应用数学和力学，第16卷 第 1 期(1995年 1 月)
Applied Mathematics and Mechanics
应用数学和力学编委会编
重 庆 出 版 社 出 版

球壳和柱壳振动中一类方程组的求解*

丁皓江　陈伟球

（浙江大学）

刘　钟

（北京结构和环境工程研究所）

摘　要

本文利用矩阵型式的Frobenius 级数方法求解了球壳和柱壳振动中一类含正则奇点的常微分方程组。考虑了指标方程根的相互关系，从而全面地得到了不同情况下解的表达形式．为解析求解工作奠定了基础．

关键词　振动　方程组　Frobenius级数　球壳　柱壳

一、引　　言

1964年，Mirsky[1] 在研究正交各向异性圆柱壳的振动问题时，得到了一组常微分方程并利用Frobenius方法求解了该方程组．Cohen等[2]后来在研究球面各向同性球壳的自由振动问题时，从三维弹性理论出发，引进两个辅助变量，把原来弹性体的运动微分方程变成如下常微分方程组

$$\left.\begin{aligned} r^2W''+2rW'+(\alpha^2r^2+k_1)W+k_2rU'+k_3U=0 \\ r^2U''+2rU'+(\beta^2r^2+k_4)U+k_5rW'+k_6W=0 \end{aligned}\right\} \tag{1.1}$$

其中，W，U 俱是关于 r 的未知函数，$\alpha^2=\rho\omega^2/A_{33}$，$\beta^2=\rho\omega^2/A_{44}$，$\rho$ 是密度，A_{33}，A_{44} 是弹性常数，ω 是圆频率，常数k_i与弹性常数有关，并存在下列关系式

$$k_2k_6+k_3k_5=k_2k_5 \tag{1.2}$$

该文也利用Frobenius 方法求解了(1.1)．但是它们都没有仔细考察指标方程的四根的相互关系，仅以标量型式给出了特殊情形下各自解的表达式，用这种方法求解全部解答是十分繁琐和困难的．

对于常微分方程的 Frobenius 方法，教科书中有求解的详细过程；但于常微分方程组该方法的描述极少．实际上，方程组的方法是相当繁杂的，有许多不同之处，并且技巧性很强．为此本文针对(1.1)的变换式，开展矩阵型式Frobenius 级数法的研究，具体分析了指标方程根的相互关系，并在不同情形下给出解的具体表达形式．本文解法另一个优点是，可推广应用于一般的常微分方程组．本文所用记号参见附录．

* 国家自然科学基金资助的课题．1993年 9 月1日收到．

二、问题与求解

作变量变换

$$W=r^{-\frac{1}{2}}w,\ U=r^{-\frac{1}{2}}u \tag{2.1}$$

则方程组(1.1)可写成如下矩阵形式

$$([H(\nabla)]+r^2[\alpha\beta])\{X\}=0 \tag{2.2}$$

式中 $\{X\}=(w,u)^T$. (2.2)是关于两个变量的二阶常微分方程组. 此时易知$r=0$是该方程组的正则奇点，故将Frobenius解法推广到矩阵形式，设

$$\{X\}=\sum_{i=0}^{\infty}\{X_i\}r^{s+i} \tag{2.3}$$

其中 s是待求指标且$\{X_0\}\neq 0$. 将上式代入(2.2)后由比较系数法可得指标方程

$$|H(s)|=s^4+\left(\nu-\frac{1}{2}\right)s^2+\delta-\frac{1}{4}\nu+\frac{1}{16}=0 \tag{2.4}$$

其四根为

$$\left.\begin{aligned} s_1&=\left[\frac{1}{4}-\frac{1}{2}\nu+\left(\frac{1}{4}\nu^2-\delta\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},\quad s_4=-s_1\\ s_2&=\left[\frac{1}{4}-\frac{1}{2}\nu-\left(\frac{1}{4}\nu^2-\delta\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},\quad s_3=-s_2\end{aligned}\right\} \tag{2.5}$$

虽然有 $\mathrm{Re}(s_1)\geqslant\mathrm{Re}(s_2)\geqslant\mathrm{Re}(s_3)\geqslant\mathrm{Re}(s_4)$. 由常微分方程理论知，对应四个根，(2.2)有四个相互独立的解. 而这四个解的线性组合构成(2.2)的通解.

在附录(A·2)中∇用s_k代入，即得(A·6). 根据(2.4)，显然有$|H(s_k)|=0$. 在下文中假设$H_{1k},H_{2k},H_{3k},H_{4k}$不全为零，并且假设$H_{1k},H_{2k}$不全为零，因为在(2.2)中将两个方程上下互换总能做到这一点. 事实上，若$H_{1k}=H_{2k}=H_{3k}=H_{4k}=0$，则有$\{X\}=\{X_0\}_k r^{s_k}$和圆频率$\omega=0$的简单解. 下面对$s_k$的各种情形来求相应的解.

1. 对应s_1的解

设

$$\{X\}_1=\sum_{i=0}^{\infty}\{X_i\}_1 r^{s_1+i} \tag{2.6}$$

将它代入(2.2)，用比较系数法得

$$[H(s_1)]\{X_0\}_1=0 \tag{2.7}$$

$$[H(s_1+1)]\{X_1\}_1=0 \tag{2.8}$$

$$[H(s_1+i)][X_i\}_1+[\alpha\beta]\{X_{i-2}\}_1=0\quad(i=2,3,\cdots) \tag{2.9}$$

由于$|H(s_1)|=0$，则(2.7)有解

$$\{X_0\}_1=e_1\{B_0\}_1 \tag{2.10}$$

由于$|H(s_1+i)|\neq 0(i\geqslant 1)$，故由(2.8)和(2.9)递推可得

$$\{X_{2i+1}\}_1=0\qquad(i=0,1,2,\cdots) \tag{2.11}$$

$$\{X_{2i}\}_1=[B(s_1+2i)]\{X_0\}_1=e_1\{B_{2i}\}_1\qquad(i=1,2,\cdots) \tag{2.12}$$

将(2.10)、(2.11)和(2.12)代入(2.6)得

$$\{X\}_1=e_1\{B(r)\}_1 \tag{2.13}$$

式中$\{B(r)\}_1$如(A.9)所示。显然$\{B(r)\}_1$是(2.2)对应于s_1的解。

2. 对应s_2的解。

要分两种情形讨论：

2.1. $s_1-s_2=A_1$（这里的A_1及以后的A_i均表示非偶数且不等于零）。显然第二解易知为$\{B(r)\}_2$。

2.2. $s_1-s_2=2m \quad (m=0,1,2,\ \cdots)$。

设

$$\{X\}_2=A\{B(r)\}_1\ln r+\sum_{i=0}^{\infty}\{X_i\}_2 r^{s_2+i} \tag{2.14}$$

将它代入(2.2)并乘r^{-s_2}，得

$$\sum_{i=0}^{\infty}([H(s_2+i)]+r^2[\alpha\beta])\{X_i\}_2 r^i$$

$$+A\sum_{i=0}^{\infty}[M(s_1+2i)]\{B_{2i}\}_1 r^{2m+2i}=0 \tag{2.15}$$

(1) 在$m=1,2\cdots$时使用比较系数法得

$$[H(s_2)]\{X_0\}_2=0 \tag{2.16}$$

$$[H(s_2+1)]\{X_1\}_2=0 \tag{2.17}$$

$$[H(s_2+2i+1)]\{X_{2i+1}\}_2+[\alpha\beta]\{X_{2i-1}\}_2=0 \quad (i=1,2,\cdots) \tag{2.18}$$

$$[H(s_2+2i)]\{X_{2i}\}_2+[\alpha\beta]\{X_{2i-2}\}_2=0 \quad (i=1,2,\cdots,m-1) \tag{2.19}$$

$$[H(s_2+2i)]\{X_{2i}\}_2+[\alpha\beta]\{X_{2i-2}\}_2 +A[M(s_1+2i-2m)]\{B_{2i-2m}\}_1=0 \quad (i=m,m+1\cdots) \tag{2.20}$$

由于$|H(s_2)|=0$，故(2.16)有解

$$\{X_0\}_2=e_2\{B_0\}_2 \tag{2.21}$$

由于$|H(s_2+2i+1)|\neq 0 \quad (i=0,1,2,\cdots)$，由(2.17)和(2.18)得

$$\{X_{2i+1}\}_2=0 \quad (i=0,1,2,\cdots) \tag{2.22}$$

递推求解(2.19)并利用(2.21)得

$$\{X_{2i}\}_2=e_2\{B_{2i}\}_2 \quad (i=1,2,\cdots,m-1) \tag{2.23}$$

在(2.20)中，当$i=m$时有

$$[H(s_1)]\{X_{2m}\}_2+[\alpha\beta]\{X_{2m-2}\}_2+A[M(s_1)]\{B_0\}_1=0 \tag{2.24}$$

由于$|H(s_1)|=0$，对$\{X_{2m}\}_2$作线性变换，即设

$$\{X_{2m}\}_2=[\{C_0\}_1\{B_0\}_1]\begin{Bmatrix} e_0 \\ e_2 \end{Bmatrix}=e_0\{C_0\}_1+e_1\{B_0\}_1 \tag{2.25}$$

再利用(2.23)计算$\{X_{2m-2}\}_2$，则(2.24)变为

$$[N(s_1)]\begin{Bmatrix} A \\ e_0 \end{Bmatrix}=-e_2[\alpha\beta]\{B_{2m-2}\}_2 \tag{2.26}$$

按(A.12)，当$m\neq 0$时，有$|N(s_1)|\neq 0$，则(2.26)有解为

$$\left\{\begin{matrix}A\\e_0\end{matrix}\right\}=e_2\left\{\begin{matrix}Q^m_{12}\\T^m_{12}\end{matrix}\right\} \tag{2.27}$$

将e_0表达式代入(2.25)，有

$$\{X_{2m}\}_2=e_2T^m_{12}\{C_0\}_1+e_1\{B_0\}_1 \tag{2.28}$$

在(2.20)中，对于 $i=m+1,m+2,\cdots$，则因$|H(s_2+2i)|=|H(s_1+2j)|\neq 0\ (j=1,2,\cdots)$，得

$$\{X_{2m+2j}\}_2=e_2(T^m_{12}\{C_{2j}\}_1+Q^m_{12}\{RB_{2j}\}_{111})+e_1\{B_{2j}\}_1\quad (j=1,2,\cdots) \tag{2.29}$$

于是将(2.21)、(2.22)、(2.23)、(2.28)和(2.29)以及(2.27)中的A代入(2.14)，得

$$\{X\}_2=e_2\{D(r)\}^m_{12}+e_2\{B(r)\}_1 \tag{2.30}$$

显然如(A·16)所示的$\{D(r)\}^m_{12}$是对应s_2的解.

(2)　在$m=0$时，使用比较系数法，得

$$[H(s_1)]\{X_0\}_2+A[M(s_1)]\{B_0\}_1=0 \tag{2.31}$$

$$[H(s_1+1)]\{X_1\}_2=0 \tag{2.32}$$

$$[H(s_1+2i+1)]\{X_{2i+1}\}_2+[\alpha\beta]\{X_{2i-1}\}_2=0\quad (i=1,2\cdots) \tag{2.33}$$

$$[H(s_1+2i)]\{X_{2i}\}_2+[\alpha\beta]\{X_{2i-2}\}_2+A[M(s_1+2i)]\{B_{2i}\}_1=0\quad (i=1,2,\cdots) \tag{2.34}$$

由于$|H(s_1)|=0$，故设

$$\{X_0\}_2=e_0\{C_0\}_1+e_1\{B_0\}_1 \tag{2.35}$$

将它代入(2.31)，得

$$[N(s_1)]\left\{\begin{matrix}A\\e_0\end{matrix}\right\}=0 \tag{2.36}$$

因$m=0$，按(A·12)知$|N(s_1)|=0$，则上式有非零解

$$\left\{\begin{matrix}A\\e_0\end{matrix}\right\}=e_2\left\{\begin{matrix}-N_{21}\\N_{11}\end{matrix}\right\} \tag{2.37}$$

因此(2.35)可写成

$$\{X_0\}_2=e_2N_{11}\{C_0\}_1+e_1\{B_0\}_1 \tag{2.38}$$

由(2.32)及(2.33)知

$$\{X_{2i+1}\}_2=0\qquad (i=0,1,2,\cdots) \tag{2.39}$$

由(2.34)递推可得

$$\{X_{2i}\}_2=e_2(N_{11}\{C_{2i}\}_1-N_{21}\{RB_{2i}\}_{111})+e_1\{B_{2i}\}_1\quad (i=1,2,\cdots) \tag{2.40}$$

最后将有关公式代入(2.14)得到

$$\{X\}_2=e_2\{D(r)\}^0_{12}+e_1\{B(r)\}_1 \tag{2.41}$$

式中$\{D(r)\}^0_{12}$是对应s_2的解，如(A·16)所示，其中$m=0$。

3.　对应s_3的解

要分五种情形讨论：

3.1.　$s_1-s_2=A_1$；$s_2-s_3=A_2$；$s_1-s_3=A_1+A_2=A_3$

则三个独立解为

$$\{B(r)\}_1;\ \{B(r)\}_2;\ \{B(r)\}_3 \tag{2.42}$$

3.2. $s_1-s_2=A_1$; $s_2-s_3=A_2$; $s_1-s_3=2l\quad(l=1,2,\cdots)$

则三个独立解为

$$\{B(r)\}_1;\ \{B(r)\}_2;\ \{D(r)\}_{13}^{l} \tag{2.43}$$

3.3. $s_1-s_2=A_1$; $s_2-s_3=2n(n=0,1,2,\cdots)$; $s_1-s_3=A_1+2n=A_3$

则三个独立解为

$$\{B(r)\}_1;\ \{B(r)\}_2;\ \{D(r)\}_{23}^{n} \tag{2.44}$$

3.4. $s_1-s_2=2m(m=0,1,2,\cdots)$; $s_2-s_3=A_2$; $s_1-s_3=2m+A_2=A_3$

则三个独立解为

$$\{B(r)\}_1;\ \{D(r)\}_{12}^{m};\ \{B(r)\}_3 \tag{2.45}$$

3.5. $s_1-s_2=2m(m=0,1,2,\cdots)$; $s_2-s_3=2n(n=0,1,2,\cdots)$; $s_1-s_3=2m+2n$

显然对应s_1和s_2的解是$\{B(r)\}_1$和$\{D(r)\}_{12}^{m}$. 对应s_3的解可设具有如下形式

$$\{X\}_3=A\{D(r)\}_{12}^{m}\ln r+\{f\}\ln r+\sum_{i=0}^{\infty}\{X_i\}_3r^{s_3+i} \tag{2.46}$$

将上式代入(2.2)，可得如下两式

$$AQ_{12}^{m}[M(\nabla)]\{B(r)\}_1+([H(\nabla)]+r_2[\alpha\beta])\{f\}=0 \tag{2.47}$$

$$2AQ_{12}^{m}\{B(r)\}_1+A[M(\nabla)]\{Y(r)\}_{12}^{m}+[M(\nabla)]\{f\}$$

$$+([H(\nabla)]+r^2[\alpha\beta])\sum_{i=0}^{\infty}\{X_i\}_3r^{s_3+i}=0 \tag{2.48}$$

比较(2.47)和(A·19)，即可写出$\{f\}$的如下级数解

$$\{f\}=A\{Y(r)\}_{12}^{m}+e\{B(r)\}_1 \tag{2.49}$$

将(2.49)、(A·9)和(A·17)等式代入(2.48)并乘r^{-s_3}后得

$$2A\sum_{i=0}^{\infty}(Q_{12}^{m}\{B_{2i}\}_1r^{2i+2m+2n}+[M(s_2+2i)]\{Y_{2i}\}_{12}^{m}r^{2i+2n})$$

$$+e\sum_{i=0}^{\infty}[M(s_1+2i)]\{B_{2i}\}_1r^{2i+2m+2n}+\sum_{i=0}^{\infty}([H(s_3+i)]+r^2[\alpha\beta])\{X_i\}_3r^i=0$$

$$(m,n=0,1,2,\cdots) \tag{2.50}$$

于是对于下列4种情形，分别用比较系数法，可各得一组方程. 仿照前面的做法对每一组都可求得相应的解$\{D(r)\}_{123}^{mn}$.

(1) $m,n=1,2,\cdots$情形. 不难求得A, e及$\{X_i\}_3(i=0,1,2,\cdots)$相应解如(A·29)所示.

(2) $m=1,2,\cdots$; $n=0$情形.不难求得A,e及$\{X_i\}_3(i=0,1,2,\cdots)$，相应 解仍如(A·29)所示，而$n=0$.

(3) $m=0$; $n=1,2\cdots$情形. 此时令$e=0$，由(2.50)得

$$[H(s_3)]\{X_0\}_3=0 \tag{2.51}$$

$$[H(s_3+1)]\{X_1\}_3=0 \tag{2.52}$$

$$[H(s_3+2i+1)]\{X_{2i+1}\}_3+[\alpha\beta]\{X_{2i-1}\}_3=0\quad(i=1,2,\cdots) \tag{2.53}$$

$$[H(s_3+2i)]\{X_{2i}\}_3+[\alpha\beta]\{X_{2i-2}\}_3=0\quad(i=1,2,\cdots,n-1) \tag{2.54}$$

$$2A(Q_{12}^{0}\{B_{2i-2n}\}_1+[M(s_1+2i-2n)]\{Y_{2i-2n}\}_{12}^{0})+[H(s_3+2i)]\{X_{2i}\}_3$$

$$+[\alpha\beta]\{X_{2i-2}\}_3=0 \quad (i=n,n+1,\cdots) \tag{2.55}$$

显然由(2.5)～(2.54)知

$$\{X_0\}_3=e_3\{B_0\}_3 \tag{2.56}$$

$$\{X_{2i+1}\}_3=0 \quad (i=0,1,2,\cdots) \tag{2.57}$$

$$\{X_{2i}\}_3=e_3\{B_{2i}\}_3 \quad (i=1,2,\cdots,n-1) \tag{2.58}$$

在式(2.55)中，当$i=n$时有

$$2A(Q_{12}^0\{B_0\}_1+[M(s_1)]\{Y_0\}_{12}^0)+[H(s_1)]\{X_{2n}\}_3+[\alpha\beta]\{X_{2n-2}\}_3=0 \tag{2.59}$$

式中$\{X_{2n-2}\}_3$可由(2.58)求得。由于$|H(s_1)|=0$，设

$$\{X_{2n}\}_3=e_0\{C_0\}_1+e_1\{B_0\}_1 \tag{2.60}$$

将上式代入(2.59)得

$$[L(s_1)]\begin{Bmatrix}2A\\e_0\end{Bmatrix}=-e_3[\alpha\beta]\{B_{2n-2}\}_3 \tag{2.61}$$

当$m=0$即$s_1=s_2$时有$|N(s_1)|=0$，按(A·25)，$|L(s_1)|=4s_1^2N_{21}^2\neq 0$（因为$s_2=n\neq 0$），故由(2.61)得

$$\begin{Bmatrix}2A\\e_0\end{Bmatrix}=e_3\begin{Bmatrix}q_{13}^n\\t_{13}^n\end{Bmatrix} \tag{2.62}$$

将e_0表达式代入(2.60)可得

$$\{X_{2n}\}_3=e_3t_{13}^n\{C_0\}_1+e_1\{B_0\}_1 \tag{2.63}$$

在(2.55)中，当$i=n+1$，$n+2\cdots$时，由于$|H(s_1+2j)|\neq 0 \quad (j=1,2,\cdots)$，故得

$$\{X_{2n+2j}\}=e_3(q_{13}^nQ_{12}^0\{RB_{2j}\}_{11}+q_{13}^n\{RY_{2j}\}_{1112}^0+t_{13}^n\{C_{2j}\}_1)+e_1\{B_{2j}\} \quad (j=1,2,\cdots) \tag{2.64}$$

于是将(2.49)以及A和$\{X_i\}_3(i=0,1,2\cdots)$的有关表达式代入(2.46)可得

$$\{X\}_3=e_3\{D(r)\}_{123}^{0n}+e_1\{B(r)\}_1 \tag{2.65}$$

其中 $\{D(r)\}_{123}^{0n}$如(A·29)所示，式中$m=0$。

(4) $m=n=0$即$s_1=s_2=s_3=s_4=0$。同样令$e=0$，可类似求解。不过这时按(A·25)有$|L(s_1)|=0$，可仿(2.36)求解。对应的解为(A·29)中$m=n=0$的情形，即$\{D(r)\}_{123}^{00}$。

4. 对应s_4的解

继续s_3的讨论，仍然分五种情形，然后考察s_4的情形。

4.1. $s_1-s_2=A_1$；$s_2-s_3=A_2$；$s_1-s_3=A_1+A_2=A_3$

4.1.1. $s_3-s_4=A_1$；$s_2-s_4=A_3$；$s_1-s_4=A_1+A_3=A_4$

则四个独立解为

$$\{B(r)\}_1;\ \{B(r)\}_2;\ \{B(r)\}_3;\ \{B(r)\}_4 \tag{2.66}$$

4.1.2. $s_3-s_4=A_1$；$s_2-s_4=A_3$；$s_1-s_4=2p \ (p=1,2,\cdots)$，

则四个独立解为

$$\{B(r)\}_1;\ \{B(r)\}_2;\ \{B(r)\}_3;\ \{D(r)\}_{14}^p \tag{2.67}$$

4.2. $s_1-s_2=A_1$；$s_2-s_3=A_2$；$s_1-s_3=A_1+A_2=2l \ (l=1,2,\cdots)$

$s_3-s_4=A_1$；$s_2-s_4=2l$；$s_1-s_4=A_1+2l=A_4$

则四个独立解为

$$\{B(r)\}_1;\ \{B(r)\}_2;\ \{D(r)\}_{13}^{l};\ \{D(r)\}_{24}^{l} \tag{2.68}$$

4.3. $s_1-s_2=A_1$; $s_2-s_3=2n(n=0,1,2,\cdots)$; $s_1-s_3=A_1+2n=A_3$

4.3.1. $s_3-s_4=A_1$; $s_2-s_4=A_3$; $s_1-s_4=A_1+A_3=A_4$

则四个独立解为

$$\{B(r)\}_1;\ \{B(r)\}_2;\ \{D(r)\}_{23}^{n};\ \{B(r)\}_4 \tag{2.69}$$

4.3.2. $s_3-s_4=A_1$; $s_2-s_4=A_3$; $s_1-s_4=2p\ (p=1,2,\cdots)$

则四个独立解为

$$\{B(r)\}_1;\ \{B(r)\}_2;\ \{D(r)\}_{23}^{n};\ \{D(r)\}_{14}^{p} \tag{2.70}$$

4.4. $s_1-s_2=2m\ (m=0,1,2,\cdots)$; $s_2-s_3=A_2$; $s_1-s_3=2m+A_2=A_3$

$s_3-s_4=2m$; $s_2-s_4=A_3$; $s_1-s_4=2m+A_3=A_4$

则四个独立解为

$$\{B(r)\}_1;\ \{D(r)\}_{12}^{m};\ \{B(r)\}_3;\ \{D(r)\}_{34}^{m} \tag{2.71}$$

4.5. $s_1-s_2=2m\ \ (m=0,1,2,\cdots)$; $s_2-s_3=2n\ (n=0,1,2,\cdots)$; $s_1-s_3=2m+2n$

$s_3-s_4=2m$; $s_2-s_4=2m+2n$; $s_1-s_4=4m+2n$

对应于s_1，s_2和s_3的解易知分别为$\{B(r)\}_1$；$\{D(r)_{12}^{m}$和$\{D(r)\}_{123}^{mn}$。对应s_4的解，设其具有如下形式

$$\{X\}_4=A\{D(r)\}_{123}^{mn}\ln r+\{f\}(\ln r)^2+\{g\}\ln r+\sum_{i=0}^{\infty}\{X_i\}_4 r^{s_4+i} \tag{2.72}$$

将上式代入(2.2)，可得如下三式

$$(H[\nabla)]+r^2[\alpha\beta])\{f\}+\frac{1}{2}AQ_{23}^{mn}Q_{12}^{m}[M(\nabla)]\{B(r)\}_1=0 \tag{2.73}$$

$$\begin{aligned}&([H(\nabla)]+r^2[\alpha\beta])\{g\}+2[M(\nabla)]\{f\}+A(2Q_{23}^{mn}Q_{12}^{m}\{B(r)\}_1\\&\quad+Q_{23}^{mn}[M(\nabla)]\{Y(r)\}_{12}^{m}+P_{123}^{mn}[M(\nabla)]\{B(r)\}_1)=0\end{aligned} \tag{2.74}$$

$$\begin{aligned}&([H(\nabla)]+r^2[\alpha\beta])\sum_{i=0}^{\infty}\{X_i\}_4 r^{s_4+i}+2\{f\}+[M(\nabla)]\{g\}+A(2Q_{23}^{mn}\{Y(r)\}_{12}^{m}\\&\quad+2P_{123}^{mn}\{B(r)\}_1+[M(\nabla)]\{z(r)\}_{123}^{mn})=0\end{aligned} \tag{2.75}$$

比较(2.73)和(A·19)，可知$\{f\}$的级数解为

$$\{f\}=\frac{1}{2}AQ_{23}^{mn}\{Y(r)\}_{12}^{m}+eQ_{12}^{m}\{B(r)\}_1+\frac{1}{2}AP_{123}^{mn}\{B(r)\}_1 \tag{2.76}$$

式中最后一项是为求解$\{g\}$方便而有意添加的(2.73)齐次方程的解。将(2.76)代入(2.74)得

$$\begin{aligned}&([H(\nabla)]+r^2[\alpha\beta])\{g\}+2A(Q_{23}^{mn}Q_{12}^{m}\{B(r)\}_1+Q_{23}^{mn}[M(\nabla)]\{Y(r)\}_{12}^{m}\\&\quad+P_{122}^{mn}[M(\nabla)]\{B(r)\}_1)+2eQ_{12}^{m}[M(\nabla)]\{B(r)\}_1=0\end{aligned} \tag{2.77}$$

将上式与(A·32)比较，立即可得$\{g\}$的级数解为

$$\{g\}=2A\{z(r)\}_{123}^{mn}+2e\{Y(r)\}_{12}^{m}+C\{B(r)\}_1 \tag{2.78}$$

将(2.76)及(2.78)代入(2.75)并乘r^{-s_4}得

$$\sum_{i=0}^{\infty}([H(s_4+i)]+r^2[\alpha\beta])\{X_i\}_4 r^i+3A\sum_{i=0}^{\infty}([M(s_3+2i)]\{z_{2i}\}_{123}^{mn}r^{2m+2i}$$

$$+Q_{23}^{mn}\{Y_{2i}\}_{12}^{m}r^{2m+2n+2i}+P_{123}^{mn}\{B_{2i}\}_{1}r^{4m+2n+2i})$$

$$+2e\sum_{i=0}^{\infty}([M(s_2+2i)]\{Y_{2i}\}_{12}^{m}\gamma^{12m+2n+2i}+Q_{12}^{m}\{B_{2i}\}_{1}r^{4m+2n+2i})$$

$$+C\sum_{i=0}^{\infty}[M(s_1+2i)]\{B_{2i}\}_{1}r^{4m+2n+2i}=0 \tag{2.79}$$

与对应s_3的解法一样，分4种情形研究，求得相应的$\{D(r)\}_{1234}^{mn}$。

(1) $m,n=1,2\cdots$情形。不难求得A、C、e及$\{X_i\}(i=0,1,2,\cdots)$，相应解如(A·45)所示。

(2) $m=1,2\cdots$；$n=0$情形。取$e=0$，不难求得A、C及$\{X_i\}_4(i=0,1,2,\cdots)$。相应解如(A·45)所示，此时$n=0$。

(3) $m=0$；$n=1,2,\cdots$情形。取$C=0$，不难求得A、e及$\{X_i\}_4(i=0,1,2,\cdots)$。相应解如(A·47)所示。

(4) $m=n=0$ 即$s_1=s_2=s_3=s_4=0$情形。取$e=C=0$，由(2.79)可得下列方程

$$[H(s_4)]\{X_0\}_4+3A([M(s_4)]\{z_0\}_{123}^{00}+Q_{123}^{00}\{Y_0\}_{12}^{0})=0 \tag{2.80}$$

$$[H(s_4+1)]\{X_1\}_4=0 \tag{2.81}$$

$$[H(s_4+2i+1)]\{X_{2i+1}\}_4+[\alpha\beta]\{X_{2i-1}\}_4=0 \quad (i=1,2,\cdots) \tag{2.82}$$

$$[H(s_4+2i)]\{X_{2i}\}_4+[\alpha\beta]\{X_{2i-2}\}_4+3A([M(s_4+2i)]\{z_{2i}\}_{123}^{00}+Q_{23}^{00}\{Y_{2i}\}_{12}^{0})=0 \quad (i=1,2,\cdots) \tag{2.83}$$

由(2.81)和(2.82)得

$$\{X_{2i+1}\}=0 \quad (i=0,1,2,\cdots) \tag{2.84}$$

对于(2.80)，因$|H(s_4)|=0$，作线性变换

$$\{X_0\}_4=e_0\{C_0\}_1+e_1\{B_0\}_1 \tag{2.85}$$

将它代入(2.80)后，可写成如下形式

$$[G(s_1)]\begin{Bmatrix}3A\\e_0\end{Bmatrix}=0 \tag{2.86}$$

按(A·42)，有$|G(s_1)|=|G(0)|=0$，则(2.86)有非零解

$$\begin{Bmatrix}3A\\e_0\end{Bmatrix}=e_4\begin{Bmatrix}-N_{21}\\G_{11}\end{Bmatrix} \tag{2.87}$$

将e_0表达式代入(2.85)有

$$\{X_0\}_4=e_4G_{11}\{C_0\}_1+e_1\{B_0\}_1 \tag{2.88}$$

由(2.83)可逆递推得到

$$\{X_{2i}\}_4=e_4(G_{11}\{C_{2i}\}_1-N_{21}(\{RZ_{2i}\}_{11123}^{00}+Q_{23}^{00}\{RY_{2i}\}_{112}^{0}))+e_1\{B_{2i}\}_1 \quad (i=1,2,\cdots) \tag{2.89}$$

于是将有关公式代入(2.72)得

$$\{X\}_4=e_4\{D(r)\}_{1234}^{00}+e_1(B(r)\}_1 \tag{2.90}$$

显然如(A·49)所示的$\{D(r)\}_{1234}^{00}$对应s_4的解。

三、结　　论

本文详细考察了指标方程四根之间的相互关系，并对于不同的情况给出了四个互相独立的解的表达式。对于更为一般的常微分方程组，如果方程个数增大或阶数提高，指标方程的根越来越多使分析更加困难。但是运用本文采用的矩阵推导和符号系统，可以在很大程度上减少许多繁琐的工作，较之用非矩阵表达的级数求解（如文[1]、[2]）要简单和清晰。对于柱壳振动的方程组可同样求解。

附　　录

$\alpha,\beta,k_1,k_2,k_3,k_5,k_6$——常数，由材料性质决定

r ——自变量

$$\nabla=r\frac{d}{dr},\quad \nabla^2=r\frac{d}{dr}r\frac{d}{dr} \tag{A.1}$$

$$[H(\nabla)]=\begin{bmatrix} H_1(\nabla) & H_2(\nabla) \\ H_3(\nabla) & H_4(\nabla)\end{bmatrix}=\begin{bmatrix} \nabla^2+k_1-\frac{1}{4} & k_2\nabla+k_3-\frac{1}{2}k_2 \\ k_5\nabla+k_6-\frac{1}{2}k_7 & \nabla^2+k_4-\frac{1}{4}\end{bmatrix} \tag{A.2}$$

$$[M(\nabla)]=\begin{bmatrix} 2\nabla & k_2 \\ k_5 & 2\nabla\end{bmatrix} \tag{A.3}$$

$$[\alpha\beta]=\begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2\end{bmatrix} \tag{A.4}$$

$k,p,t=1,2,3,4$ ——脚标，

S_k ——指标方程的根，

$\{X\}_k$ ——对应于S_k的原方程之解，

$\{X_i\},\{X_i\}_k$ $(i=0,1,2,\cdots)$ ——列阵，待求，

$$\gamma=k_1+k_4-k_2k_5,\quad \delta=k_1k_4-k_3k_6 \tag{A.5}$$

$$[H(S_k)]=\begin{bmatrix} H_{1k} & H_{2k} \\ H_{3k} & H_{4k}\end{bmatrix} \tag{A.6}$$

$$\{B_0\}_k=\begin{bmatrix} -H_{2k} \\ H_{1k}\end{bmatrix},\{C_0\}_k=\begin{bmatrix} \overline{H}_{1k} \\ \overline{H}_{2k}\end{bmatrix} \tag{A.7}$$

（$\overline{H}_{1k}$和$\overline{H}_{2k}$分别为H_{1k}和H_{2k}的共轭复数）

$A,c,e,e_j(j=0,1,\ \ldots,\ 6)$——任意常数

$$[B(S_k+2i)]=(-1)^i[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\cdots[H(S_k+2)]^{-1}[\alpha\beta] \quad (i=1,2,\ \cdots) \tag{A.8}$$

$$\{B_{2i}\}_k=[B(S_k+2i)]\{B_0\}_k,\qquad \{B(r)\}_k=\sum_{i=0}^{\infty}\{B_{2i}\}_k r^{S_k+2i} \tag{A.9}$$

$$\{C_{2i}\}_k=[B(S_k+2i)]\{C_0\}_k,\qquad \{C(r)\}_k=\sum_{i=0}^{\infty}\{C_{2i}\}_k r^{S_k+2i} \tag{A.10}$$

$$[N(S_k)]=[[M(S_k)]\{B_0\}_k\quad [H(S_k)]\{C_0\}_k]=\begin{bmatrix} N_{1k} & N_{2k} \\ N_{3k} & N_{4k}\end{bmatrix} \tag{A.11}$$

（显然，$N_{2k}=H_{1k}\overline{H}_{1k}+H_{2k}\overline{H}_{2k}>0$）

$$|N(S_k)|=2S_kN_{2k}(S_1^2+S_2^2-2S_k^2) \tag{A·12}$$

$$\begin{Bmatrix} Q_{kt}^m \\ T_{kt}^m \end{Bmatrix}=\begin{cases} \begin{Bmatrix} -N_{2k} \\ N_{1k} \end{Bmatrix} & (m=0) \\ -[N(S_k)]^{-1}[\alpha\beta]\{B_{2m-2}\}_t & (m=1,2,\cdots) \end{cases} \tag{A·13}$$

$$\{HMB_{2i}\}_{hhh}=[H(S_k+2i)]^{-1}[M(S_k+2i)]\{B_{2i}\}_k \quad (i=1,2,\cdots) \tag{A·14}$$

$$\begin{aligned}\{RB_{2i}\}_{hhh}=&-\{HMB_{2i}\}_{hhh}+[H(S_k+2i)]^{-1}[\alpha\beta]\{HMB_{2i-2}\}_{hhh}\\&-[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\{HMB_{2i-4}\}_{hhh}+\cdots\\&+(-1)^i[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\cdots\\&\cdot[H(S_k+4)]^{-1}[\alpha\beta]\{HMB_2\}_{hhh} \quad (i=1,2,\cdots)\end{aligned} \tag{A·15}$$

$$\{D(r)\}_{kt}^m=Q_{kt}^m\{B(r)\}_k\ln r+\{Y(r)\}_{kt}^m \quad (m=0,1,2,\cdots) \tag{A·16}$$

$$\{Y(r)\}_{kt}^m=\sum_{i=0}^{\infty}\{Y_{2i}\}_{kt}^m r^{s_t+2i} \quad (m=0,1,2\cdots,) \tag{A·17}$$

$$\{Y_{2i}\}_{kt}^m=\begin{cases}\{B_{2i}\}_t & (i=0,1,2,\ldots,m=1;m\neq 0) \\ T_{kt}^m\{C_0\}_k & (i=m) \\ T_{kt}^m\{C_{2i-2m}\}_k+Q_{tk}^m\{RB_{2i-2m}\}_{hhh} & (i=m+1,m+2,\cdots)\end{cases} \tag{A·18}$$

$$([H(\nabla)]+r^2[\alpha\beta])\{Y(r)\}_{kt}^m+Q_{kt}^m[M(\nabla)]\{B(r)\}_k=0 \tag{A·19}$$

$\{f\},\{g\}$——函数列阵

$$\{HMY_{2i}\}_{kkpt}^m=[H(S_k+2i)]^{-1}[M(S_k+2i)]\{Y_{2i}\}_{pt}^m \quad (i=1,2,\cdots) \tag{A·20}$$

$$\begin{aligned}\{RY_{2i}\}_{kkpt}^m=&-\{HMY_{2i}\}_{kkpt}^m+[H(S_k+2i)]^{-1}[\alpha\beta]\{HMY_{2i-2}\}_{kkpt}^m\\&-[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\{HMY_{2i-4}\}_{kkpt}^m+\cdots\\&+(-1)^i[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\cdots\\&\cdot[H(S_k+4)^{-1}[\alpha\beta]\{HMY_2\}_{kkpt}^m \quad (i=1,2,\cdots)\end{aligned} \tag{A·21}$$

$$\{HB_{2i}\}_{hh}=[H(S_k+2i)]^{-1}\{B_{2i}\}_k \quad (i=1,2,\cdots) \tag{A·22}$$

$$\begin{aligned}\{RB_{2i}\}_{hh}=&-\{HB_{2i}\}_{hh}+[H(S_k+2i)]^{-1}[\alpha\beta]\{HB_{2i-2}\}_{hh}\\&-[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\{HB_{2i-4}\}_{hh}+\cdots\\&+(-1)^i[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\cdots\\&\cdot[H(S_k+4)]^{-1}[\alpha\beta]\{HB_2\}_{hh} \quad (i=1,2,\cdots)\end{aligned} \tag{A·23}$$

$$[L(S_k)]=[N_{1k}[M(S_k)\{C_0\}_k-N_{2k}\{B_0\}_k \qquad [H(S_k)]\{C_0\}_k]=\begin{bmatrix} L_{1k} & N_{2k} \\ L_{3k} & N_{4k} \end{bmatrix} \tag{A·24}$$

$$|L(S_k)|=N_{2k}^2(6S_k^2-S_1^2-S_2^2) \quad (当|N(S_k)|=0) \tag{A·25}$$

$$\begin{Bmatrix} q_{kt}^m \\ t_{kt}^m \end{Bmatrix}=\begin{cases} \begin{Bmatrix} -N_{2k} \\ L_{1k} \end{Bmatrix} & (m=0) \\ -[L(S_k)]^{-1}[\alpha\beta]\{B_{2m-2}\}_t & (m=1,2,\cdots) \end{cases} \tag{A·26}$$

$$\begin{Bmatrix} Q_{23}^{mn} \\ T_{23}^{mn} \end{Bmatrix}=\begin{cases} \begin{Bmatrix} -N_{21} \\ 0 \end{Bmatrix} & (m=n=0) \\ \begin{Bmatrix} q_{13}^n \\ 0 \end{Bmatrix} & (m=0;n=1,2,\cdots) \\ \begin{Bmatrix} Q_{23}^n \\ T_{23}^n \end{Bmatrix} & (m,n=1,2,\cdots) \end{cases} \tag{A·27}$$

$$\begin{Bmatrix} P_{123}^{mn} \\ B_{123}^{mn} \end{Bmatrix} = \begin{cases} \begin{Bmatrix} 0 \\ L_{11} \end{Bmatrix} & (m=n=0) \\ \begin{Bmatrix} 0 \\ t_{13}^{n} \end{Bmatrix} & (m=0;n=1,2\cdots\cdots) \\ -[N(S_1)]^{-1}([\alpha\beta](T_{23}^{n}\ \{C_{2m-2}\}_2+Q_{23}^{n}\ \{RY_{2m-2}\}_{2m12}^{m}) \\ \quad +Q_{23}^{n}\ ([(S_1)]\{Y_{2m}\}_{12}^{m}+Q_{12}^{m}\{B_0\}_1)) & (m=1,2\cdots;n=0,1\cdots) \end{cases} \quad (A\cdot 28)$$

$$\{D(r)\}_{123}^{mn}=\frac{1}{2}Q_{23}^{mn}(\{D(r)\}_{12}^{m}+\{Y(r)\}_{12}^{m})\ln r+P_{123}^{mn}\ \{B(r)\}_1\ln r+\{Z(r)\}_{123}^{mn} \quad (A\cdot 29)$$

$$\{Z(r)\}_{123}^{mn}=\sum_{i=0}^{\infty}\ \{Z_{2i}\}_{123}^{mn}r^{S_3+i} \quad (A\cdot 30)$$

$$\{Z_{2i}\}_{123}^{mn}=\begin{cases} \{B_{2i}\}_3 & (i=0,1,\cdots,n-1;n\neq 0) \\ T_{23}^{mn}\{C_0\}_2 & (i=n;m\neq 0) \\ T_{23}^{mn}\{C_{2i-2n}\}_2+Q_{23}^{mn}\ \{RY_{2i-2n}\}_{2212}^{m} & (i=n+1,n+2,\cdots,m+n-1;m\neq 0) \\ B_{123}^{mn}\{C_0\}_1 & (i=m+n) \\ B_{123}^{mn}\ \{C_{2i-2m-2n}\}_1+P_{123}^{mn}\{RB_{2i-2m-2n}\}_{111} \\ \quad +Q_{23}^{mn}\ (\ Q_{12}^{m}\ \{RB_{2i-2m-2n}\}_{11}+\ \{RY_{2i-2n}\}_{2212}^{m}) & (i=m+n+1,m+n+2,\cdots) \end{cases} \quad (A\cdot 31)$$

$$([H(\nabla)]+r^2[\alpha\beta])\{Z(r)\}_{123}^{mn}+P_{123}^{mn}\ [M(\nabla)]\{B(r)\}_1+Q_{23}^{mn}\ (Q_{12}^{m}\{B(r)\}_1 +[M(\nabla)]\{Y(r)\}_{12}^{m})=0 \quad (A\cdot 32)$$

$$\{HMZ_{2i}\}_{kk123}^{mn}\ [H(S_k+2i)]^{-1}[M(S_k+2i)]\{Z_{2i}\}_{123}^{mn} \qquad (i=1,2,\cdots) \quad (A\cdot 33)$$

$$\begin{aligned}\{RZ_{2i}\}_{kk123}^{m} &= -\{HMZ_{2i}\}_{kk123}^{mn}+[H(S_k+2i)]^{-1}[\alpha\beta]\{HNZ_{2i-2}\}_{kk123}^{mn} \\ &-[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\{HMZ_{2i-4}\}_{kk123}^{mn}+\cdots \\ &+(-1)^iH[(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\cdots \\ &\ [H(S_k+4)]^{-1}[\alpha\beta]\{HMZ_2\}_{kk123}^{mn} \qquad (i=1,2,\ \cdots)\end{aligned} \quad (A\cdot 34)$$

$$\begin{Bmatrix} P_{234}^{nm} \\ B_{234}^{nm} \end{Bmatrix}=-[N(S_2)]^{-1}([\alpha\beta](T_{34}^{m}\ \{C_{2n-2}\}_3+Q_{34}^{n}\{RZ_{2n-2}\}_{33123}^{mn}) + Q_{34}^{n}([M(S_2)]\{Z_{2n}\}_{123}^{mn}+Q_{23}^{mn}\{B_0\}_2)) \qquad (m,n=1,2,\ \cdots) \quad (A\cdot 35)$$

$$\{HY_{2i}\}_{kpi}^{m}=[H(S_k+2i)]^{-1}\{Y_{2i}\}_{pi}^{m} \qquad (i=1,2,\cdots) \quad (A\cdot 36)$$

$$\begin{aligned}\{RY_{2i}\}_{kpi}^{m} &= -\{HY_{2i}\}_{kpi}^{m}+[H(S_k+2i)]^{-1}[\alpha\beta]\{HY_{2i-2}\}_{kpi}^{m} \\ &-[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\{HY_{2i-4}\}_{kpi}^{m}+\cdots \\ &+(-1)^i[H(S_k+2i)]^{-1}[\alpha\beta][H(S_k+2i-2)]^{-1}[\alpha\beta]\cdots \\ &\cdot[H(S_k+4)]^{-1}[\alpha\beta]\{HY_2\}_{kpi}^{m} \qquad (i=1,2,\cdots)\end{aligned} \quad (A\cdot 37)$$

$$\left\{\begin{matrix} P_{1234}^{mn} \\ B_{1234}^{mn} \end{matrix}\right\} = -[N(S_1)]^{-1}([\alpha\beta](B_{234}^{nm}\{C_{2m-2}\}_2 + Q_{34}^m \{RZ_{2m+2n-2}\}_{33123}^{mn}$$

$$+ Q_{34}^m Q_{23}^n \{RY_{2m-2}\}_{212}^m + P_{234}^{nm} \{RY_{2m-2}\}_{2212}^m)$$

$$+ Q_{34}^m ([M(S_1)]\{Z_{2m+2n}\}_{123}^{mn} + Q_{23}^n \{Y_{2m}\}_{12}^m + P_{123}^{mn}\{B_0\}_1)$$

$$+ P_{234}^{nm} ([M(S_1)]\{Y_{2m}\}_{12}^m + Q_{12}^m\{B_0\}_1)) \qquad (m,n=1,2,\cdots) \qquad \text{(A·38)}$$

$$\left\{\begin{matrix} P_{1234}^{m0} \\ B_{1234}^{m0} \end{matrix}\right\} = -[N(S_1)]^{-1}([\alpha\beta](t_{34}^m\{C_{2m-2}\}_3 + q_{34}^m \{RZ_{2m-2}\}_{33123}^{m0}$$

$$- N_{22}\{RY_{2-2}\}_{212}^m) + q_{34}^m ([M(S_1)]\{Z_{2m}\}_{123}^{m0} - N_{22}\{Y_{2m}\}_{12}^m$$

$$+ P_{123}^{m0}\{B_0\}_1)) \qquad (m=1,2,\cdots) \qquad \text{(A·39)}$$

$$\left\{\begin{matrix} P_{1234}^{0n} \\ B_{1234}^{0n} \end{matrix}\right\} = -[L(S_1)]^{-1}([\alpha\beta](N_{13}\{C_{2n-2}\}_3 - N_{23}\{RZ_{2n-2}\}_{33123}^{0n})$$

$$-N_{23}([M(S_3)\{Z_{2n}\}_{123}^{0n} + q_{13}^n\{Y_0\}_{12}^0)) \qquad (n=1,2\cdots,)$$

$$[G(S_1)] = [L_{11}[M(S_1)]\{C_0\}_1 - N_{11}N_{11}\{C_0\}_1 \qquad [H(S_1)]\{C_0\}_1] = \begin{bmatrix} G_{11} & N_{21} \\ G_{31} & N_{41} \end{bmatrix} \qquad \text{(A·41)}$$

$$|G(S_1)| = 0 \qquad (当S_1=0) \qquad \text{(A·42)}$$

$$\left\{\begin{matrix} P_{234}^{mn} \\ B_{234}^{mn} \end{matrix}\right\} = \begin{cases} \left\{\begin{matrix} 0 \\ t_{34}^m \end{matrix}\right\} & (m=1,2,\cdots;n=0) \\ \left\{\begin{matrix} P_{234}^{nm} \\ B_{234}^{nm} \end{matrix}\right\} & (m,n=1,2,\cdots) \end{cases} \qquad \text{(A·43)}$$

$$\left\{\begin{matrix} Q_{34}^{mn} \\ T_{34}^{mn} \end{matrix}\right\} = \begin{cases} \left\{\begin{matrix} q_{34}^m \\ 0 \end{matrix}\right\} & (m=1,2,\cdots;n=0) \\ \left\{\begin{matrix} Q_{34}^m \\ T_{34}^m \end{matrix}\right\} & (m,n=1,2,\cdots) \end{cases} \qquad \text{(A·44a)}$$

$$Q_{34} = \begin{cases} 0 & (m=1,2,\cdots;n=0) \\ Q_{34}^m & (m,n=1,2,\cdots) \end{cases} \qquad \text{(A·44b)}$$

$$\{D(r)\}_{1234}^{mn} = (\tfrac{1}{3}Q_{34}^{mn}\{D(r)\}_{233}^{mn} + \tfrac{1}{6}Q_{34}^{mn}Q_{23}^{mn}\{Y(r)\}_{12}^m \ln r + \tfrac{1}{6}Q_{34}^{mn}P_{123}^{mn}\{B(r)\}_1 \ln r$$

$$+ \tfrac{1}{2}Q_{12}^m P_{234}^{mn}\{B(r)_1 \ln r + \tfrac{2}{3}Q_{34}^{mn}Q_{34}^{mn} \{Z(r)\}_{123}^{mn} + P_{234}^{mn} \{Y(r)\}_{21}^m$$

$$+ P_{1234}^{mn}\{B(r)\}_1)\ln r + \{J(r)\}^{mn} \qquad (m=1,2\cdots;n=0,1,2,\cdots) \qquad \text{(A·45)}$$

$$\{J(r)\}^{mn} = \sum_{i=0}^{m-1} \{B_{2i}\}_4 r^{s_4+2i} + T_{34}^{mn}\sum_{i=0}^{n-1} \{C_{2i}\}_3 r^{s_3+2i} + Q_{34}\sum_{i=1}^{n-1} \{RZ_{2i}\}_{33123}^{mn} r^{s_3+2i}$$

$$+ B_{234}^{mn}\sum_{i=0}^{m-1} \{C_{2i}\}_2 r^{s_2+2i} + B_{1234}^{mn} \{C(r)\}_1$$

$$+ \sum_{i=1}^{m-1} (Q_{34}^{mn} (\{RZ_{2i+2n}\}_{33123}^{mn} + Q_{23}^{mn} \{RY_{2i}\}_{212}^m) + P_{234}^{mn} \{RY_{2i}\}_{2212}^m) r^{s_2+2i}$$

$$+\sum_{i=1}^{\infty}(Q_{34}^{mn}(\{RZ_{2i+2m+2n}\}_{33123}^{mn}+Q_{23}^{mn}\{RY_{2i+2m}\}_{212}^{m})$$

$$+P_{234}^{mn}\{RY_{2i+2m}\}_{2212}^{m}+(Q_{34}^{mn}P_{123}^{mn}+Q_{12}^{m}P_{234}^{mn})\{RB_{2i}\}_{11}$$

$$+P_{1234}^{mn}\{RB_{2i}\}_{111})r^{s_1+2i}\qquad(m=1,2\cdots;n=0,1,2\cdots)\qquad(A\cdot 46)$$

$$\{D(r)\}_{1234}^{0n}=\frac{1}{2}(P_{1234}^{0n}Q_{12}^{0}\{B(r)\}_1-\frac{1}{3}N_{23}(Q_{23}^{2n}\{Y(r)\}_{12}^{0}+P_{123}^{0n}\{B(r)_1\}))(\ln r)^2$$

$$+(P_{1234}^{0n}\{Y(r)\}_{12}^{0}-\frac{1}{3}N_{23}(\{D(r)\}_{123}^{0n}+2\{Z(r)\}_{123}^{0n}))\ln r+\{J(r)\}^{0n}$$

$$(n=1,2\cdots)\qquad(A\cdot 47)$$

$$\{J(r)\}^{0n}=\sum_{i=0}^{n-1}\{C_{2i}\}_3r^{s_3+2i}-N_{23}\sum_{i=1}^{n-1}\{RZ_{2i}\}_{33123}^{0n}r^{s_3+2i}+B_{1234}^{0n}\{C(r)\}_1$$

$$+\sum_{i=1}^{\infty}(P_{1234}^{0n}\{RY_{2i}\}_{1112}^{0}+Q_{12}^{0}\{RB_{2i}\}_{11})-N_{23}(\{RZ_{2i+2n}\}_{33123}^{0n}$$

$$-Q_{23}^{0n}\{RY_{2i}\}_{112}^{0}+P_{123}^{0n}\{RB_{2i}\}_{11}))r^{s_1+2i}\qquad(n=1,2,\cdots)\qquad(A\cdot 48)$$

$$\{D(r)\}_{1234}^{00}=-\frac{1}{6}N_{21}(2\{D(r)\}_{123}^{00}+Q_{23}^{00}\{Y(r)\}_{12}^{0}\ln r+4\{Z(r)\}_{123}^{00})\ln r+\{J(r)\}^{00}$$

$$(A\cdot 49)$$

$$\{J(r)\}^{00}=G_{11}\{C(r)\}_1-N_{21}\sum_{i=1}^{\infty}(\{RZ_{2i}\}_{11123}^{00}+Q_{23}^{00}\{RY_{2i}\}_{112}^{0})r^{2i}\qquad(A\cdot 50)$$

参 考 文 献

[1] Mirsky, I., Axisymmetric vibrations of orthotropic cylinders, *J. Acoust. Soc. Am.*, 36(1964), 2106—2112.

[2] Cohen, H. and A. H. Shan, Free vibrations of a spherically isotropic hollow sphere, *Acustica*, 26(1972), 329—333.

Solutions to Equations of Vibrations of Spherical and Cylindlical Shells

Ding Hao-jiang　　Chen Wei-qiu

(Zhejiang University, Hangzhou)

Liu Zhong

(Beijing Institute of Structure & Environment Engineering, Beijing)

Abstract

The governing equations of the free vibrations of spherical and cylindrical shells with a regular singularity are solved by Frobenius Seris Method in the form of matrix. Considering the relatonship of the roots of the indicial equation, we get some various expressions of solution according to different cases. This work lays a foundation of solving certain elastic problems by analytical method.

Key words vibration, equations, Frobeniu Sories, spherical shell, cylindrical shell