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关于大几何参数的开顶扁球壳屈曲问题的奇摄动*

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摘 要

本文利用修正多重尺度法研究了大几何参数的具有刚性中心的边缘铰接的开顶扁球壳，在复合载荷作用下的非线性稳定问题。求得了扁壳几何参数k值较大时，本问题的一致有效的渐近解。

关键词 扁球壳 边界层 奇异摄动

一、引 言

在近代航空工程、精密仪器工程、自动控制和建筑结构等领域中，经常使用扁球壳。按照设计要求，需要研究它的稳定性，从理论上导出尽量精确可靠的计算公式或图表。

由于扁球壳屈曲问题的基本方程是非线性方程，求出这些方程的精确解在数学上存在很大困难。所以，多年来人们大都采用某种近似方法讨论几何参数k值较小时扁球壳、圆柱壳等的稳定性问题的近似解[2~4]，而对于几何参数k值较大的开顶扁球壳的非线性稳定问题的讨论却很少见，作者曾在文[5～6]中首先利用文[1]提出的奇异摄动方法研究了在单一载荷作用下大几何参数的开顶扁球壳的非线性稳定问题。

本文利用修正多重尺度法研究当几何参数k值较大时，外边缘铰链支承受均布载荷和中心集中力联合作用下，具有硬中心的开顶圆底扁球壳的非线性稳定问题，克服了在内、外边缘同时出现边界层现象的困难，导出了此边值问题的一致有效渐近解，进行了余项误差估计。这为决定临界载荷提供了较精确可靠的计算公式。

二、基本方程和边界条件

考虑图1所示的具有硬中心的开顶扁球壳，厚度为h，跨度为$2a$，内缘半径为b，中曲面半径为R。在中心集中载荷p和均布载荷q的联合作用下，扁球壳的大挠度弯曲方程为[7]：

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$$\left.\begin{aligned} &D\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}\right)-\frac{1}{2\pi r}[p+\pi q(r^2-b^2)]-N_r\left(\frac{r}{R}+\frac{dw}{dr}\right)=0\\ &\frac{1}{Eh}r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r^2N_r)\right)+\frac{r}{R}\cdot\frac{dw}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2=0 \end{aligned}\right\} \tag{2.1}$$

式中

$D=\dfrac{Eh^3}{12(1-\nu^2)}$ 为抗弯刚度，ν为泊松比，

E为弹性模量，N_r为径向薄膜内力，

w为球壳中曲面的挠度。

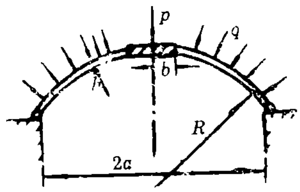

图 1

求得N_r，$\dfrac{dw}{dr}$之后，可利用以下公式计算径向薄膜位移u，径向弯矩M_r，径向剪力Q_r：

$$\left.\begin{aligned} &u=\frac{r}{hE}\left\{(1-\nu)N_r+r\frac{dN_r}{dr}\right\}\\ &M_r=-D\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)\\ &Q_r=-D\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) \end{aligned}\right\} \tag{2.2}$$

假定球壳外边缘铰链支承，而内边缘固定在可上、下移动的无变形的硬中心上，则w，u满足下列边界条件：

$$\text{当}r=a\text{时，}\ w=0,\ N_r=0,\ \frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}=0 \tag{2.3a}$$

$$\text{当}r=b\text{时，}\ \frac{dw}{dr}=0,\ u=0 \tag{2.3b}$$

为了简化计算，我们引进下列无量纲量：

$$\rho=\frac{r}{a},\quad \alpha=\frac{b}{a},\quad y=\sqrt{12(1-\nu^2)}\frac{w}{h},$$

$$\theta=-\frac{dy}{d\rho},\quad \bar{N}_r=\frac{a^2}{D}N_r,\quad S=\rho\bar{N}_r,$$

$$k=\sqrt{12(1-\nu^2)}\frac{a^2}{Rh},\quad \bar{\alpha}Q=[12(1-\nu^2)]^{3/2}a^2p/2\pi Eh^4,$$

$$\bar{\beta}Q=[12(1-\nu^2)]^{3/2}a^4q/2Eh^4.$$

将基本方程(2.1)和边界条件(2.3)化为无量纲边值问题

$$\left.\begin{aligned} &\varepsilon^2\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}(\rho\theta)\right)+\varepsilon^2Q[(\bar{\alpha}-\bar{\beta}\alpha^2)\rho^{-1}+\bar{\beta}\rho]-\varepsilon^2\rho^{-1}S\theta+S=0\\ &\varepsilon^2\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}(\rho S)\right)+\varepsilon^2\theta^2/2\rho-\theta=0 \end{aligned}\right\} \tag{2.4}$$

$$\text{当}\rho=1\text{时，}\ y=0,\ S=0,\ \frac{d\theta}{d\rho}+\nu\theta=0 \tag{2.5a}$$

$$\text{当}\rho=\alpha\text{时，}\ \theta=0,\ \alpha\frac{dS}{d\rho}-\nu S=0 \tag{2.5b}$$

其中 $\varepsilon^2=\dfrac{Rh}{a^2\sqrt{12(1-\nu^2)}}$.

这样，我们的问题就化为在边界条件(2.5)下求解带小参数$\varepsilon>0$的变系数的非线性微分方程组(2.4).

显然，当$\bar{\alpha}>0$，$\bar{\beta}>0$时，均布载荷与集中载荷方向相同；而$\bar{\alpha}<0$，$\bar{\beta}>0$时，二者方向相反. $\bar{\alpha}=0$，$\bar{\beta}\neq 0$时，扁壳受均布载荷作用，$\bar{\alpha}\neq 0$，$\bar{\beta}=0$时，扁壳仅受集中载荷作用. 后两种情形的渐近解，作者均已在文[5～6]中讨论过.

三、非线性摄动问题的求解

1. 外部解

先应用正则摄动法求其外部解. 假设边值问题(2.4)和(2.5)的外部展开式为

$$\theta^o=\sum_{n=0}^{\infty}\varepsilon^n\theta_n(\rho),\quad S^o=\sum_{n=0}^{\infty}\varepsilon^n S_n(\rho) \tag{3.1}$$

代入方程(2.4)，令ε的各次幂系数为零，得到关于$\theta_n(\rho)$，$S_n(\rho)$ $(n=0,1,2,\cdots)$ 的递推方程，从而容易求得

$$\theta^o=0,\quad S^o=-Q[(\bar{\alpha}-\bar{\beta}\alpha^2)\rho^{-1}+\bar{\beta}\rho]\varepsilon^2 \tag{3.2}$$

显然，(3.2)不满足两端边界条件(2.5)，故在$\rho=1$和$\rho=\alpha$近旁出现边界层. 下面应用"两变量"展开程序在$\rho=1$及$\rho=\alpha$的邻域内构造边界层校正项.

2. 边界层项

在$\rho=1$的邻域内引进两变量ξ和η：

$$\xi=\frac{u(\rho)}{\varepsilon},\qquad \eta=\rho,$$

则有

$$\frac{\partial}{\partial\rho}=\varepsilon^{-1}(\delta_{1,0}+\varepsilon\delta_{1,1}),$$

$$\frac{\partial^2}{\partial\rho^2}=\varepsilon^{-2}(\delta_{2,0}+\varepsilon\delta_{2,1}+\varepsilon^2\delta_{2,2}),$$

其中 $$\delta_{1,0}=u'\frac{\partial}{\partial\xi},\quad \delta_{1,1}=\frac{\partial}{\partial\eta},\quad \delta_{2,0}=(u')^2\frac{\partial^2}{\partial\xi^2},$$

$$\delta_{2,1}=2u'\frac{\partial^2}{\partial\xi\partial\eta}+u''\frac{\partial}{\partial\xi},\quad \delta_{2,2}=\frac{\partial^2}{\partial\eta^2}.$$

把(2.4)所对应的齐次方程变换成

$$\left.\begin{aligned}&(D_0+\varepsilon D_1+\varepsilon^2D_2)\theta-\varepsilon^2\eta S\theta+S\eta^2=0\\&(D_0+\varepsilon D_1+\varepsilon^2D_2)S+\varepsilon^2(2\eta)^{-1}\theta^2-\eta^2\theta=0\end{aligned}\right\} \tag{3.3}$$

其中 $D_0=\eta^2\delta_{2,0}$，$D_1=\eta^2\delta_{2,1}+\eta\delta_{1,0}$，$D_2=\delta_{2,2}+\eta\delta_{1,1}-1$.

设在$\rho=1$的邻域内边界层校正项的N阶近似式为

$$\left.\begin{aligned} V^{(1)}(\xi,\eta;\ \varepsilon) &= \sum_{n=0}^{N} \varepsilon^{n+1} v_n(\xi,\eta) \\ \theta^{(1)}(\xi,\eta;\ \varepsilon) &= \sum_{n=0}^{N} \varepsilon^{n+1} h_n(\xi,\eta) \end{aligned}\right\} \tag{3.4}$$

其中v_n和h_n是在$\rho=1$的邻域内的待求的边界层型函数。

将(3.4)代入(3.3)式得

$$\left.\begin{aligned} &(D_0+\varepsilon D_1+\varepsilon^2 D_2)\sum_{n=0}^{N}\varepsilon^{n+1}h_n(\xi,\eta)-\eta\sum_{n=2}^{2N}\sum_{k=0}^{n-2}\varepsilon^{n+2}v_k h_{n-2-k}+\eta^2\sum_{n=0}^{N}\varepsilon^{n+1}v_n=0 \\ &(D_0+\varepsilon D_1+\varepsilon^2 D_2)\sum_{n=0}^{N}\varepsilon^{n+1}v_n(\xi,\eta)+\sum_{n=2}^{2N}\varepsilon^{n+2}\sum_{k=0}^{n-2}(2\eta)^{-1}h_k h_{n-2-k}-\eta^2\sum_{n=0}^{N}\varepsilon^{n+1}h_n=0 \end{aligned}\right\} \tag{3.5}$$

在上式中逐次地比较ε的同次幂的系数，得到v_n，h_n的递推方程：

$$D_0h_0+\eta^2 v_0=0 \tag{3.6}$$

$$D_0h_1+D_1h_0+\eta^2 v_1=0 \tag{3.7}$$

$$D_0h_2+D_1h_1+D_2h_0+\eta^2 v_2=0 \tag{3.8}$$

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$$D_0h_{n-1}+D_1h_{n-2}+D_2h_{n-3}-\eta\sum_{k=0}^{n-4}v_k h_{n-4-k}+\eta^2 v_{n-1}=0 \qquad (n=4,5,\cdots) \tag{3.9}$$

$$D_0v_0-\eta^2 h_0=0 \tag{3.10}$$

$$D_0v_1+D_1v_0-\eta^2 h_1=0 \tag{3.11}$$

$$D_0v_2+D_1v_1+D_2v_0-\eta^2 h_2=0 \tag{3.12}$$

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$$D_0v_{n-1}+D_1v_{n-2}+D_2v_{n-3}+(2\eta)^{-1}\sum_{k=0}^{n-4}h_k h_{n-4-k}-\eta^2 h_{n-1}=0 \qquad (n=4,5,\cdots) \tag{3.13}$$

由(3.6)和(3.10)得

$$\begin{cases} D_0h_0+\eta^2 v_0=0 & (3.14)\\ D_0v_0-\eta^2 h_0=0 & (3.15)\end{cases}$$

则有

$$[u'(\eta)]^4\frac{\partial^4 v_0}{\partial\xi^4}+v_0=0 \tag{3.16}$$

在(3.16)中，若取$u'(\eta)=-1$，即取$u(\eta)=1-\eta$，则得

$$\frac{\partial^4 v_0}{\partial\xi^4}+v_0=0 \tag{3.17}$$

容易求得(3.17)具有边界层性质的解为

$$v_0=C_0(\eta)\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{cc}. \tag{3.18}$$

其中 cc. 表示前面表达式的共轭复量.

把(3.18)式代入(3.15)式可得

$$h_0=-iC_0(\eta)\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{cc}. \tag{3.19}$$

将(3.18)和(3.19)代入(3.7)和(3.11)得

$$\left.\begin{aligned}&D_0h_1+\eta^2v_1=-i\left[-\frac{\sqrt{2}}{2}(1-i)\right][2\eta^2C_0'(\eta)\\&\qquad+\eta C_0(\eta)]\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{cc}.\\&D_0v_1-\eta^2h_1=\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right][2\eta^2C_0'(\eta)+\eta C_0(\eta)]\\&\qquad\left[-\frac{\sqrt{2}}{2}(1-i)\right]+\text{cc}.\end{aligned}\right\} \tag{3.20}$$

由消除h_1，v_1中的长期项，可得$C_0=0$，从而得

$$v_0=0,\quad h_0=0 \tag{3.21}$$

方程(3.20)化为

$$\left.\begin{aligned}&D_0h_1+\eta^2v_1=0\\&D_0v_1-\eta^2h_1=0\end{aligned}\right\} \tag{3.22}$$

再由以后导出的关于v_i，h_i的边界条件，可逐步求得v_i，$h_i(i=1,\ 2,\cdots,N)$。

类似地，在$\rho=a$的邻域内引进两变量

$$\tilde{\xi}=\frac{\tilde{u}(\rho)}{\varepsilon},\quad \tilde{\eta}=\rho,$$

可以把(2.4)对应的齐次方程变换成

$$\left.\begin{aligned}&(\tilde{D}_0+\varepsilon\tilde{D}_1+\varepsilon^2\tilde{D}_2)\theta-\varepsilon^2\tilde{\eta}S\theta+S\tilde{\eta}^2=0\\&(\tilde{D}_0+\varepsilon\tilde{D}_1+\varepsilon^2\tilde{D}_2)S+\varepsilon^2(2\tilde{\eta})^{-1}\theta^2-\tilde{\eta}^2\theta=0\end{aligned}\right\} \tag{3.5}$$

其中 $\tilde{D}_0=\tilde{\eta}^2\tilde{\delta}_{2,0}$，$\tilde{D}_1=\tilde{\eta}^2\tilde{\delta}_{2,1}+\tilde{\eta}\tilde{\delta}_{1,0}$，$\tilde{D}_2=\tilde{\delta}_{2,2}+\tilde{\eta}\tilde{\delta}_{1,1}-1$，

$$\tilde{\delta}_{1,0}=\tilde{u}'\frac{\partial}{\partial\tilde{\xi}},\quad \tilde{\delta}_{1,1}=\frac{\partial}{\partial\tilde{\eta}},\quad \tilde{\delta}_{2,0}=(\tilde{u}')^2\frac{\partial^2}{\partial\tilde{\xi}^2},$$

$$\tilde{\delta}_{2,1}=2\tilde{u}'\frac{\partial^2}{\partial\tilde{\xi}\partial\tilde{\eta}}+\tilde{u}''\frac{\partial}{\partial\tilde{\eta}},\quad \tilde{\delta}_{2,2}=\frac{\partial^2}{\partial\tilde{\eta}^2}.$$

假设在$\rho=a$的邻域的边界层项具有下列形式的N阶近似式

$$\left.\begin{aligned}&V^{(a)}(\tilde{\xi},\tilde{\eta};\ \varepsilon)=\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{v}_n(\tilde{\xi},\ \tilde{\eta})\\&\theta^{(a)}(\tilde{\xi},\ \tilde{\eta};\ \varepsilon)=\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{h}_n(\tilde{\xi},\ \tilde{\eta})\end{aligned}\right\} \tag{3.23}$$

其中 $\tilde{v}_n$和$\tilde{h}_n$是在$\rho=a$的邻域内的待求的边界层型函数.

与前面讨论步骤相同，可得$\tilde{v}_n$，$\tilde{h}_n$的递推方程

$$D_0\tilde{h}_0+\tilde{\eta}^2\tilde{v}_0=0 \tag{3.24}$$

$$\tilde{D}_0\tilde{h}_1+\tilde{D}_1\tilde{h}_0+\tilde{\eta}^2\tilde{v}_1=0 \tag{3.25}$$

$$\tilde{D}_0\tilde{h}_2+\tilde{D}_1\tilde{h}_1+\tilde{D}_2\tilde{h}_0+\tilde{\eta}^2\tilde{v}_2=0 \tag{3.26}$$

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$$\tilde{D}_0\tilde{h}_{n-1}+\tilde{D}_1\tilde{h}_{n-2}+\tilde{D}_2\tilde{h}_{n-3}-\tilde{\eta}\sum_{k=0}^{n-4}\tilde{v}_k\tilde{h}_{n-4-k}+\tilde{\eta}^2\tilde{v}_{n-1}=0 \qquad (n=4,5,\cdots) \tag{3.27}$$

$$\tilde{D}_0\tilde{v}_0-\tilde{\eta}^2\tilde{h}_0=0 \tag{3.28}$$

$$\tilde{D}_0\tilde{v}_1+\tilde{D}_1\tilde{v}_0-\tilde{\eta}^2\tilde{h}_1=0 \tag{3.29}$$

$$\tilde{D}_0\tilde{v}_2+\tilde{D}_1\tilde{v}_1+\tilde{D}_2\tilde{v}_0-\tilde{\eta}^2\tilde{h}_2=0 \tag{3.30}$$

…………

$$\tilde{D}_0\tilde{v}_{n1}+\tilde{D}_1\tilde{v}_{n-2}+\tilde{D}_2\tilde{v}_{n-3}+(2\eta)^{-1}\sum_{k=0}^{n-4}\tilde{h}_k\tilde{h}_{n-4-k}-\tilde{\eta}^2\tilde{h}_{n-1}=0 \qquad (n=4,5,\cdots) \tag{3.31}$$

同样地，若取 $\tilde{u}'(\tilde{\eta})=1$，即取 $\tilde{u}(\tilde{\eta})=\tilde{\eta}-\alpha$，则可逐次求出上述递推方程的具有边界层性质的解为

$$\tilde{v}_0=\tilde{C}_0(\tilde{\eta})\exp\left[-\frac{\sqrt{2}}{2}(1-i)\tilde{\xi}\right]+cc. \tag{3.32}$$

$$\tilde{h}_0=-i\tilde{C}_0(\tilde{\eta})\exp\left[-\frac{\sqrt{2}}{2}(1-i)\tilde{\xi}\right]+cc. \tag{3.33}$$

由取 $\tilde{C}_0=0$，可得

$$\tilde{v}_0=0,\quad \tilde{h}_0=0 \tag{3.34}$$

而 $\tilde{v}_1$ 和 $\tilde{h}_1$ 由下列方程

$$\left.\begin{aligned}&\tilde{D}_0\tilde{h}_1+\tilde{\eta}^2\tilde{v}_1=0\\&\tilde{D}_0\tilde{v}_1-\tilde{\eta}^2\tilde{h}_1=0\end{aligned}\right\} \tag{3.35}$$

和以后导出的关于 $\tilde{v}_1$，$\tilde{h}_1$ 的边界条件确定．类似地，可逐步求得 $\tilde{v}_i$，$\tilde{h}_i(i=1,2,\cdots,N)$．

假设边值问题(2.4)和(2.5)的解 S，θ 的 N 阶近似式为

$$\left.\begin{aligned}S_N&=\sum_{n=0}^{N}\varepsilon^nS_n(\rho)+\sum_{n=0}^{N}\varepsilon^{n+1}v_n(\xi,\eta)+\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{v}_n(\tilde{\xi},\tilde{\eta})\\\theta_N&=\sum_{n=0}^{N}\varepsilon^n\theta_n(\rho)+\sum_{n=0}^{N}\varepsilon^{n+1}h_n(\xi,\eta)+\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{h}_n(\tilde{\xi},\tilde{\eta})\end{aligned}\right\} \tag{3.36}$$

其中 S_n，θ_n，v_n，h_n，$\tilde{v}_n$ 和 $\tilde{h}_n$ 分别由相应的递推方程所确定．

将(3.36)式代入边界条件(2.5)，考虑到 $v_n(\tilde{v}_n)$ 和 $h_n(\tilde{h}_n)(n=0,1,\cdots,N)$ 的边界层性质，得到关系式：

$$\sum_{n=0}^{N}\varepsilon^nS_n(1)+\sum_{n=0}^{N}\varepsilon^{n+1}v_n(\xi,\eta)|_{\eta=1}=0 \tag{3.37}$$

$$\sum_{n=0}^{N}\varepsilon^n\theta_n'(1)+\nu\sum_{n=0}^{N}\varepsilon^n\theta_n(1)+\sum_{n=0}^{N}(\delta_{1,0}+\varepsilon\delta_{1,1})\varepsilon^nh_n|_{\eta=1}+\nu\sum_{n=0}^{N}\varepsilon^{n+1}h_n|_{\eta=1}=0 \tag{3.38}$$

$$\sum_{n=0}^{N}\varepsilon^n\theta_n(\alpha)+\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{h}_n(\tilde{\xi},\tilde{\eta})|_{\tilde{\eta}=\alpha}=0 \tag{3.39}$$

$$\alpha\sum_{n=0}^{N}\varepsilon^n S_n'(\alpha)-\nu\sum_{n=0}^{N}\varepsilon^n S_n(\alpha)+\alpha\sum_{n=0}^{N}(\tilde{\delta}_{1,0}+\varepsilon\tilde{\delta}_{1,1})\varepsilon^n\tilde{v}_n(\tilde{\xi},\tilde{\eta})\,|_{\tilde{\eta}=\alpha}$$

$$-\sum_{n=0}^{N}\varepsilon^{n+1}\tilde{v}_n(\tilde{\xi},\tilde{\eta})\,|_{\tilde{\eta}=\alpha}=0 \tag{3.40}$$

从关于v_1，h_1的方程(3.22)和边界条件(3.37)、(3.38)

$$v_1\Big|_{\eta=1}=A,\quad \frac{\partial h_1}{\partial\xi}\Big|_{\eta=1}=0,\ \text{其中}A=Q[(\bar{\alpha}-\bar{\beta}\alpha^2)+\bar{\beta}] \tag{3.41}$$

解得

$$\left.\begin{aligned}v_1&=A\sqrt{\eta}\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\left(\cos\frac{\sqrt{2}}{2}\xi-\sin\frac{\sqrt{2}}{2}\xi\right)\\ h_1&=A\sqrt{\eta}\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\left(\sin\frac{\sqrt{2}}{2}\xi+\cos\frac{\sqrt{2}}{2}\xi\right)\end{aligned}\right\} \tag{3.42}$$

代入(3.8)和(3.12)以及边界条件得

$$\left.\begin{aligned}&D_0h_2+\eta^2v_2=0,\quad v_2|_{\eta=1}=0\\ &D_0v_2-\eta^2h_2=0,\quad \frac{\partial h_2}{\partial\xi}\Big|_{\eta=1}=-\frac{A}{2}(1+2\nu)\end{aligned}\right\} \tag{3.43}$$

容易求得(3.43)具有边界层性质的解为

$$\left.\begin{aligned}h_2&=-iC_2(\eta)\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{cc.}\\ v_2&=C_2(\eta)\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{cc.}\end{aligned}\right\} \tag{3.44}$$

把(3.44)代入(3.9)和(3.13)得

$$\left.\begin{aligned}D_0h_3+\eta^2v_3&=i\frac{\sqrt{2}}{2}(1-i)[2\eta^2C_2'(\eta)+\eta C_2(\eta)]\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{cc.}\\ D_0v_3-\eta^2h_3&=-\left[\frac{\sqrt{2}}{2}(1-i)\right][2\eta^2C_2'(\eta)+\eta C_2(\eta)]\exp\left[-\frac{\sqrt{2}}{2}(1-i)\xi\right]+\text{cc.}\end{aligned}\right\} \tag{3.45}$$

由消除(3.45)的解h_3，v_3中出现长期项和(3.43)中的边界条件，可定出

$$C_2(\eta)=-i\frac{A}{2\sqrt{2}}(1+2\nu)\sqrt{\eta} \tag{3.46}$$

将(3.46)代入(3.44)得

$$\left.\begin{aligned}v_2&=-\sqrt{\frac{\eta}{2}}A(1+2\nu)\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\sin\frac{\sqrt{2}}{2}\xi\\ h_2&=\sqrt{\frac{\eta}{2}}A(1+2\nu)\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\cos\frac{\sqrt{2}}{2}\xi\end{aligned}\right\} \tag{3.47}$$

再从关于$\tilde{v}_1$和$\tilde{h}_1$的方程(3.35)以及边界条件(3.39)、(3.40)

$$\tilde{h}_1\Big|_{\tilde{\eta}=\alpha}=0,\ \frac{\partial\tilde{v}_1}{\partial\tilde{\xi}}\Big|_{\tilde{\eta}=\alpha}=0 \tag{3.48}$$

解得

$$\tilde{h}_1=0,\quad \tilde{v}_1=0 \tag{3.49}$$

把(3.44)、(3.49)代入(3.26)和(3.30)以及边界条件(3.39)、(3.40)得

$$\left.\begin{array}{l} D_0\tilde{h}_2+\tilde{\eta}^2\tilde{v}_2=0,\ \tilde{h}_2\Big|_{\tilde{\eta}=a} \\ D_0\tilde{v}_2-\tilde{\eta}^2\tilde{h}_2=0,\ \dfrac{\partial\tilde{v}_2}{\partial\xi}\Big|_{\tilde{\eta}=a}=B \end{array}\right\} \tag{3.50}$$

其中 $B=Q(\nu-1)[\bar{\beta}-(\bar{\alpha}-\bar{\beta}\alpha^2)\alpha^{-2}]$.

容易求得(3.50)具有边界层性质的解为

$$\left.\begin{array}{l} \tilde{h}_2=-i\tilde{C}_2(\tilde{\eta})\exp\left[-\dfrac{\sqrt{2}}{2}(1-i)\xi\right]+\text{cc.} \\ \tilde{v}_2=\tilde{C}_2(\tilde{\eta})\exp\left[-\dfrac{\sqrt{2}}{2}(1-i)\xi\right]+\text{cc.} \end{array}\right\} \tag{3.51}$$

类似地，可定出 $\tilde{C}_2=-\dfrac{\sqrt{2}}{2}B\sqrt{\tilde{\eta}}$

从而得

$$\left.\begin{array}{l} \tilde{v}_2=-\sqrt{2}B\sqrt{\tilde{\eta}}\exp\left[-\dfrac{\sqrt{2}}{2}\xi\right]\cos\dfrac{\sqrt{2}}{2}\xi \\ \tilde{h}_2=\sqrt{2}B\sqrt{\tilde{\eta}}\exp\left[-\dfrac{\sqrt{2}}{2}\xi\right]\sin\dfrac{\sqrt{2}}{2}\xi \end{array}\right\} \tag{3.52}$$

于是

$$\begin{aligned} S_N=&-Q[(\bar{\alpha}-\bar{\beta}\alpha^2)\rho^{-1}+\bar{\beta}\rho]\varepsilon^2+\varepsilon^2\left\{A\sqrt{\eta}\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\right. \\ &\left.\cdot\left(\cos\frac{\sqrt{2}}{2}\xi-\sin\frac{\sqrt{2}}{2}\xi\right)\right\}+\varepsilon^3\left\{\sqrt{\frac{\eta}{2}}A(1+2\nu)\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\right. \\ &\left.\cdot\sin\frac{\sqrt{2}}{2}\xi-\sqrt{2\tilde{\eta}}B\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\cos\frac{\sqrt{2}}{2}\xi\right\}+O(\varepsilon^4) \\ =&\varepsilon^2\left\{A\sqrt{\rho}\exp\left[-\frac{\sqrt{2}}{2}\frac{(1-\rho)}{\varepsilon}\right]\left(\cos\frac{\sqrt{2}}{2}\frac{(1-\rho)}{\varepsilon}\right.\right. \\ &\left.\left.-\sin\frac{\sqrt{2}}{2}\frac{(1-\rho)}{\varepsilon}\right)-Q[(\bar{\alpha}-\bar{\beta}\alpha^2)\rho^{-1}+\bar{\beta}\rho]\right\} \\ &+\varepsilon^3\left\{\sqrt{\frac{\rho}{2}}A(1+2\nu)\exp\left[-\frac{\sqrt{2}}{2}\frac{(1-\rho)}{\varepsilon}\right]\sin\frac{\sqrt{2}}{2}\frac{(1-\rho)}{\varepsilon}\right. \\ &\left.-\sqrt{2\rho}B\exp\left[-\frac{\sqrt{2}}{2}\frac{(\rho-\alpha)}{\varepsilon}\right]\cos\frac{\sqrt{2}}{2}\frac{(\rho-\alpha)}{\varepsilon}\right\} \\ &+O(\varepsilon^4) \end{aligned} \tag{3.53}$$

$$\begin{aligned} \theta_N=&\varepsilon^2\left\{A\sqrt{\eta}\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\left(\sin\frac{\sqrt{2}}{2}\xi+\cos\frac{\sqrt{2}}{2}\xi\right)\right\} \\ &+\varepsilon^3\left\{-\sqrt{\frac{\eta}{2}}A(1+2\nu)\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\cos\frac{\sqrt{2}}{2}\xi\right. \\ &\left.+\sqrt{2\tilde{\eta}}B\exp\left[-\frac{\sqrt{2}}{2}\xi\right]\sin\frac{\sqrt{2}}{2}\xi\right\}+O(\varepsilon^4) \\ =&\varepsilon^2A\sqrt{\rho}\exp\left[-\frac{\sqrt{2}}{2}\frac{(1-\rho)}{\varepsilon}\right]\left(\sin\frac{\sqrt{2}}{2}\frac{(1-\rho)}{\varepsilon}\right. \end{aligned}$$

$$+\cos\frac{\sqrt{2}}{2\varepsilon}(1-\rho)\Big)+\varepsilon^3\Big\{-\sqrt{\frac{\rho}{2}}A(1+2\nu)\exp\Big[-\frac{\sqrt{2}}{2\varepsilon}(1-\rho)\Big]$$

$$\cdot\cos\frac{\sqrt{2}}{2\varepsilon}(1-\rho)+\sqrt{2\rho}\,B\exp\Big[-\frac{\sqrt{2}}{2\varepsilon}(\rho-a)\Big]$$

$$\cdot\sin\frac{\sqrt{2}}{2\varepsilon}(\rho-a)\Big\}+O(\varepsilon^4) \tag{3.54}$$

式中 $A=Q[(\bar{a}-\bar{\beta}a^2)+\bar{\beta}]$，$B=Q(\nu-1)[\bar{\beta}-(\bar{a}-\bar{\beta}a^2)a^{-2}]$.

3. 余项估计

我们以R_N，Z_N分别表示边值问题(2.4)和(2.5)的真解θ_ε，S_ε与形式逐近解θ_N，S_N的余项，即 $R_N=\theta_\varepsilon-\theta_N$，$Z_N=S_\varepsilon-S_N$.

且记 $R_N=\varepsilon^{N+1}R^N$，$Z_N=\varepsilon^{N+1}Z^N$，$a(\rho)=\varepsilon^2\rho^{-1}$，$b(\rho)=\varepsilon^2\rho^{-1}$，

$$f(\rho,\theta,S)=\varepsilon^2\{\theta\rho^{-1}-Q[(\bar{a}-\bar{\beta}a^2)\rho^{-1}+\bar{\beta}\rho]+\rho^{-1}S\theta\}-S$$

$$g(\rho,\theta,S)=\varepsilon^2\rho^{-2}S-\varepsilon^2\theta^2(2\rho)^{-1}+\theta.$$

将$\theta_\varepsilon=\theta_N+\varepsilon^{N+1}R^N$，$S_\varepsilon=S_N+\varepsilon^{N+1}Z^N$代入边值问题(2.4)和(2.5)得到$R^N$，$Z^N$满足下列边值问题

$$\left.\begin{aligned}&\varepsilon^2\frac{d^2R^N}{d\rho^2}+a(\rho)\frac{dR^N}{d\rho}=F(R^N,Z^N)+p(\rho,\varepsilon)\\&\varepsilon^2\frac{d^2Z^N}{d\rho^2}+b(\rho)\frac{dZ^N}{d\rho}=G(R^N,Z^N)+q(\rho,\varepsilon)\\&Z^N\Big|_{\rho=1}=O(1),\ \Big(\rho\frac{dZ^N}{d\rho}-\nu Z^N\Big)\Big|_{\rho=a}=O(1)\\&\Big(\frac{dR^N}{d\rho}+\nu R^N\Big)\Big|_{\rho=1}=O(1),\quad R^N\Big|_{\rho=a}=O(1)\end{aligned}\right\} \tag{3.55}$$

其中 $F(R^N,Z^N)=\varepsilon^{-(N+1)}[f(\rho,\theta_N+\varepsilon^{N+1}R^N,\ S_N+\varepsilon^{N+1}Z^N)-f(\rho,\theta_N,Z_N)]$,

$G(R^N,Z^N)=\varepsilon^{-(N+1)}[g(\rho,\theta_N+\varepsilon^{N+1}R^N,\ S_N+\varepsilon^{N+1}Z^N)-g(\rho,\theta_N,Z_N)]$,

$p(\rho,\varepsilon)=O\Big(1+\frac{1}{\varepsilon}\exp[-\kappa(1-\rho)/\varepsilon]\Big)$ （对$\kappa>0$）

$q(\rho,\varepsilon)=O\Big(1+\frac{1}{\varepsilon}\exp[-\kappa(\rho-a)/\varepsilon]\Big)$ （对$\kappa>0$）

为了估计余项R^N和Z^N，我们把边值问题(3.55)化为以下非线性积分方程组。下面为简便起见省去了R^N和Z^N的上角标.

$$\left.\begin{aligned}R(\rho,\varepsilon)=&R_0(\rho,\varepsilon)+\frac{1}{\varepsilon(1-\varepsilon)}\int_\rho^1(u^\varepsilon-u)F(R(u),\ Z(u))du\\&+\int_a^1B(\rho,u,\varepsilon)F(R(u),Z(u))du\\Z(\rho,\varepsilon)=&Z_0(\rho,\varepsilon)+\frac{1}{\varepsilon(1-\varepsilon)}\int_\rho^1(u^\varepsilon-u)G(R(u),\ Z(u))du\\&+\int_a^1B(\rho,u,\varepsilon)G(R(u),Z(u))du\end{aligned}\right\} \tag{3.56}$$

其中

$$R_0(\rho,\varepsilon)=(R(1,\varepsilon)+\varepsilon R_\rho(1,\varepsilon))-\varepsilon R_\rho(\alpha,\varepsilon)\left\{\alpha^\varepsilon+\frac{1}{\varepsilon(1-\varepsilon)}\alpha^\varepsilon(1-\rho^{1-\varepsilon})\right\}$$

$$-\int_\alpha^1 u^\varepsilon p(u,\varepsilon)du-\frac{1}{\varepsilon}\int_\rho^1\int_\alpha^v\left(\frac{u}{v}\right)^\varepsilon p(u,\varepsilon)dudv,$$

$$Z_0(\rho,\varepsilon)=(S(1,\varepsilon)+\varepsilon S_\rho(1,\varepsilon))-\varepsilon S_\rho(\alpha,\varepsilon)\left\{\alpha^\varepsilon+\frac{1}{\varepsilon(1-\varepsilon)}\alpha^\varepsilon(1-\rho^{1-\varepsilon})\right\}$$

$$-\int_\alpha^1 u^\varepsilon q(u,\varepsilon)\,du-\frac{1}{\varepsilon}\int_\rho^1\int_\alpha^v\left(\frac{u}{v}\right)^\varepsilon q(u,\varepsilon)dudv,$$

$$B(\rho,u,\varepsilon)=-\exp\left[-\frac{1}{\varepsilon}\int_u^1 a(t)dt\right]-\frac{1}{\varepsilon}\eta(\rho-u)\int_\rho^1\exp\left[-\frac{1}{\varepsilon}\int_u^v a(t)dt\right]dv,$$

$$\eta(\lambda)=\begin{cases}0 & (\lambda<0)\\ 1 & (\lambda\geqslant 0).\end{cases}$$

显然 $$\int_\alpha^1 B(\rho,u,\varepsilon)du=O(\varepsilon).$$

现在，我们把(3.56)式第二项积分中的F，G线性化得到

$$\left.\begin{aligned}R(\rho)&=R_0(\rho,\varepsilon)+\int_\rho^1 K_1(u,\varepsilon)R(u)du+\int_\rho^1 K_2(u,\varepsilon)Z(u)du\\&\quad+\int_\alpha^1 B(\rho,u,\varepsilon)F(R(u),Z(u))du+\varepsilon^{N+1}H(\rho,R(\rho),Z(\rho))\\Z(\rho)&=Z_0(\rho,\varepsilon)+\int_\rho^1 K_3(u,\varepsilon)R(u)du+\int_\rho^1 K_4(u,\varepsilon)Z(u)du\\&\quad+\int_\alpha^1 B(\rho,u,\varepsilon)G(R(u),Z(u))du+\varepsilon^{N+1}M(\rho,R(\rho),Z(\rho))\end{aligned}\right\}\tag{3.57}$$

其中

$$(K_1(u,\varepsilon),K_2(u,\varepsilon))=-(f_\theta(u,R_N,Z_N),\ f_S(u,R_N,Z_N))\cdot\frac{u^\varepsilon-u}{\varepsilon(1-\varepsilon)},$$

$$(K_3(u,\varepsilon),K_4(u,\varepsilon))=-(g_\theta(u,R_N,Z_N),\ g_S(u,R_N,Z_N))\cdot\frac{u^\varepsilon-u}{\varepsilon(1-\varepsilon)}.$$

当R,Z有界时，H，M是有界函数.

将(3.57)式写成下列向量形式

$$R^*=R_0^*+J_1R^*+J_2R^*\tag{3.58}$$

其中 $$R^*=\begin{pmatrix}R\\Z\end{pmatrix},\quad R_0^*=\begin{pmatrix}R_0\\Z_0\end{pmatrix},$$

$$J_1R^*=\int_\rho^1 K^*(u,\varepsilon)R^*(u,\varepsilon)du,\quad J_2R^*=\int_\alpha^1 M^*(R^*,\rho,u,\varepsilon)du,$$

而

$$K^*=\begin{pmatrix}K_1 & K_2\\K_3 & K_4\end{pmatrix},\quad M^*=\begin{pmatrix}B & F\\B & G\end{pmatrix}+\varepsilon^{N+1}\begin{pmatrix}H\\M\end{pmatrix}.$$

因为核K^*是有界的，所以向量积分算子J_1是可逆的，即$(I-J_1)^{-1}$存在.从而（3.58）可化为

$$R^*=(I-J_1)^{-1}R_0^*+(I-J_1)^{-1}J_2R^*\tag{3.59}$$

其中 $$(I-J_1)^{-1}\phi=\phi+\int_\rho^1 W^*(\rho,u,\varepsilon)\phi(u)du$$

是对任何ϕ中K^*的的预解核．当ε充分小时，它在$\alpha\leqslant\rho,u\leqslant 1$上是有界的（参见文献[8]）．

引进范数　$\|R^*\|=\sup(|R|,\ |Z|;\ \alpha\leqslant\rho\leqslant 1,\ 0<\varepsilon\leqslant\varepsilon_0)$.

我们有

$$\left|\int_\alpha^1 B(\rho,u,\varepsilon)(F(R_1(u),Z_1(u))-F(R_2(u),Z_2(u))du\right|$$

$$\leqslant O(1)\left(\int_\alpha^1 B(\rho,u,\varepsilon)du\right)\|R_1^*-R_2^*\|=O(\varepsilon)\|R_1^*-R_2^*\|,$$

同理

$$\left|\int_\alpha^1 B(\rho,u,\varepsilon)(G(R_1(u),Z_1(u))-G(R_2(u),Z_2(u))du\right|\leqslant O(\varepsilon)\|R_1^*-R_2^*\|.$$

又因为

$$(I-J_1)^{-1}J_2R^*=\int_\alpha^1 M^*(R^*,\rho,u,\varepsilon)\,du+\int_\rho^1 W^*(\rho,u,\varepsilon)\int_\alpha^1 M^*(R^*,u,v,\varepsilon)dvdu,$$

由M^*的定义，则有

$$\|(I-J_1)^{-1}(J_2R_1^*-J_2R_2^*)\|\leqslant O(\varepsilon_0)\|R_1^*-R_2^*\|.$$

利用 Banach 不动点定理，可知在$C[\alpha,1]\times C[\alpha,1]$上存在唯一的不动点，即对充分小的$\varepsilon$，在$\alpha\leqslant\rho\leqslant 1$上存在唯一的连续函数组$(R,Z)$满足积分方程(3.56)．

综合上述讨论，我们有下面的定理．

定理1　当ε充分小时，边值问题(2.4)和(2.5)在$\alpha\leqslant\rho\leqslant 1$上存在唯一的解（$S(\rho,\varepsilon)$，$\theta(\rho,\varepsilon)$），且对每个整数$N\geqslant 0$解可表为

$$S(\rho,\varepsilon)=S_N(\rho,\varepsilon)+\varepsilon^{N+1}Z^N(\rho,\varepsilon),$$
$$\theta(\rho,\varepsilon)=\theta_N(\rho,\varepsilon)+\varepsilon^{N+1}R^N(\rho,\varepsilon),$$

其中　S_N和θ_N由(3.36)式给出，R^N和Z^N在$\alpha\leqslant\rho\leqslant 1$上一致有界．即问题(2.4)和(2.5)在$\alpha\leqslant\rho\leqslant 1$上的一致有效渐近解为

$$S=\varepsilon^2Q\left\{[(\bar\alpha-\bar\beta\alpha^2)+\bar\beta]\sqrt{\rho}\exp\left[-\frac{\sqrt{2}}{2\varepsilon}(1-\rho)\right]\cdot\left(\cos\frac{\sqrt{2}}{2\varepsilon}(1-\rho)-\sin\frac{\sqrt{2}}{2\varepsilon}(1-\rho)\right)-[(\bar\alpha-\bar\beta\alpha^2)\rho^{-1}+\bar\beta\rho]\right\}$$
$$+\varepsilon^3Q\sqrt{\rho}\left\{\frac{\sqrt{2}}{2}(1+2\nu)[(\bar\alpha-\bar\beta\alpha^2)+\bar\beta]\exp\left[-\frac{\sqrt{2}}{2\varepsilon}(1-\rho)\right]\cdot\sin\frac{\sqrt{2}}{2\varepsilon}(1-\rho)-\sqrt{2}(\nu-1)[\bar\beta-(\bar\alpha-\bar\beta\alpha^2)\alpha^{-2}]\right.$$
$$\left.\cdot\exp\left[-\frac{\sqrt{2}}{2\varepsilon}(\rho-\alpha)\right]\cos\frac{\sqrt{2}}{2\varepsilon}(\rho-\alpha)\right\}+O(\varepsilon^4),$$

$$\theta=\varepsilon^2Q\sqrt{\rho}[(\bar\alpha-\bar\beta\alpha^2)+\bar\beta]\exp\left[-\frac{\sqrt{2}}{2\varepsilon}(1-\rho)\right]\left(\sin\frac{\sqrt{2}}{2\varepsilon}(1-\rho)+\cos\frac{\sqrt{2}}{2\varepsilon}(1-\rho)\right)+\varepsilon^3Q\sqrt{\rho}\left\{\sqrt{2}(\nu-1)[\bar\beta-(\bar\alpha-\bar\beta\alpha^2)\alpha^{-2}]\right.$$
$$\cdot\exp\left[-\frac{\sqrt{2}}{2\varepsilon}(\rho-\alpha)\right]\sin\frac{\sqrt{2}}{2\varepsilon}(\rho-\alpha)-\frac{\sqrt{2}}{2}(1+2\nu)[(\bar\alpha-\bar\beta\alpha^2)+\bar\beta]$$

$$\cdot \exp\left[-\frac{\sqrt{2}}{2\varepsilon}(1-\rho)\right]\cos\frac{\sqrt{2}}{2\varepsilon}(1-\rho)\Bigg\}+O(\varepsilon^4)$$

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The Singular Perturbation for the Buckling of a Truncated Shallow Spherical Shell with the Large Geometrical Parameter

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Abstract

A problem of practical interest for nonlinear axisymmetrical stability of a truncated shallow spherical shell of the large geometrical parameter with an articulated external edge and a nondeformable rigid body at the center under compound loads is investigated in this paper. By using modified method of multiple scales, the uniformly valid assymptotic solutions of this boundary value problem are obtained when the geometrical parameter k is large.

Key words shallow shell, boundary layer, singular perturbation