系(2.5)式和(3.5)，(3.9)式则有

$$\left.\begin{aligned}\sigma_\varphi^1&=\mu(\sigma_r^1+\sigma_\varphi^1)+\frac{8M_0}{\varDelta}(1-\mu^2)(b^2-a^2)r\varphi\sin\varphi-A^0\\&-(1+\mu)\frac{4M_0}{\varDelta}\Big[-\frac{a^2b^2}{r^2}\ln\frac{a}{b}+(1-2\mu)\Big(b^2r\ln\frac{r}{b}\\&-a^2r\ln\frac{r}{a}\Big)-(1-\mu)(b^2-a^2)r\Big]\cos\varphi\end{aligned}\right\}\qquad(3.11)$$

由几何关系(3.5)式和边界条件(3.1)式求得位移为:

$$\left.\begin{aligned}u^1&=(1+\mu)\Big\{2(1-\mu)A_1\ln r+(1-4\mu)B_1r^2\\&+C_1\frac{1}{r^2}+D_1(1-2\mu)\ln r\Big\}\cos\varphi\\&-\frac{M_0}{\varDelta}(1+\mu)\Big(\frac{1}{2}-\mu\Big)(b^2-a^2)\Big\{(1-4\mu)\Big[r^2\ln\frac{r}{a}\\&+r^2\ln\frac{r}{b}\Big]+\frac{1}{2}(-5+16\mu)r^2\Big\}\cos\varphi\\&+\frac{M_0}{\varDelta}(1+\mu)\Big\{2(1-2\mu)(1-\mu)\Big[b^2r^2\ln\frac{r}{b}-a^2r^2\ln\frac{r}{a}\Big]\\&-(1-\mu)(1+2\mu)(b^2-a^2)r^2\Big\}\cos\varphi\\&+\frac{M_0}{\varDelta}(1+\mu)\Big\{2a^2b^2\ln\frac{a}{b}-4\mu(1-\mu)(b^2-a^2)r^2\Big\}\varphi\sin\varphi\\&+(1+\mu)[(1-2\mu)A_1+2(1-\mu)D_1]\varphi\sin\varphi\\&+\mu A^0r-A^1\cos\varphi\\w^1&=(1+\mu)\Big\{-A_1[1+2(1-\mu)\ln r]+(5-4\mu)B_1r^2\\&+C_1\frac{1}{r^2}-D_1[1+(1-2\mu)\ln r]\Big\}\sin\varphi\\&-\frac{M_0}{\varDelta}\Big(\frac{1}{2}-\mu\Big)(1+\mu)(b^2-a^2)\Big\{(5-4\mu)\Big(r^2\ln\frac{r}{a}\\&+r^2\ln\frac{r}{b}\Big)-\frac{5}{2}r^2\Big\}\sin\varphi\\&+\frac{M_0}{\varDelta}(1+\mu)\Big\{2(1-2\mu)(1-\mu)\Big[b^2r^2\ln\frac{r}{b}-a^2r^2\ln\frac{r}{a}\Big]\\&+2a^2b^2\ln\frac{a}{b}+5(1-2\mu)(1-\mu)(b^2-a^2)r^2\Big\}\sin\varphi\\&+\frac{M_0}{\varDelta}(1+\mu)\Big\{2a^2b^2\ln\frac{a}{b}+4\mu(1-\mu)(b^2-a^2)r^2\Big\}\varphi\cos\varphi\\&+(1+\mu)[(1-2\mu)A_1+2(1-\mu)D_1]\varphi\cos\varphi+A^1\sin\varphi\end{aligned}\right\}\qquad(3.12)$$

A^1为新的待定常数。

3. 常数A^0，A^1的确定

α^{p+2}级以上的近似解十分冗繁，不再列出．在此仅设法求出α^p，α^{p+1}级中出现的位移待定常数A^0，A^1．

在α^{p+2}级中，根据x方向体积力的合力应为零的条件来确定A^0，则得

$$A^0=\frac{-8}{3\pi(b^2-a^2)}\Big\{(2-3\mu)A_1(b-a)+(1-4\mu)B_1(b^3-a^3)$$
$$-C_1\left(\frac{1}{b}-\frac{1}{a}\right)+\left(\frac{1}{2}-3\mu\right)D_1(b-a)$$
$$+\frac{M_0}{\varDelta}\left(\frac{1}{2}-\mu\right)(b^2-a^2)\left[(1-4\mu)\left(b^3\ln\frac{a}{b}+a^3\ln\frac{a}{b}\right)\right.$$
$$\left.+\left(\frac{3}{2}-\frac{20}{3}\mu\right)(b^3-a^3)\right]$$
$$+\frac{2M_0}{\varDelta}\Big[3a^2b^2(b-a)\ln\frac{a}{b}+(3-7\mu+2\mu^2)(b-a)a^2b^2\ln\frac{a}{b}$$
$$-\frac{10}{3}(1-\mu^2)(b^2-a^2)(b^3-a^3)\Big]\Big\} \tag{3.13}$$

根据α^{p+3}级中x方向体积力的合力为零的条件可以确定A^1，略去小量后有

$$A^1=A_1\left[\frac{2(1-\mu^2)}{b^2-a^2}(b^2\ln b-a^2\ln a)-\frac{1}{2}(1+\mu)(1-2\mu)\right]$$
$$-B_1(1+\mu)(a^2+b^2)$$
$$+D_1\left[\frac{(1+\mu)(1-2\mu)}{b^2-a^2}(b^2\ln b-a^2\ln a)+\mu(1+\mu)\right]$$
$$+\frac{4}{3\pi}(1+\mu)\frac{b^3-a^3}{b^2-a^2}A^0$$
$$+\frac{M_0}{\varDelta}(1+\mu)\Big\{\Big[\left(\frac{-1}{2}+\mu\right)(a^4+b^4)+2(1-\mu)a^2b^2\Big]\ln\frac{a}{b}$$
$$+\left(-\frac{17}{4}+4\mu+3\mu^2\right)(b^4-a^4)\Big\} \tag{3.14}$$

这样由(3.3)，(3.13)和(3.12)，(3.14)式给出了α^p和α^{p+1}级的位移解．由(3.3)，(3.9)，(3.11)，(3.13)式得到α^p和α^{p+1}级的应力解．在此α^p，α^{p+1}的解是精确求出的．为判断解的收敛性，也可根据α^{p+2}级的体积力，用简单的曲杆公式求α^{p+2}级的应力，以便与前两级应力解作比较．

表1和图2列出了作用M_0时，$\varphi=0$面上应力的各级渐近解．计算结果表明，收敛非常迅速．当$r_0/R_0=1/15$，$b=2a$时，一级近似误差为10%，二级近似解误差为4%，三级近似解的误差小于1%．沿截面厚度σ_φ，σ_r，σ_θ呈非线性分布，与曲杆的应力分布规律十分相近．因此对这种弯环壳，工程上用曲杆公式估算其应力是合理的．

表 1　作用M_0时，弯环$\varphi=0$截面上的应力分布（无量纲）

r	各级σ_φ近似			$r_0/R_0=1/15$，σ_r，σ_φ，σ_θ各级近似解*.单位：M_0				
	$a^p\times M_0$	$a^{p+1}\times\left(\frac{r_0}{R_0}\right)M_0$	$a^{p+2}\times\left(\frac{r_0}{R_0}\right)^2M_0$	$\sigma_{\varphi I}$	$\sigma_{\varphi II}$	$\sigma_{\varphi III}$	σ_{rII}	$\sigma_{\theta II}$
1.0	7.755	−6.454	−61.08	7.755	7.325	7.054	0	1.641
1.2	3.507	−2.573	−27.90	3.507	3.335	3.211	0.850	0.734
1.4	0.542	0.306	−4.23	0.542	0.562	0.543	.0984	−0.015
1.6	−1.689	2.605	13.56	−1.689	−1.516	−1.456	0.788	−0.651
1.8	−3.474	4.533	27.39	−3.474	−3.172	−3.051	0.417	−1.234
2.0	−4.917	6.205	28.44	−4.917	−4.503	−4.332	0	−1.706
误差				10%	4%	/		

* 注：$\sigma_{(\ )I}=a^p$级，$\sigma_{(\ )II}=a^p+a^{p+1}$级，$\sigma_{(\ )III}=a^p+a^{p+1}+a^{p+2}$级

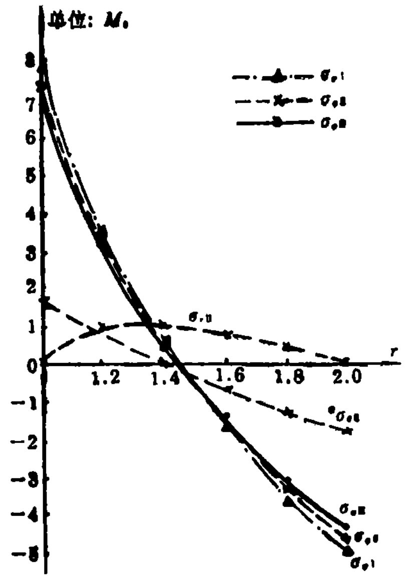

图 2　弯环$\varphi=0$截面上的应力分布

参 考 文 献

[1] Flügge, W., *Stresses in Shells*, (1960).
[2] 钱伟长、郑思梁，轴对称圆环壳的一般解，应用数学和力学，1(3)(1980).
[3] 钱伟长、吴明德，U型波纹管的非线性特性摄动法计算，应用数学和力学，4(5)(1983).
[4] Timoshenko, S. and J. N. Goodier, *Theory of Elasticity*, (1951).
[5] 赵兴华、蒋智翔、景士都，弹性波形厚壳的应力和变形分析，清华大学研究报告，(1966).
[6] Chien Wei-zang and Ho Shui-tsing, Asymptotic method on the problems of thin elastic ring shell with rotational symmetrical load, *The Engineering Reports of Tsinghua University*, 3(2) (1948).

The Asymptotic Solving Equations of Thick Ring Shell and it's Solution under Moment M_0

Zhao Xing-hua

(Shanghai Unirersity; Shanghai Institute of Applied Mathematics and Mechanics, Shanghai)

Abstract

In this paper, from the fundamental equations of three dimensional elastic mechanics, I have found a sequence of asymptotic solution equations of thick ring shells (or body) applied arbitrary loads by the perturbation method based upon a geometric small parameter $a=r_0/R_0$, which may be divided into two indepondent equation groups which are similar to the equation groups for plane strain and torsional problems. Using these equations, I have aslo found the first order and second order approximate solutions of thick ring shell applied moment M_0 *are obtained*.

Key words thick ring shell, perturbation, asymptotic equations, stress analysis