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不用Kirchhoff-Love假设的三维弹性板二级近似理论及其边界条件*

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摘 要

前文[1]给出了不用Kirchhoff-Love假设的三维弹性板的一级近似理论及其边界条件. 这个理论有6个微分方程求解6个待定平面函数，即 u_0，u_α，$A_{(0)}$，$S_{(2)\alpha}$，其中有3个方程为一组求解3个待定平面函数u_0，$S_{(2)\alpha}$，而另一组3个方程求解另外3个待定平面函数u_α，$A_{(0)}$. 它们的边界条件和这些方程一样，可以从本问题的广义变分原理的泛函变分的驻值条件求得. 当板厚h和板宽a之比h/a很小时，这种解接近于经典薄板解. 当h/a值较大时（如$h/a\approx 0.3$），这种解和经典薄板解，就有较大差别. 但这种差别在h/a值的什么范围内是合理的这一问题，并不清楚，为了解决这一问题，我们必须研究本问题的二级近似理论. 本文是前文的继续，我们将用本问题的广义变分原理的泛函变分驻值条件，导出9个微分方程和有关边界条件，用以求解9个二级近似解的待定平面函数 u_0，u_α，$A_{(0)}$，$A_{(1)}$，$S_{(2)\alpha}$，$S_{(3)\alpha}$，把二级近似理论解和一级近似理论以及经典理论的解相比较，就能明确一级近似理论的适用范围. 这里必须指出，二级近似理论也可以分成两组方程求解，求解过程也并不过分复杂. 有关符号和前文相同，这里将不再重复.

关键词 三维弹性板 克希霍夫-拉夫假设 二级近似理论

一、不用克希霍夫-拉夫假设的三维弹性板的二级近似理论的变形近似表示

在二级近似条件下，e_{00}，$e_{0\alpha}$取

$$e_{00}=A_{(0)}+A_{(1)}x_0 \tag{1.1a}$$

$$e_{0\alpha}=\left(\frac{1}{4}h^2-x_0^2\right)\{S_{(2)\alpha}+x_0S_{(3)\alpha}\} \tag{1.1b}$$

和一级近似理论相比较，e_{00}和$e_{0\alpha}$的x_0幂级数表近式中各增加了一项，即$A_{(1)}$和$S_{(3)\alpha}$.

于是有关的位移近似表达式U_0，U_α为

$$U_0(x_1,x_2,x_0)=u_0(x_1,x_2)+A_{(0)}x_0+\frac{1}{2}A_{(1)}x_0^2 \tag{1.2a}$$

$$U_\alpha(x_1,x_2,x_0)=u_\alpha(x_1,x_2)-u_{0,\alpha}x_0-\frac{1}{2}A_{(0),\alpha}x_0^2-\frac{1}{6}A_{(1),\alpha}x_0^3$$

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$$+2\left\{\left[\frac{h^2}{4}-\frac{1}{3}x_0^2\right]x_0S_{(2)\alpha}+\left[\frac{1}{4}h^2-\frac{1}{2}x_0^2\right]\frac{1}{2}x_0^2S_{(3)\alpha}\right\} \tag{1.2b}$$

应变分量除(1.1a,b)外，可以写成

$$e_{\alpha\beta}=\frac{1}{2}(U_{\alpha,\beta}+U_{\beta,\alpha})=\frac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha})-u_{0,\alpha\beta}x_0-\frac{1}{2}A_{(0),\alpha\beta}x_0^2$$

$$-\frac{1}{6}A_{(1),\alpha}x_0^3+\left\{\left[-\frac{h^2}{4}-\frac{1}{3}x_0^2\right]x_0(S_{(2)\alpha,\beta}+S_{(2)\beta,\alpha})\right]$$

$$+\left[\frac{1}{4}h^2-\frac{1}{2}x_0^2\right]\frac{1}{2}x_0^2[S_{(3)\alpha,\beta}+S_{(3)\beta,\alpha}]\right\} \tag{1.3}$$

于是，应力分量可以写成，

$$\sigma_{\alpha\beta}=\frac{E_1}{1-\nu_1^2}\left\{\frac{1}{2}(1-\nu_1)(u_{\alpha,\beta}+u_{\beta,\alpha})+\nu_1u_{\gamma,\gamma}\delta_{\alpha\beta}-[(1-\nu_1)u_{0,\alpha\beta}\right.$$

$$+\nu_1u_{0,\gamma\gamma}\delta_{\alpha\beta}]x_0+\nu_1A_{(0)}\delta_{\alpha\beta}+\nu_1A_{(1)}\delta_{\alpha\beta}x_0-\frac{1}{2}\{(1-\nu_1)A_{(0),\alpha\beta}$$

$$+\nu_1A_{(0),\gamma\gamma}\delta_{\alpha\beta}]x_0^2-\frac{1}{6}[(1-\nu_1)A_{(1),\alpha\beta}+\nu_1A_{(1),\gamma\gamma}\delta_{\alpha\beta}]x_0^3$$

$$+\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0[(1-\nu_1)(S_{(2)\alpha,\beta}+S_{(2)\beta,\alpha})+2\nu_1S_{(2)\gamma,\gamma}\delta_{\alpha\beta}]$$

$$+\left(\frac{1}{4}h^2-\frac{1}{2}x_0^2\right)\frac{1}{2}x_0^2[(1-\nu_1)(S_{(3)\alpha,\beta}+S_{(3)\beta,\alpha})+2\nu_1S_{(3)\gamma,\gamma}\delta_{\alpha\beta}] \tag{1.4a}$$

$$\sigma_{0\alpha}=\frac{E_1}{1+\nu_1}\left(\frac{1}{4}h^2-x_0^2\right)\{S_{(2)\alpha}+x_0S_{(3)\alpha}\} \tag{1.4b}$$

$$\sigma_{00}=\frac{E_1}{1-\nu_1^2}\left\{\nu_1u_{\alpha,\alpha}-\nu_1\nabla^2u_0x_0+A_{(0)}+A_{(1)}x_0\right.$$

$$-\nu_1\left[\frac{1}{2}\nabla^2A_{(0)}x_0^2+\frac{1}{6}\nabla^2A_{(1)}x_0^3\right]+2\nu_1\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0S_{(2)\alpha,\alpha}$$

$$\left.+\nu_1\left(\frac{1}{4}h^2-\frac{1}{2}x_0^2\right)x_0^2S_{(3)\alpha,\alpha}\right\} \tag{1.4c}$$

下面将利用本题的广义变分泛函Π的驻值条件$\delta\Pi=0$，其中

$$\delta\Pi=-\iiint_\tau(\sigma_{ij,j}+\bar{f}_i\rho)\delta U_idx_0dx_1dx_2+\iint_{\Omega_0}[(\sigma_{00}^+-\bar{p}_+)\delta U_0^+$$

$$-(\sigma_{00}^--\bar{p}_-)\delta U_0^-]dx_1dx_2+\iint_{\Omega_0}(\sigma_{\alpha0}^+\delta U_\alpha^+-\sigma_{\alpha0}^-\delta U_\alpha^-)dx_1dx_2$$

$$-\iint_{\Omega_u}(U_i-\bar{U}_i)n_\alpha\delta\sigma_{i\alpha}dsdx_0-\iint_{\Omega_\sigma}[\bar{\tau}_{ni}-\sigma_{i\alpha}n_\alpha]\delta U_idx_0ds \tag{1.5}$$

$\delta\Pi$ 可以通过(1.2a,b)，(1.4a,b)用u_i，$A_{(0)}$，$A_{(1)}$，$S_{(2)\alpha}$，$S_{(3)\alpha}$表示，由于δu_i，$\delta A_{(0)}$，$\delta A_{(1)}$，$\delta S_{(2)\alpha}$，$\delta S_{(3)\alpha}$等在τ内和边界各线段s_u,s_σ，以及Ω_0各点上都是独立变分，所以(1.5)式$\int(\cdots)dx_0$积分进行后，就可以求出二级近似的各微分方程和边界条件。

二、二级近似理论的方程和边界条件

二级近似理论的方程和边界条件的推导过程与前文[1]所述的一级近似理论相同。现在以δu_α有关各项为例，称$\delta\Pi$中有关δu_α的诸项的和为$\delta\Pi_{u\alpha}$，从(1.5)式，用(1.2a,b)中的U_i和(1.4a,b)中的$\sigma_{i\alpha}$代入δU_i和$\delta\sigma_{i\alpha}$后，即可求得

$$\delta\Pi_{u\alpha}=-\iiint_\tau(\sigma_{\alpha i,i}+\bar{f}_\alpha\rho)\delta u_\alpha dx_0dx_1dx_2+\iint_{\Omega_0}(\sigma_{\alpha 0}^+-\sigma_{\alpha 0}^-)\delta u_\alpha dx_1dx_2$$
$$-\frac{E_1}{1-\nu_1^2}\iint_{\Omega_u}(U_\beta-\bar{U}_\beta)n_\alpha\left\{\frac{1}{2}(1-\nu_1)(\delta u_{\alpha,\beta}+\delta u_{\beta,\alpha})+\nu_1\delta_{\alpha\beta}\delta u_{\gamma,\gamma}\right\}\cdot dx_0ds$$
$$-\iint_{\Omega_\sigma}[\bar{\sigma}_{n\beta}-\sigma_{\beta\alpha}n_\alpha]\delta u_\beta dx_0ds \tag{2.1}$$

对x_0积分，并注意到

$$\int_h\sigma_{\alpha0,0}\delta u_\alpha dx_0=\int_h(\sigma_{\alpha0}\delta u_\alpha)_{,0}dx_0=(\sigma_{\alpha0}^+-\sigma_{\alpha0}^-)\delta u_\alpha \tag{2.2}$$

同时引进薄膜张力分量$N_{\alpha\beta}$和$N_{(h\sigma)\alpha\beta}$

$$N_{\alpha\beta}=\int_h\sigma_{\alpha\beta}dx_0,\quad N_{(h_\sigma)\alpha\beta}=\int_{h_\sigma}\sigma_{\alpha\beta}dx_0 \tag{2.3a,b}$$

于是(2.1)可以简化为

$$\delta\Pi_{u_\alpha}=-\iint_{\Omega_0}(N_{\alpha\beta,\beta}+\int_h\bar{f}_\alpha\rho dx_0)\delta u_\alpha dx_1dx_2$$
$$-\frac{E_1}{1-\nu_1^2}\int_{s_u}\left[\int_{h_u}(U_\beta-\bar{U}_\beta)dx_0\right]n_\alpha\left\{\frac{1}{2}(1-\nu_1)(\delta u_{\alpha,\beta}+\delta u_{\beta,\alpha})\right.$$
$$\left.+\nu_1\delta_{\alpha\beta}\delta u_{\gamma,\gamma}\right\}ds-\int_{s_\sigma}\left\{\int_{h_\sigma}\bar{\sigma}_{n\alpha}dx_0-N_{(h_\sigma)\alpha\beta}n_\beta\right\}\delta u_\alpha ds \tag{2.4}$$

其中，有关x_σ的积分号和s_u,s_σ的定义分别见前文[1]的(1.15)，(1.16)，(1.17)式。式(2.4)的δu_α在Ω_0中和在s_σ上，$\delta u_{\alpha,\beta}$在边界s_u上都是独立变分。因此$\delta\Pi_{u\alpha}$的变分驻值条件给出

$$N_{\alpha\beta,\beta}+\int_h\bar{f}_\alpha\rho dx_0=0\qquad(\text{在}\Omega_0\text{中}) \tag{2.5}$$

$$\int_{h_u}(U_\alpha-\bar{U}_\alpha)dx_0=0\qquad(\text{在}s_u\text{中}) \tag{2.6}$$

$$\int_{h_\sigma}\bar{\sigma}_{n\alpha}dx_0-N_{(h_\sigma)\alpha\beta}n_\beta=0\qquad(\text{在}s_\sigma\text{中}) \tag{2.7}$$

下面研究有关δu_0各项的泛函变分，称$\delta\Pi$中涉及δu_0的部份为$\delta\Pi_{u_0}$，从(1.5)式，我们有

$$\delta\Pi_{u_0}=\iiint_\tau\{(\sigma_{\alpha j,j}+\bar{f}_\alpha\rho)x_0\delta u_{0,\alpha}-(\sigma_{0j,j}+\bar{f}_0\rho)\delta u_0\}dx_0dx_1dx_2$$
$$+\iint_{\Omega_0}\{(\sigma_{00}^+-\sigma_{00}^-)-(\bar{p}_+-\bar{p}_-)\}\delta u_0dx_1dx_2$$
$$-\iint_{\Omega_0}(\sigma_{\alpha0}^++\sigma_{\alpha0}^-)\delta u_{0,\alpha}\frac{h}{2}dx_1dx_2+\frac{E_1}{1-\nu_1^2}\iint_{\Omega_u}(U_\alpha-\bar{U}_\alpha)$$

$$\cdot[(1-\nu_1)\delta u_{0,\alpha\beta}+\nu_1\delta_{\alpha\beta}\delta u_{0,\gamma\gamma}]x_0 n_\beta ds dx_0+\iint_{\Omega_\sigma}\{(\bar{\sigma}_{n\alpha}-\sigma_{\alpha\beta}n_\beta)x_0\delta u_{0,\alpha}$$
$$-(\bar{\sigma}_{n0}-\sigma_{0\alpha}n_\alpha)\delta u_0\}ds dx_0 \qquad (2.8)$$

对x_0积分，并用格林公式进行简化，其结果为

$$\delta\Pi u_0=-\iint_{\Omega_0}\left\{M_{\alpha\beta,\alpha\beta}+\bar{p}_+-\bar{p}_-+\int_h\bar{f}_0\rho dx_0+\int_h\bar{f}_{\alpha,\alpha}\rho x_0 dx_0\right\}\delta u_0 dx_1 dx_2$$
$$+\int_{s_u}\left\{M_{(h_u)\alpha\beta,\beta}-Q_{(h_u)\alpha}+\int_{h_u}\bar{f}_\alpha\rho x_0 dx_0\right\}n_\alpha\delta u_0 ds$$
$$+\frac{E_1}{1-\nu_1^2}\int_{s_u}\left\{\int_{h_u}(U_\alpha-\bar{U}_\alpha)x_0 dx_0\right\}[(1-\nu_1)\delta u_{0,\alpha\beta}+\nu_1\delta_{\alpha\beta}\delta u_{0,\gamma\gamma}]n_\beta ds$$
$$+\int_{s_\sigma}\left\{\left[\int_{h_\sigma}\bar{\sigma}_{n\alpha}x_0 dx_0-M_{(h_\sigma)\alpha\beta}n_\beta\right]\delta u_{0,\alpha}-\left[\int_{h_\sigma}\bar{\sigma}_{n0}dx_0+(M_{(h_\sigma)\alpha\beta,\beta}\right.\right.$$
$$\left.\left.+\int_{h_\sigma}\bar{f}_\alpha\rho x_0 dx_0)n_\alpha\right]\delta u_0\right\}ds \qquad (2.9)$$

其中Q_α为横剪，$M_{\alpha\beta}$为弯矩，$Q_{(h_\sigma)\alpha}$，$M_{(h_\sigma)\alpha\beta}$为外力已给边界面($h_\sigma$)上的横剪和弯矩，$Q_{(h_u)\alpha}$，$M_{(h_u)\alpha\beta}$为位移已给边界面($h_u$)上的横剪和弯矩．它们分别是

$$Q_\alpha=\int_h\sigma_{0\alpha}dx_0,\quad M_{\alpha\beta}=\int_h\sigma_{\alpha\beta}x_0 dx_0 \qquad (2.10a,b)$$

$$Q_{(h_\sigma)\alpha}=\int_{h_\sigma}\sigma_{0\alpha}dx_0,\quad M_{(h_\sigma)\alpha\beta}=\int_{h_\sigma}\sigma_{\alpha\beta}x_0 dx_0 \qquad (2.10c,d)$$

$$Q_{(h_u)\alpha}=\int_{h_u}\sigma_{0\alpha}dx_0,\quad M_{(h_u)\alpha\beta}=\int_{h_u}\sigma_{\alpha\beta}x_0 dx_0 \qquad (2.10e,f)$$

δu_0，$\delta u_{0,\alpha}$，$\delta u_{0,\alpha\beta}$在有关各点上都是独立变分．所以，当$\delta\Pi u_0$是驻值时，(2.9)式给出

$$M_{\alpha\beta,\alpha\beta}+\bar{p}_+-\bar{p}_-+\int_h\bar{f}_0\rho dx_0+\int_h\bar{f}_{\alpha,\alpha}\rho x_0 dx_0=0 \quad (在\Omega_0中) \qquad (2.11)$$

$$n_\alpha\left[M_{(h_u)\alpha\beta,\beta}-Q_{(h_u)\alpha}+\int_{h_u}\bar{f}_\alpha\rho x_0 dx_0\right]=0 \quad (在s_u中) \qquad (2.12a)$$

$$\int_{h_\sigma}\bar{\sigma}_{n\alpha}x_0 dx_0-M_{(h_\sigma)\alpha\beta}n_\beta=0 \quad (在s_u中) \qquad (2.12b)$$

$$\int_{h_\sigma}\bar{\sigma}_{n\alpha}x_0 dx_0-M_{(h_\sigma)\alpha\beta}n_\beta=0 \quad (在s_\sigma中) \qquad (2.13a)$$

$$\int_{h_\sigma}\bar{\sigma}_{n0}dx_0-\left(M_{(h_\sigma)\alpha\beta,\beta}+\int_{h_\sigma}\bar{f}_\alpha\rho x_0 dx_0\right)n_\alpha=0 \quad (在s_\sigma中) \qquad (2.13b)$$

现在让我们研究涉及δA_{00}各项的变分泛函$\delta\Pi A_{00}$，把δU_j，$\delta\sigma_{ij}$中有关δA_{00}的各项代入(1.5)式，得

$$\delta\Pi A_{00}=\iiint_\tau\{(\sigma_{\alpha\beta,\beta}+\sigma_{\alpha0,0}+\bar{f}_\alpha\rho)\frac{1}{2}x_0^2\delta A_{00),\alpha}$$
$$-[\sigma_{0\alpha,\alpha}+\sigma_{00,0}+\bar{f}_0\rho]x_0\delta A_{00}\}dx_0 dx_1 dx_2$$
$$+\iint_{\Omega_0}\left\{[\sigma_{00}^+ +\sigma_{00}^- -\bar{p}_+ -\bar{p}_-]\frac{1}{2}h\delta A_{00}-\frac{1}{8}(\sigma_{\alpha0}^+ -\sigma_{\alpha0}^-)h^2\delta A_{00,\alpha}\right\}\cdot dx_1 dx_2$$

$$-\frac{1}{2}\frac{E_1}{1-\nu_1^2}\int_{s_u}\int_{h_u}(U_\alpha-\bar{U}_\alpha)x_0^2[(1-\nu_1)\delta A_{(0)},_{\alpha\beta}$$

$$+\nu_1\delta_{\alpha\beta}\delta A_{(0)},_{\gamma\gamma}]\,n_\beta dx_0ds_0+\int_{s_\sigma}\int_{h_\sigma}\Big\{[\bar{\sigma}_{n\alpha}-\sigma_{\alpha\beta}n_\beta)\frac{1}{2}\delta A_{(0)},_\alpha x_0^2$$

$$-[\bar{\sigma}_{n0}-\sigma_{0\alpha}n_\alpha]\,\delta A_{(0)}\,x_0\Big\}dx_0ds \qquad (2.14)$$

对x_0积分，然后利用格林公式，可简化得

$$\delta\Pi A_{(0)}=-\iint_{\Omega_0}\Big\{\Big[M^{(2)}_{\alpha\beta,\beta}-2H^{(0)}+(\bar{p}_++\bar{p}_-)\,h+2\int_h\bar{f}_0\rho x_0dx_0$$

$$+\int_h\bar{f}_{\alpha,\alpha}\rho x_0^2dx_0\Big\}\,\delta A_{(0)}\,dx_1dx_2+\int_{s_u}\frac{1}{2}\Big\{M^{(2)}_{(h_u)\alpha\beta,\beta}-2Q^{(1)}_{(h_u)\alpha}$$

$$+\int_{h_u}\bar{f}_\alpha\rho x_0^2\,dx_0\Big\}n_\alpha\delta A_{(0)}\,ds-\frac{1}{2}\frac{E_1}{1-\nu_1^2}\int_{s_u}\Big[\int_{h_u}(U_\alpha-\bar{U}_\alpha)x_0^2dx_0\Big]$$

$$\cdot[(1-\nu_1)\delta A_{(0)},_{\alpha\beta}+\nu_1\delta_{\alpha\beta}\delta A_{(0)},_{\gamma\gamma}]n_\beta ds$$

$$+\frac{1}{2}\int_{s_0}\Big\{\Big[\int_{h_\sigma}\bar{\sigma}_{n\alpha}x_0^2dx_0-M^{(2)}_{(h_\sigma)\alpha\beta}n_\beta\Big]\delta A_{(0)},_\alpha$$

$$-\Big[2\int_{h_\sigma}{}_{n0}x_0dx_0-\Big(M^{(2)}_{\alpha\beta,\beta}+\int_h\bar{f}_\alpha\rho x_0^2dx_0\Big)n_\alpha\Big]\delta A_{(0)}\Big\}ds \qquad (2.15)$$

其中

$$M^{(2)}_{\alpha\beta}=\int_h\sigma_{\alpha\beta}x_0^2\,dx_0,\quad Q^{(1)}_\alpha=\int_h\sigma_{0\alpha}x_0\,dx_0,\quad H^{(0)}=\int_h\sigma_{00}dx_0 \qquad (2.16\text{a,b,c})$$

$$M^{(2)}_{(h_u)\alpha\beta}=\int_{h_u}\sigma_{\alpha\beta}x_0^2\,dx_0,\quad Q^{(1)}_{(h_u)\alpha}=\int_{h_u}\sigma_{0\alpha}x_0dx_0 \qquad (2.16\text{d,e})$$

$$M^{(3)}_{(h_\sigma)\alpha\beta}=\int_{h_\sigma}\sigma_{\alpha\beta}x_0^2\,dx_0,\quad Q^{(1)}_{(h_\sigma)\alpha}=\int_{h_\sigma}\sigma_{0\alpha}x_0dx_0 \qquad (2.16\text{f,g})$$

$\delta\Pi A_{(0)}$ 的驻值条件给出

$$M^{(2)}_{(h_u)\alpha\beta,\alpha\beta}-2H^{(0)}+(\bar{p}_++\bar{p}_-)h+2\int_h\bar{f}_0\rho x_0\,dx_0+\int_h\bar{f}_{\alpha,\alpha}\rho x_0^2dx_0=0$$

$$(\text{在}\Omega_0\text{中}) \qquad (2.17)$$

$$n_\alpha\Big[M^{(2)}_{(h_u)\alpha\beta,\beta}-2Q^{(1)}_{(h_u)\alpha}+\int_{h_u}\bar{f}_\alpha\rho x_0^2dx_0\Big]=0 \qquad (\text{在}s_u\text{中}) \qquad (2.18\text{a})$$

$$\int_{h_u}(U_\alpha-\bar{U}_\alpha)x_0^2dx_0=0 \qquad (\text{在}s_u\text{中}) \qquad (2.18\text{b})$$

$$\int_{h_\sigma}\bar{\sigma}_{n\alpha}x_0^2dx_0-M^{(2)}_{(h_\sigma)\alpha\beta}n_\beta=0 \qquad (\text{在}s_\mu\text{中}) \qquad (2.19\text{a})$$

$$2\int_{h_\sigma}\bar{\sigma}_{n0}x_0dx_0-\Big(M^{(2)}_{(h_\sigma)\alpha\beta,\beta}+\int_{h_\sigma}\bar{f}_\alpha\rho x_0^2dx_0\Big)n_\alpha=0 \qquad (\text{在}s_\sigma\text{中}) \qquad (2.19\text{b})$$

其次，涉及$\delta A_{(1)}$的泛函变分为

$$\delta\Pi A_{(1)}=-\iiint_\tau\Big\{(\sigma_{0i,i}+\bar{f}_0\rho)\frac{1}{2}x_0^2\delta A_{(1)}-(\sigma_{\beta i,i}+\bar{f}_\beta\rho)\frac{1}{6}x_0^3\delta A_{(1)},_\beta\Big\}dx_0dx_1dx_2$$

$$+\iint_{\Omega_0}[\sigma^+_{00}-\sigma^-_{00}-\bar{p}_++\bar{p}_-]\frac{1}{8}h^2\delta A_{(1)}dx_1dx_2$$

$$-\iint_{\Omega_0}(\sigma^+_{0\alpha}+\sigma^-_{0\alpha})\frac{1}{48}h^2A_{(1),\alpha}dx_1dx_2$$

$$-\iint_{\Omega_u}\frac{E_1}{1-\nu_1^2}(U_\beta-\bar{U}_\beta)n_\alpha\Big\{\nu_1\delta_{\alpha\beta}x_0\delta A_{(1)}$$

$$-\frac{1}{6}[(1-\nu_1)\delta A_{(1),\alpha\beta}+\gamma_1\delta_{\alpha\beta}\delta A_{(1),\gamma\gamma}]x_0^3\Big\}dx_0ds$$

$$-\iint_{\Omega_\sigma}(\bar{\sigma}_{n0}-\sigma_{\alpha0}n_\alpha)\frac{1}{2}x_0^2\delta A_{(1)}dx_0ds$$

$$+\iint_{\Omega_0}(\bar{\sigma}_{n\beta}-\sigma_{\alpha\beta}n_\alpha)\frac{1}{6}x_0^3\delta A_{(1),\alpha}dx_0ds \qquad (2.20)$$

对x_0积分，并用格林公式进行简化，其结果为

$$\delta\Pi A_{(1)}=-\frac{1}{6}\iint_{\Omega_0}\Big\{M^{(3)}_{\alpha\beta,\alpha\beta}-6H^{(1)}+\frac{3}{4}h^2(\bar{p}_+-\bar{p}_-)+3\int_h\bar{f}_0\rho x_0^2dx_0$$

$$+\int_h(\bar{f}_{\beta,\beta})\rho x_0^3dx_0\Big\}\delta A_{(1)}dx_1dx_2-\frac{E_1}{1-\nu_1^2}$$

$$\int_{s_u}\Big\{\int_{h_u}(U_\beta-\bar{U}_\beta)x_0dx_0\Big\}\nu_1n_\alpha\delta_{\alpha\beta}\delta A_{(1)}ds-\frac{E_1}{6(1-\nu_1^2)}$$

$$\cdot\int_{s_u}\Big\{\int_{h_u}(U_\beta-\bar{U}_\beta)x_0^3dx_0\Big\}\{(1-\nu_1)\delta A_{(1),\beta\alpha}-\nu_1\delta_{\alpha\beta}\delta A_{(1),\gamma\gamma}\}ds$$

$$+\int_{s_u}\Big\{-\frac{1}{2}Q^{(2)}_{(h_u)\alpha}+\frac{1}{6}M^{(3)}_{(h_u)\alpha\beta,\beta}+\frac{1}{6}\int_{h_u}\bar{f}_\alpha\rho x_0^3dx_0\Big\}n_\alpha\delta A_{(1)}ds$$

$$-\int_{s_\sigma}\Big\{\frac{1}{2}\int_{h_\sigma}\bar{\sigma}_{n0}x_0^2dx_0-\frac{1}{6}\Big[M^{(3)}_{(h_\sigma)\alpha\beta,\alpha}+\int_{h_\sigma}\bar{f}_\beta\rho x_0^3dx_0\Big]n_\beta\delta A_{(1)}\Big\}ds$$

$$+\frac{1}{6}\int_{s_\sigma}\Big\{\int_{h_\sigma}\bar{\sigma}_{n\alpha}x_0^3dx_0-M^{(3)}_{(h_\sigma)\alpha\beta}n_\beta\Big\}\delta A_{(1),\alpha}ds \qquad (2.21)$$

于是，从$\delta\Pi A_{(1)}$的变分驻值条件求得

$$M^{(3)}_{\alpha\beta,\alpha\beta}-6H^{(1)}+\frac{3}{4}h^2(\bar{p}_+-\bar{p}_-)+3\int_h\bar{f}_0\rho x_0^2dx_0$$

$$+\int_h\bar{f}_{\beta,\beta}\rho x_0^3dx_0=0 \qquad (\text{在}\Omega_0\text{中}) \qquad (2.22)$$

$$n_\alpha\Big\{M^{(3)}_{(h_u)\alpha\beta,\beta}-3Q^{(2)}_{(h_u)\alpha}+\int_{h_u}\bar{f}_\alpha\rho x_0^3dx_0\Big\}=0 \qquad (\text{在}s_u\text{中}) \qquad (2.23a)$$

$$\int_{h_u}(U_\alpha-\bar{U}_\alpha)x_0^3dx_0=0 \qquad (\text{在}s_u\text{中}) \qquad (2.23b)$$

$$n_\alpha\int_{h_u}(U_\alpha-\bar{U}_\alpha)x_0dx_0=0 \qquad (\text{在}s_u\text{中}) \qquad (2.23c)$$

$$n_\alpha\Big\{\int_{h_\sigma}\bar{f}_\alpha\rho x_0^3dx_0+M^{(3)}_{(h_\sigma)\alpha\beta,\beta}-3Q^{(2)}_{(h_\sigma)\alpha}\Big\}=0 \qquad (\text{在}s_\sigma\text{中}) \qquad (2.24a)$$

$$3\int_{h_\sigma}\bar{\sigma}_{n0}x_0^2dx_0-n_\beta\left\{M^{(3)}_{(h_\sigma)\beta\alpha,\alpha}+\int_{h_\sigma}\bar{f}_\beta\rho x_0^3dx_0\right\}=0 \qquad (在s_\sigma中) \tag{2.24b}$$

$$\int_{h_\sigma}\bar{\sigma}_{n\alpha}x_0^3dx_0-n_\beta M^{(3)}_{(h_\sigma)\alpha\beta}=0 \qquad (在s_\sigma中) \tag{2.24c}$$

其中

$$M^{(3)}_{\alpha\beta}=\int_h\sigma_{\alpha\beta}x_0^3dx_0,\quad Q^{(2)}_\alpha=\int_h\sigma_{\alpha0}x_0^2dx_0,\quad H^{(1)}=\int\sigma_{00}x_0dx_0 \tag{2.25a,b,c}$$

$$M^{(3)}_{(h_u)\alpha\beta}=\int_{h_u}\sigma_{\alpha\beta}x_0^3dx_0,\quad Q^{(2)}_{(h_u)\alpha}=\int_{h_u}\sigma_{\alpha0}x_0^2dx_0 \tag{2.25d,e}$$

$$M^{(3)}_{(h_u)\alpha\beta}=\int_{h_\sigma}\sigma_{\alpha\beta}x_0^3dx_0 \tag{2.25f}$$

其次，有关$\delta S_{(2)\alpha}$的泛函变分为

$$\begin{aligned}\delta\Pi S_{(2)\alpha}=&-\iiint_\tau\left\{(\sigma_{\alpha\beta,\beta}+\sigma_{\alpha0,0}+\bar{f}_\alpha\rho)\delta S_{(2)\alpha}2\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0dx_0dx_1dx_2\right.\\&+\iint_{\Omega_0}(\sigma^+_{\alpha0}+\sigma^-_{\alpha0})\delta S_{(2)\alpha}\frac{1}{6}h^3dx_1dx_2\\&-\frac{E_1}{1-\nu_1^2}\iint_{\Omega_u}\{(U_\alpha-\bar{U}_\alpha)[(1-\nu_1)(\delta S_{(2)\alpha,\beta}+\delta S_{(2)\beta,\alpha})\\&+2\nu_1\delta_{\alpha\beta}S_{(2)\gamma,\gamma\beta}]n_\beta\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0\\&\left.+(U_0-\bar{U}_0)n_\alpha\delta S_{(2)\alpha}(1-\nu_1)\left(\frac{1}{4}h^2-x_0^2\right)\right\}dx_0ds\\&-\iint_{\Omega_\sigma}\{(\bar{\sigma}_{n\alpha}-\sigma_{\alpha\beta}n_\beta)\delta S_{(2)\alpha}\}2\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0dx_0ds\end{aligned} \tag{2.26}$$

对x_0积分，并引进内力素

$$\Sigma^{(3)}_{\alpha\beta}=\int_h\sigma_{\alpha\beta}2\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0dx_0 \tag{2.27a}$$

$$\chi^{(2)}_\alpha=\int_h\sigma_{0\alpha}2\left(\frac{1}{4}h^2-x_0^2\right)dx_0 \tag{2.27b}$$

$$\Sigma^{(3)}_{(h_\sigma)\alpha\beta}=\int_{h_\sigma}\sigma_{\alpha\beta}2\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0dx_0 \tag{2.27c}$$

(2.26)式可以写成

$$\begin{aligned}\delta\Pi S_{(2)\alpha}=&-\iint_{\Omega_0}\left\{\Sigma^{3}_{\alpha\beta,\beta}-\chi^{(2)}_\alpha+\int f_\alpha\rho2\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0dx_0\right\}dx_1dx_2\\&-\frac{E_1}{1-\nu_1^2}\int_{s_u}\left\{\int_{h_u}(U_\alpha-\bar{U}_\alpha)\left(\frac{1}{4}h^2-\frac{1}{2}x_0^2\right)x_0dx_0\right\}n_\beta\\&\cdot\{(1-\nu_1)(\delta S_{(2)\alpha,\beta}+\delta S_{(2)\beta,\alpha})+\nu_1\delta_{\alpha\beta}\delta S_{(2)\gamma,\gamma}\}n_\beta ds\\&-\frac{E}{1-\nu_1^2}\int_{s_u}\left\{\int_{h_u}(U_0-\bar{U}_0)\left(\frac{1}{4}h^2-x_0^2\right)dx_0\right\}n_\alpha\{(1-\nu_1)\delta S_{(2)\alpha}\}ds\end{aligned}$$

$$-\int_{s_\sigma}\left\{2\int_h\sigma_{\alpha n}\left(\frac{1}{4}h^2-x_0^2\right)x_0dx_0-\Sigma^{(3)}_{(h_\sigma)\alpha\beta}n_\beta\right\}\delta S_{(2)\alpha}ds \qquad (2.28)$$

于是，得到$\delta\Pi S_{2\alpha}$的驻值条件

$$\Sigma^{(3)}_{\alpha\beta,\beta}-\chi^{(2)}_\alpha+\int_h 2\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0\bar{f}_\alpha\rho dx_0=0 \qquad (\text{在}\Omega_0\text{中}) \qquad (2.29)$$

$$\int_{h_u}(U_\alpha-\bar{U}_\alpha)\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0dx_0=0 \qquad (\text{在}s_u\text{中}) \qquad (2.30a)$$

$$\int_{h_u}(U_0-\bar{U}_0)\left(\frac{1}{4}h^2-x_0^2\right)dx_0=0 \qquad (\text{在}s_u\text{中}) \qquad (2.30b)$$

$$2\int_{h_\sigma}\bar{\sigma}_{n\alpha}\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0dx_0-\Sigma^{(3)}_{(h_\sigma)\alpha\beta}n_\beta=0 \qquad (\text{在}s_\sigma\text{中}) \qquad (2.31)$$

最后，用相同的方法，研究有关$\delta S_{3\alpha}$的泛函变分，它是

$$\begin{aligned}\delta\Pi S_{3\alpha}=&-\iiint_\tau\left\{(\sigma_{\alpha\beta,\beta}+\sigma_{\alpha0,0}+\bar{f}_\alpha\rho)\delta S_{(3)\alpha}\left(\frac{1}{4}h^2-\frac{1}{2}x_0^2\right)x_0^2dx_0dx_1dx_2\right.\\&+\iint_{\Omega_0}\{(\sigma^+_{\alpha0}-\sigma^-_{\alpha0})\delta S_{3\alpha}\}\frac{1}{32}h^4dx_1dx_2\\&-\frac{E_1}{1-\nu_1^2}\iint_{\Omega_u}\left\{(U_\alpha-\bar{U}_\alpha)[(1-\nu_1)(\delta S_{3\alpha,\beta}+\delta S_{3\beta,\alpha})\right.\\&+2\nu_1\delta S_{(3)\gamma,\gamma}]n_\beta\left(\frac{1}{4}h^2-\frac{1}{2}x_0^2\right)x_0^2\\&\left.+(U_0-\bar{U}_0)n_\alpha\delta S_{(3)\alpha}(1-\nu_1)\left(\frac{1}{4}h^2-x_0^2\right)x_0\right\}dx_0ds\\&-\iint_{\Omega_0}(\bar{\sigma}_{n\alpha}-\sigma_{\alpha\beta}n_\beta)\delta S_{3\alpha}\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0^2dx_0ds\end{aligned} \qquad (2.32)$$

对x_0积分，得

$$\begin{aligned}\delta\Pi S_{3\alpha}=&-\iint_{\Omega_0}\left\{\Sigma^{(4)}_{\alpha\beta,\beta}-\chi^{(3)}_\alpha+\int_h\bar{f}_\alpha\rho\left(\frac{1}{4}h^2-\frac{1}{2}x_0^2\right)x_0^2dx_0\right\}\delta S_{(3)\alpha}dx_1dx_2\\&-\frac{E_1}{1-\nu_1^2}\int_{s_u}\left\{\int_{h_u}(U_\alpha-\bar{U}_\alpha)\left(\frac{1}{4}h^2-\frac{1}{2}x_0^2\right)x_0^2dx_0\right.\\&\left.\cdot[(1-\nu_1)(\delta S_{3\alpha,\beta}+\delta S_{3\beta,\alpha})+\nu_1\delta_{\alpha\beta}\delta S_{(3)\gamma,\gamma}]n_\beta\right\}ds\\&-\frac{E_1}{1-\nu_1^2}\int_{s_u}\left\{\int_{h_u}(U_0-\bar{U}_0)\left(\frac{1}{4}h^2-x_0^2\right)x_0dx_0[n_\alpha(1-\nu_1)\delta S_{(3)\alpha}]\right\}ds\\&-\int_{s_\sigma}\left\{\int_{h_\sigma}\bar{\sigma}_{n\alpha}\left(\frac{1}{4}h^2-\frac{1}{3}x_0^2\right)x_0^2dx_0-\Sigma^{(4)}_{(h_\sigma)\alpha\beta}n_\beta\right\}\delta S_{(3)\alpha}ds\end{aligned} \qquad (2.33)$$

其中

$$\Sigma^{(4)}_{\alpha\beta}=\int_h\sigma_{\alpha\beta}\left(\frac{1}{4}h^2-\frac{1}{2}x_0^2\right)x_0^2dx_0,\quad \chi^{(3)}_\alpha=\int_h\sigma_{\alpha0}\left(\frac{1}{4}h^2-x_0^2\right)x_0dx_0 \qquad (2.34a,b)$$

$$\Sigma^{(4)}_{(h_u)\alpha\beta}=\int_{h_\sigma}\sigma_{\alpha\beta}\left(\frac{1}{4}h^2-\frac{1}{2}x_0^2\right)x_0^2dx_0 \qquad (2.34c)$$

于是，泛函变分的驻值条件给出

$$\Sigma^{(4)}_{\alpha\beta,\beta}-\chi^{(3)}_{\alpha}+\int_{h}\bar{f}_{\alpha}\rho\left(\frac{1}{4}h^{2}-\frac{1}{2}x_{0}^{2}\right)x_{0}^{2}dx_{0}=0 \quad (\text{在}\Omega_{0}\text{中}) \qquad (2.35)$$

$$\int_{h_u}(U_{\alpha}-\bar{U}_{\alpha})\left(\frac{1}{4}h^{2}-\frac{1}{2}x_{0}^{2}\right)x_{0}^{2}dx_{0}=0 \quad (\text{在}s_u\text{中}) \qquad (2.36a)$$

$$\int_{h_u}(U_{0}-\bar{U}_{0})\left(\frac{1}{4}h^{2}-x_{0}^{2}\right)x_{0}dx_{0}=0 \quad (\text{在}s_u\text{中}) \qquad (2.36b)$$

$$\int_{h_\sigma}\bar{\sigma}_{n\alpha}\left(\frac{1}{4}h^{2}-\frac{1}{2}x_{0}^{2}\right)x_{0}^{2}dx_{0}-\Sigma^{(4)}_{(h_\sigma)\alpha\beta}n_{\beta}=0 \quad (\text{在}s_\sigma) \qquad (2.37)$$

从上面的讨论中，我们导出了求解u_i，$A_{(0)}$，$A_{(1)}$，$S_{(2)\alpha}$，$S_{(3)\alpha}$等9个待定函数的9个二维微分方程组：即 (2.5)，(2.11)，(2.17)，(2.22)，(2.29)，(2.35)．其余方程为近似边界条件，计有在s_u中必须满足的 (2.6)，(2.12a,b)，(2.18a,b)，(2.23a,b,c)，(2.30a,b)，(2.36a,b)等18个位移已给边界条件，和在s_σ中必需满足的(2.7)，(2.13a,b)，(2.19a,b)，(2.24a,b,c)，(2.31)，(2.37)等16个外力已给的边界条件．

三、用9个待定函数表达的微分方程和边界条件

把 (1.4a,b,c) 代入 (2.3a,b)，(2.10a,b,c,d,e,f)，(2.16a,b,c,d,e,f,g)，(2.25a,b,c,d,e,f)，(2.34a,b,c)，就可以求得用待定函数表达的诸内力素．它们是

$$\begin{aligned}N_{\alpha\beta}=B_1\Big\{&\frac{1}{2}(1-\nu_1)(u_{\alpha,\beta}+u_{\beta,\alpha})+\nu_1u_{\gamma,\gamma}\delta_{\alpha\beta}+\nu_1A_{(0)}\delta_{\alpha\beta}\\&-\frac{1}{24}h^2[(1-\nu_1)A_{(0),\alpha\beta}+\nu_1A_{(0),\gamma\gamma}\delta_{\alpha\beta}]\\&+\frac{7}{480}h^4\Big[\frac{1}{2}(1-\nu_1)(S_{(3)\alpha,\beta}+S_{(3)\beta,\alpha})+\nu_1S_{(3)\gamma,\gamma}\delta_{\alpha\beta}\Big]\Big\}\end{aligned} \qquad (3.1a)$$

$$\begin{aligned}M_{\alpha\beta}=D_1\{&-[(1-\nu_1)u_{0,\alpha\beta}+\nu_1u_{0,\gamma\gamma}\delta_{\alpha\beta}]+\nu_1A_{(1)}\delta_{\alpha\beta}\\&-\frac{1}{40}h^2[(1-\nu_1)A_{(1),\alpha\beta}+\nu_1A_{(1),\gamma\gamma}\delta_{\alpha\beta}]\\&+\frac{2}{5}h^2\Big[\frac{1}{2}(1-\nu_1)(S_{(2)\alpha,\beta}+S_{(2)\beta,\alpha})+\nu_1S_{(2)\gamma,\gamma}\delta_{\alpha\beta}\Big]\Big\}\end{aligned} \qquad (3.1b)$$

$$Q_{\alpha}=2D_1(1-\nu_1)S_{(2)\alpha} \qquad (3.1c)$$

$$\begin{aligned}M^{(2)}_{\alpha\beta}=D_1\Big\{&\frac{1}{2}[(1-\nu_1)(u_{\alpha,\beta}+u_{\beta,\alpha})+\nu_1u_{\gamma,\gamma}\delta_{\alpha\beta}+\nu_1A_{(0)}\delta_{\alpha\beta}\\&-\frac{3}{40}h^2[(1-\nu_1)A_{(0),\alpha\beta}+\nu_1A_{(0),\gamma\gamma}\delta_{\alpha\beta}]\\&+\frac{27}{1120}h^4\Big[\frac{1}{2}(1-\nu_1)(S_{(3)\alpha,\beta}+S_{(3)\beta,\alpha})+\nu_1S_{(3)\gamma,\gamma}\delta_{\alpha\beta}\Big]\Big\}\end{aligned} \qquad (3.1d)$$

$$Q^{(1)}_{\alpha}=\frac{2}{3}D_1(1-\nu_1)S_{(3)\alpha} \qquad (3.1e)$$

$$H^{(0)}=B_1\left\{\nu_1u_{\alpha,\alpha}+A_{(0)}-\frac{1}{24}\nu_1h^2\nabla^2A_{(0)}-\frac{7}{480}\nu_1h^4S_{(3)\alpha,\alpha}\right\} \tag{3.1f}$$

$$\begin{aligned}M_{\alpha\beta}^{(3)}=D_1^{(3)}\Big\{&-[(1-\nu_1)u_{0,\alpha\beta}+\nu_1u_{0,\gamma\gamma}\delta_{\alpha\beta}]+\nu_1A_{(1)}\delta_{\alpha\beta}\\&-\frac{5}{168}h^2[(1-\nu_1)A_{(1),\alpha\beta}+\nu_1A_{(1),\gamma\gamma}\delta_{\alpha\beta}]\\&+\frac{8}{21}h^2\left[\frac{1}{2}(1-\nu_1)\ (S_{(2)\alpha,\beta}+S_{(2)\beta,\alpha})+\nu_1S_{(2)\gamma,\gamma}\delta_{\alpha\beta}\right]\Big\}\end{aligned} \tag{3.1g}$$

$$Q_\alpha^{(2)}=\frac{2}{3}(1-\nu_1)D_1^{(3)}S_{(2)\alpha} \tag{3.1h}$$

$$H^{(1)}=D_1\left\{-\nu_1\nabla^2u_0+A_{(1)}-\frac{1}{40}\nu_1h^2\nabla^2A_{(1)}+\frac{2}{5}\nu_1h^2S_{(2)\alpha,\alpha}\right\} \tag{3.1i}$$

$$\begin{aligned}\Sigma_{\alpha\beta}^{(3)}=D_1^{(3)}\Big\{&-\frac{3}{8}[(1-\nu_1)u_{0,\alpha\beta}+\nu_1u_{0,\gamma\gamma}\delta_{\alpha\beta}]+\frac{3}{8}\nu_1A_{(1)}\delta_{\alpha\beta}\\&-\frac{4}{63}h^2[(1-\nu_1)A_{(1),\alpha\beta}+\nu_1A_{(1),\gamma\gamma}\delta_{\alpha\beta}]\\&+\frac{68}{63}h^2\left[\frac{1}{2}(1-\nu_1)\ (S_{(2)\alpha,\beta}+S_{(2)\beta,\alpha})+\nu_1S_{(2)\gamma,\gamma}\delta_{\alpha\beta}\right]\Big\}\end{aligned} \tag{3.1j}$$

$$\chi_\alpha^{(2)}=\frac{16}{3}(1-\nu_1)D_1^{(3)}S_{(2)\alpha} \tag{3.1k}$$

$$\begin{aligned}\Sigma_{\alpha\beta}^{(1)}=D_1^{(3)}\Big\{&\frac{7}{6}\left[\frac{1}{2}(1-\nu_1)\ (u_{\alpha,\beta}+u_{\beta,\alpha})+\nu_1u_{\gamma,\gamma}\delta_{\alpha\beta}\right]+\frac{7}{6}\nu_1A_{(0)}\delta_{\alpha\beta}\\&-\frac{9}{112}h^2[(1-\nu_1)A_{(0),\alpha\beta}+\nu_1A_{(0),\gamma\gamma}\delta_{\alpha\beta}]\\&+\frac{107}{4032}h^4\left[\frac{1}{2}(1-\nu_1)\ (S_{(3)\alpha,\beta}+S_{(3)\beta,\alpha})+\nu_1S_{(3),\gamma\gamma}\delta_{\alpha\beta}\right]\Big\}\end{aligned} \tag{3.1l}$$

$$\chi_\alpha^{(3)}=\frac{4}{21}D_1^{(3)}(1-\nu_1)h^2S_{(3)\alpha} \tag{3.1m}$$

其中　B_1，D_1，$D_1^{(3)}$分别为板的平面应变问题的抗拉刚度、抗弯刚度、和高阶抗弯刚度

$$B_1=\frac{E_1h}{1-\nu_1^2},\ D_1=\frac{E_1h^3}{12(1-\nu_1^2)},\ D_1^{(3)}=\frac{E_1h^5}{80(1-\nu_1^2)} \tag{3.2}$$

把上述30个内力素代入(2.5)，(2.11)，(2.17)，(2.22)，(2.29)，(2.35)，即得用u_i，$A_{(0)}$，$A_{(1)}$，$S_{(2)\alpha}$，$S_{(3)\alpha}$等9个场函数所表达的9个微分方程组．这里应该指出，所有这些场函数都是(x_1, x_2)的函数．所以，这些微分方程也都是(x_1, x_2)平面的偏微分方程．这些微分程组可以分成两组，第一组只含有u_1，u_2，$A_{(0)}$，$S_{(3)1}$，$S_{(3)2}$等五个待定场函数．它们是分别从(2.5)，(2.17)，(2.35)等五个方程导出的：

$$\begin{aligned}&\frac{1}{2}(1-\nu_1)\ u_{\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)u_{\beta,\alpha\beta}+\nu_1A_{(0),\alpha}-\frac{1}{24}h^2A_{(0),\beta\beta\alpha}\\&\quad+\frac{7}{480}h^4\left[\frac{1}{2}(1-\nu_1)S_{(3)\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)S_{(3)\beta,\alpha\beta}\right]+\frac{1}{B_1}\int_h f_\alpha\rho dx_0=0\end{aligned} \tag{3.3a}$$

$$\nu_1 u_{\alpha,\alpha}-\frac{1}{24}h^2\nabla^2 u_\alpha+A_{(0)}-\frac{1}{12}\nu_1 h^2\nabla^2 A_{(0)}+\frac{1}{320}h^4\nabla^4 A_{(0)}$$

$$-\frac{7}{480}\nu_1 h^4 S_{(3)\alpha,\alpha}-\frac{9}{8960}h^6\nabla^2 S_{(3)\alpha,\alpha}+\frac{1}{2B_1}\Big\{h(\bar{p}_++\bar{p}_-)$$

$$+2\int_h \bar{f}_0\rho x_0\,dx_0+\int_h \bar{f}_{\alpha,\alpha}\rho x_0^2 dx_0\Big\}=0 \tag{3.3b}$$

$$\frac{1}{2}(1-\nu_1)\nabla^2 u_\alpha+\frac{1}{2}(1+\nu_1)u_{\beta,\alpha\beta}+\nu_1 A_{(0),\alpha}-\frac{27}{392}h^2\nabla^2 A_{(0),\alpha}$$

$$+\frac{107}{4704}h^4\Big[\frac{1}{2}(1-\nu_1)\nabla^2 S_{(3)\alpha}+\frac{1}{2}(1+\nu_1)S_{(3)\beta,\alpha\beta}\Big]-\frac{12}{49}(1-\nu_1)S_{(3)\alpha}$$

$$+\frac{6}{7D_1 h^3}\int_{h_\sigma}\bar{f}_\alpha\rho\Big(\frac{1}{4}h^2-x_0^2\Big)x_0^2 dx_0=0 \tag{3.3c}$$

第二组只含有u_0，$A_{(1)}$，$S_{(2)1}$，$S_{(2)2}$等四个待定函数．它们是分别从(2.11)，(2.22)，(3.29)等四个方程导出的．

$$-\nabla^4 u_0+\nu_1\nabla^2 A_{(1)}-\frac{1}{40}h^2\nabla^4 A_{(1)}+\frac{2}{5}h^2\nabla^2 S_{(2)\alpha,\alpha}$$

$$+\frac{1}{D_1}\Big\{\bar{p}_+-\bar{p}_-+\int_h \bar{f}_0\rho dx_0+\int_h \bar{f}_{\alpha,\alpha}\rho x_0 dx_0\Big\}=0 \tag{3.4a}$$

$$\nu_1\nabla^2 u_0-\frac{1}{40}h^2\nabla^4 u_0-A_{(1)}+\frac{1}{20}\nu_1 h^2\nabla^2 A_{(1)}-\frac{1}{1344}h^4\nabla^4 A_{(1)}-\frac{2}{5}h^2 S_{(2)\alpha,\alpha}$$

$$+\frac{1}{105}h^4\nabla^2 S_{(2)\alpha,\alpha}+\frac{1}{6D_1}\Big\{\frac{3}{4}h^2(\bar{p}_+-\bar{p}_-)+3\int_h \bar{f}_0\rho x_0^2 dx_0+\int_h \bar{f}_{\alpha,\alpha}\rho x_0^3 dx_0\Big\}=0 \tag{3.4b}$$

$$-\nabla^2 u_{0,\alpha}+\nu_1 A_{(1),\alpha}-\frac{1}{42}h^2\nabla^2 A_{(1),\alpha}+\frac{17}{42}h^2\Big[\frac{1}{2}(1-\nu_1)\nabla^2 S_{(2)\alpha}$$

$$+\frac{1}{2}(1+\nu_1)S_{(2)\beta,\alpha\beta}\Big]-2(1-\nu_1)S_{(2)\alpha}$$

$$+\frac{3}{4D_1 h^3}\int_h\Big(\frac{1}{4}h^2-\frac{1}{3}x_0^2\Big)x_0\bar{f}_\alpha\rho dx_0=0 \tag{3.4c}$$

(3.3a，b)和前文[1]的(4.10)，(4.11)相比，在(3.3a,b)中增加了$S_{(3)\alpha}$各项．(3.4e，c)和前文[1]的(4.12)，(4.13)相比，在(3.4a，c)中增加了$A_{(1)}$各项．这就基本上证实了(3.3a，b，c)，(3.4a，b，c)为本题的二级近似微分方程．

在研究边界条件时，我们重点研究位移已知和外力已知的两种边界区域的交接线垂直于中面的情况（见前文[1]的图3）．在这种情况下，s_u，s_σ不重叠而且相接；同时h_u，h_σ都等于h，或

$$s_u+s_\sigma=s,\qquad \text{（整个中面的边长）} \tag{3.5a}$$

$$h_u=h_\sigma=h,\qquad \text{（板厚，并设等厚）} \tag{3.5b}$$

把(1.2a，b)(1.4a，b，c)和(3.1a，b，…l，m)代入(2.6)，(2.12a，b)，(2.18a，b)，(2.23a，b,c)，(2.30a，b)，(2.36a，b)，积分后，得u_0，u_α，$A_{(0)}$，$A_{(1)}$，$S_{(2)\alpha}$，$S_{(3)\alpha}$的18个位移已给边界条件(在s_u中)

$$u_\alpha-\frac{1}{24}h^2A_{(0),\alpha}-\frac{7}{480}h^4S_{(3)\alpha}-\frac{1}{h}\int\bar{U}_\alpha dx_0=0 \tag{3.6a}$$

$$n_\alpha\Big\{-\nabla^2u_{0,\alpha}+\nu_1A_{(1),\alpha}-\frac{1}{40}h^2\nabla^2A_{(1),\alpha}-2(1-\nu_1)S_{(2)\alpha}$$
$$+\frac{2}{5}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(2)\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)S_{(2)\beta,\alpha\beta}\Big]+\frac{1}{D_1}\int_h\bar{f}_\alpha\rho x_0dx_0\Big\}=0 \tag{3.6b}$$

$$-u_{0,\alpha}-\frac{1}{40}h^2A_{(1),\alpha}+\frac{2}{5}S_{(2)\alpha}h^2-\frac{12}{h^3}\int\bar{U}_\alpha x_0dx_0=0 \tag{3.6c}$$

$$n_\alpha\Big\{\frac{1}{2}(1-\nu_1)u_{\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)u_{\beta,\alpha\beta}+\nu_1A_{(0),\alpha}$$
$$-\frac{3}{40}h^2\nabla^2A_{(0),\alpha}-\frac{1}{5}h^2(1-\nu_1)S_{(3)\alpha}+\frac{27}{1120}h^4\Big[\frac{1}{2}(1-\nu_1)S_{(3)\alpha,\beta\beta}$$
$$+\frac{1}{2}(1+\nu_1)S_{(3)\beta,\alpha\beta}\Big]-\frac{1}{D_1}\int\bar{f}_\alpha x_0^2dx_0\Big\}=0 \tag{3.6d}$$

$$u_\alpha-\frac{3}{40}h^2A_{(0),\alpha}+\frac{27}{1120}h^4S_{(3)\alpha}-\frac{12}{h^3}\int\bar{U}_\alpha x_0^2dx_0=0 \tag{3.6e}$$

$$n_\alpha\Big\{-\nabla^2u_{0,\alpha}+\nu_1A_{(1),\alpha}-\frac{5}{168}h^2\nabla^2A_{(1),\alpha}-2(1-\nu_1)S_{(2)\alpha}$$
$$+\frac{8}{21}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(2)\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)S_{(2)\beta,\alpha\beta}\Big]$$
$$+\frac{1}{D_1^{(3)}}\int_h\bar{f}_\alpha\rho x_0^3dx_0\Big\}=0 \tag{3.6f}$$

$$-u_{0\alpha}-\frac{5}{168}h^2A_{(1),\alpha}+\frac{8}{21}h^2S_{(2)\alpha}-\frac{80}{h^5}\int\bar{U}_\alpha x_0^3dx_0=0 \tag{3.6g}$$

$$n_\alpha-u_{0\alpha}-\frac{1}{40}h^2A_{(1),\alpha}+\frac{2}{5}h^2S_{(2)\alpha}-\frac{12}{h^3}\int\bar{U}_\alpha x_0dx_0=0 \tag{3.6h}$$

$$-u_{0,\alpha}-\frac{1}{42}h^2A_{(1),\alpha}+\frac{17}{42}h^2S_{(2)\alpha}-\frac{60}{5}\int\bar{U}_\alpha\Big(\frac{1}{4}h^2-\frac{1}{3}x_0^2\Big)x_0dx_0=0 \tag{3.6i}$$

$$u_0+\frac{1}{40}A_{(1)}h^2-\frac{6}{h^3}\int_h\bar{U}_0\Big(\frac{1}{4}h^2-x_0^2\Big)dx_0=0 \tag{3.6j}$$

$$u_0-\frac{27}{392}h^2A_{(0),\alpha}-\frac{107}{9408}S_{(3)\alpha}h^4-\frac{480}{7h^5}\int_h\bar{U}_\alpha\Big(\frac{1}{4}h^2-\frac{1}{2}x_0^2\Big)x_0^2dx_0=0 \tag{3.6k}$$

$$A_{(0)}-\frac{120}{h^5}\int_h\bar{U}_0\Big(\frac{1}{4}h^2-x_0^2\Big)x_0dx_0=0 \tag{3.6l}$$

[以上各式在s_u中适用]

把（1.2a, b），（1.4a, b, c）和（3.1a, b,…, l, m）代入（2.7），（2.13a, b),(2.19a, b),（2.24a, k, c),（2.31),（2.37),积分后，得u_0, u_1, u_2, $A_{(0)}$, $A_{(1)}$, $S_{(2)\alpha}$, $S_{(3)\alpha}$, 的16个外力已给的边界条件（在S_σ中）：

$$n_\beta\Big\{\frac{1}{2}(1-\nu_1)u_{\alpha,\beta}+u_{\beta,\alpha})+\nu_1 u_{\gamma,\gamma}\delta_{\alpha\beta}+\nu_1 A_{(0)}\delta_{\alpha\beta}-\frac{1}{24}h^2[(1-\nu_1)A_{(0),\alpha\beta}$$

$$+\nu_1 A_{(0),\gamma\gamma}\delta_{\alpha\beta}]+\frac{7}{480}h^4\Big[\frac{1}{2}(1-\nu_1)(S_{(3)\alpha,\beta}+S_{(3)\beta,\alpha})+\nu_1 S_{(3)\gamma,\gamma}\delta_{\alpha\beta}\Big\}$$

$$-\frac{1}{B}\int\bar{\sigma}_{n\alpha}dx_0=0 \tag{3.7a}$$

$$n_\beta\Big\{-[(1-\nu_1)u_{0,\alpha\beta}+\nu_1 u_{0,\gamma\gamma}\delta_{\alpha\beta}]+\nu_1 A_{(1)}\delta_{\alpha\beta}-\frac{1}{40}h^2[(1-\nu_1)A_{(1),\alpha\beta}$$

$$+\nu_1 A_{(1),\gamma\gamma}\delta_{\alpha\beta}]+\frac{2}{5}h^2\Big[\frac{1}{2}(1-\nu_1)(S_{(2)\alpha,\beta}+S_{(2)\beta,\alpha})+\nu_1 S_{(2)\gamma,\gamma}\delta_{\alpha\beta}\Big]\Big\}$$

$$-\frac{1}{D_1}\int_h\bar{\sigma}_{n\alpha}x_0 dx_0=0 \tag{3.7b}$$

$$n_\beta\Big\{-\nabla^2 u_{0,\beta}+\nu_1 A_{(1),\beta}-\frac{1}{40}h^2\nabla^2 A_{(1),\beta}+\frac{2}{5}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(2)\beta,\alpha\alpha}$$

$$+\frac{1}{2}(1+\nu_1)S_{(2)\alpha,\beta\alpha}\Big]+\frac{1}{D_1}\int_h\bar{f}_\beta\rho x_0 dx_0\Big\}-\frac{1}{D_1}\int_h {}_{n\beta}x_0 dx_0 \tag{3.7c}$$

$$n_\beta\Big\{\frac{1}{2}(1-\nu_1)(u_{\alpha,\beta}+u_{\beta,\alpha})+\nu_1 u_{\gamma,\gamma}\delta_{\alpha\beta}+\nu_1 A_{(0)}\delta_{\alpha\beta}-\frac{3}{40}h^2[(1-\nu_1)A_{(0),\alpha\beta}$$

$$+\nu_1 A_{(0),\gamma\gamma}\delta_{\alpha\beta}]+\frac{27}{1120}h^4\Big[\frac{1}{2}(1-\nu_1)(S_{(3)\alpha,\beta}+S_{(3)\beta,\alpha})+\nu_1 S_{(3)\gamma,\gamma}\delta_{\alpha\beta}\Big]\Big\}$$

$$-\frac{1}{D_1}\int_h\bar{\sigma}_{n\alpha}x_0^2 dx_0=0 \tag{3.7d}$$

$$n_\beta\Big\{\frac{1}{2}(1-\nu_1)\Big(u_{\beta,\alpha\alpha}+\frac{1}{2}(1+\nu_1)u_{\alpha,\alpha\beta}+\nu_1 A_{(0),\beta}-\frac{3}{40}h^2\nabla^2 A_{(0),\beta}$$

$$+\frac{27}{1120}h^4\Big[\frac{1}{2}(1-\nu_1)S_{(3)\beta,\alpha\alpha}+\frac{1}{2}(1+\nu_1)S_{(3)\gamma,\gamma\beta}\Big]$$

$$+\frac{1}{D_1}\int_h\bar{f}_\beta\rho x_0^2 dx\Big\}-\frac{2}{D_1}\int_h {}_{n0}x_0 dx_0=0 \tag{3.7e}$$

$$n_\beta\Big\{-\nabla^2 u_{0,\beta}+\nu_1 A_{(1),\beta}-\frac{5}{168}h^2\nabla^2 A_{(1),\beta}+\frac{8}{21}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(2)\beta,\alpha\alpha}$$

$$+\frac{1}{2}(1+\nu_1)S_{(2)\alpha,\alpha\beta}\Big]-\frac{4}{3}(1-\nu_1)S_{(2)\beta}+\frac{1}{D_1^{(3)}}\int_h\bar{f}_\beta\rho x_0^3 dx_0\Big\}=0 \tag{3.7f}$$

$$n_\beta\Big\{-\nabla^2 u_{0,\beta}+\nu_1 A_{(0),\beta}-\frac{5}{158}h^2\nabla^2 A_{(1),\beta}+\frac{8}{21}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(2)\beta,\alpha\alpha}$$

$$+\frac{1}{2}(1+\nu_1)S_{(2)\alpha,\beta\alpha}\Big]+\frac{1}{D_1^{(3)}}\int_h\bar{f}_\beta\rho x_0^2 dx_0\Big\}$$

$$-\frac{3}{D_1^{(3)}}\int_h\bar{\sigma}_{n0}x_0^2 dx_0=0 \tag{3.7g}$$

$$n_\beta\Big\{-[(1-\nu_1)u_{0,\alpha\beta}+\nu_1 u_{0,\gamma\gamma}\delta_{\alpha\beta}]+\nu_1 A_{(1)}\delta_{\alpha\beta}-\frac{5}{168}h^2[(1-\nu_1)A_{(1),\alpha\beta}$$

$$+\nu_1 A_{(1),\gamma\gamma}\delta_{\alpha\beta}]+\frac{8}{21}h^2\Big[\frac{1}{2}(1-\nu_1)(S_{(2)\alpha,\beta}+S_{(2)\beta,\alpha})+\nu_1 S_{(2)\gamma,\gamma}\delta_{\alpha\beta}\Big]\Big\}$$

$$-\frac{1}{D_1^{(3)}}\int_h \bar{\sigma}_{n\alpha}x_0^3 dx_0=0 \tag{3.7h}$$

$$n_\beta\Big\{-\frac{8}{3}[(1-\nu_1)u_{0,\alpha\beta}+\nu_1 u_{0,\gamma\gamma}\delta_{\alpha\beta}]+\frac{8}{3}\nu_1 A_{(1)}\delta_{\alpha\beta}-\frac{4}{63}h^2[(1-\nu_1)A_{(2),\alpha\beta}$$

$$+\nu_1 A_{(1),\gamma\gamma}\delta_{\alpha\beta}]\Big\}-\frac{2}{D_1^{(3)}}\int_h \bar{\sigma}_{n\alpha}\Big(\frac{1}{4}h^2-\frac{1}{3}x_0^2\Big)x_0 dx_0 \tag{3.7i}$$

$$n_\beta\Big\{\frac{7}{6}\Big[\frac{1}{2}(1-\nu_1)(u_{\alpha,\beta}+u_{\beta,\alpha})+\nu_1 u_{\gamma,\gamma}\delta_{\alpha\beta}\Big]+\frac{7}{6}\nu_1 A_{(0)}\delta_{\alpha\beta}$$

$$-\frac{9}{112}h^2[(1-\nu_1)A_{(0)\alpha,\beta}+\nu_1 A_{(0),\gamma\gamma}\delta_{\alpha\beta}]$$

$$+\frac{107}{4032}h^4\Big[\frac{1}{2}(1-\nu_1)(S_{(3)\alpha,\beta}+S_{(3)\beta,\alpha})+\nu_1 S_{(3)\gamma,\gamma}\delta_{\alpha\beta}\Big]\Big\}$$

$$-\frac{1}{D_1^{(3)}}\int_{h_\sigma}\bar{\sigma}_{n\alpha}\Big(\frac{1}{4}h^2-\frac{1}{2}x_0^2\Big)x_0^2 dx_0=0 \tag{3.7j}$$

[以上各式在s_σ中适用]

到此为止，我们业已建立了u_0, u_1, u_2, $A_{(0)}$, $A_{(1)}$, $S_{(2)1}$, $S_{(2)2}$, $S_{(3)1}$, $S_{(3)2}$等9个待定(x_1, x_2)函数的9个微分程组(3.3a, b, c), (3.4a, b, c)。和相关的34个边界条件(3.6a, b, c,…, k, l) (3.7a, b, c,…, i, j)。这种边界条件只适用于位移已给区域和外力已给区域的边界交接线垂直于中面的情况。

在下面我们将在常见的特定边界条件下，简化这些复杂的边界条件。

四、在特定的常规边界条件下简化二级近似的边界条件(3.6)和(3.7)

现在我们将在固定边界、自由边界，铰支边界等三种常见的情况下，研究简化边界条件(3.6)和(3.7)。

(A) 固定边界中的边界条件

当边界面各点固定时，在边界面Ω_u上，

$$\bar{U}_1=0,\ \bar{U}_2=0,\ \bar{U}_0=0 \qquad \Big(-\frac{1}{2}h\leqslant x_0\leqslant\frac{1}{2}h\Big) \tag{4.1}$$

在同一边界区域内，外力未给，所以，只有和s_u有关的边界条件(3.6)式，才是有效的。在(3.6)式的18个边界条件中，涉及u_1, u_2, $A_{(0)}$, $S_{(3)1}$, $S_{(3)2}$者有(3.6a, d, e, k, l)等8个，涉及u_0, $A_{(1)}$, $S_{(2)1}$, $S_{(2)2}$者有(3.6b, c, f, g, h, i, j)10个。

把(4.1)代入涉及u_1, u_2, $A_{(0)}$, $S_{(3)2}$的边界条件(3.6a, d, e, k, l)，得

$$u_\alpha-\frac{1}{24}h^2 A_{(0),\alpha}-\frac{7}{480}h^4 S_{(3)\alpha}=0 \tag{4.2a}$$

$$u_\alpha-\frac{3}{40}h^2A_{(0),\alpha}+\frac{27}{1120}h^4S_{(3)\alpha}=0 \tag{4.2b}$$

$$u_\alpha-\frac{27}{392}h^2A_{(0),\alpha}-\frac{107}{9408}S_{(3)\alpha}h^4=0 \tag{4.2c}$$

$$A_{(0)}=0 \tag{4.2d}$$

$$\begin{aligned}n_\alpha\Big\{&\frac{1}{2}(1-\nu_1)u_{\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)u_{\beta,\alpha\beta}+\nu_1A_{(0),\alpha}-\frac{3}{40}h^2\nabla^2A_{(0),\alpha}\\&-\frac{1}{5}h^2(1-\nu_1)S_{(3)\alpha}+\frac{27}{1120}h^4\Big[\frac{1}{2}(1-\nu_1)S_{(3)\alpha,\beta\beta}\\&+\frac{1}{2}(1+\nu_1)S_{(3)\beta,\alpha\beta}\Big]-\frac{1}{D_1}\int_h\bar{f}_\alpha\rho x_0^2dx_0\Big\}=0\end{aligned} \tag{4.2e}$$

$$-u_{0,\alpha}-\frac{1}{40}A_{(1),\alpha}+\frac{2}{5}h^2S_{(2)\alpha}=0 \tag{4.2f}$$

$$-u_{0,\alpha}-\frac{1}{42}A_{(1),\alpha}+\frac{17}{42}h^2S_{(2)\alpha}=0 \tag{4.2g}$$

$$-u_{0,\alpha}-\frac{5}{168}h^2A_{(1),\alpha}+\frac{8}{21}h^2S_{(2)\alpha}=0 \tag{4.2h}$$

$$u_0+\frac{1}{40}h^2A_{(1)}=0 \tag{4.2i}$$

$$\begin{aligned}n_\alpha\Big\{&-\nabla^2u_{0,\alpha}+\nu_1A_{(1),\alpha}-\frac{5}{168}h^2\nabla^2A_{(1),\alpha}-2(1-\nu_1)S_{(2)\alpha}\\&+\frac{8}{21}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(2)\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)S_{(2)\beta,\alpha\beta}\Big]+\frac{1}{D_1^{(3)}}\int_h\bar{f}_\alpha\rho x_0^3dx_0\Big\}=0\end{aligned} \tag{4.2j}$$

$$\begin{aligned}n_\alpha\Big\{&-\nabla^2u_{0,\alpha}+\nu_1A_{(1),\alpha}-\frac{1}{40}h^2\nabla^2A_{(1),\alpha}-2(1-\nu_1)S_{(2)\alpha}\\&+\frac{2}{5}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(2)\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)S_{(2)\beta,\alpha\beta}\Big]+\frac{1}{D_1}\int_h\bar{f}_\alpha\rho x_0dx_0\Big\}=0\end{aligned} \tag{4.2k}$$

根据(4.2f)，条件(3.6h)业已满足.

从(4.2a, b, c, d)和(4.2f, g, h, i)，我们可以证明

$$\begin{aligned}&u_\alpha=A_{(0)}=A_{(0),\alpha}=S_{(3)\alpha}=0\\&u_{0,\alpha}=A_{(1),\alpha}=S_{(2)\alpha}=0\\&u_0+\frac{1}{40}h^2A_{(1)}=0\end{aligned} \tag{4.3}$$

在利用了(4.3)后，(4.2e, j, k)可以写成

$$\begin{aligned}n_\alpha\Big\{&\frac{1}{2}(1-\nu_1)u_{\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)u_{\beta,\alpha\beta}-\frac{3}{40}h^2\nabla^2A_{(0),\alpha}\\&+\frac{27}{1120}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(3)\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)S_{(3)\beta,\alpha\beta}\Big]\end{aligned}$$

$$u_\alpha-\frac{3}{40}h^2A_{(0),\alpha}+\frac{27}{1120}h^4S_{(3)\alpha}=0 \tag{4.2b}$$

$$u_\alpha-\frac{27}{392}h^2A_{(0),\alpha}-\frac{107}{9408}S_{(3)\alpha}h^4=0 \tag{4.2c}$$

$$A_{(0)}=0 \tag{4.2d}$$

$$n_\alpha\Big\{\frac{1}{2}(1-\nu_1)u_{\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)u_{\beta,\alpha\beta}+\nu_1A_{(0),\alpha}-\frac{3}{40}h^2\nabla^2A_{(0),\alpha}$$
$$-\frac{1}{5}h^2(1-\nu_1)S_{(3)\alpha}+\frac{27}{1120}h^4\Big[\frac{1}{2}(1-\nu_1)S_{(3)\alpha,\beta\beta}$$
$$+\frac{1}{2}(1+\nu_1)S_{(3)\beta,\alpha\beta}\Big]-\frac{1}{D_1}\int_h\bar{f}_\alpha\rho x_0^2dx_0\Big\}=0 \tag{4.2e}$$

$$-u_{0,\alpha}-\frac{1}{40}A_{(1),\alpha}+\frac{2}{5}h^2S_{(2)\alpha}=0 \tag{4.2f}$$

$$-u_{0,\alpha}-\frac{1}{42}A_{(1),\alpha}+\frac{17}{42}h^2S_{(2)\alpha}=0 \tag{4.2g}$$

$$-u_{0,\alpha}-\frac{5}{168}h^2A_{(1),\alpha}+\frac{8}{21}h^2S_{(2)\alpha}=0 \tag{4.2h}$$

$$u_0+\frac{1}{40}h^2A_{(1)}=0 \tag{4.2i}$$

$$n_\alpha\Big\{-\nabla^2u_{0,\alpha}+\nu_1A_{(1),\alpha}-\frac{5}{168}h^2\nabla^2A_{(1),\alpha}-2(1-\nu_1)S_{(2)\alpha}$$
$$+\frac{8}{21}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(2)\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)S_{(2)\beta,\alpha\beta}\Big]+\frac{1}{D_1^{(3)}}\int_h\bar{f}_\alpha\rho x_0^3dx_0\Big\}=0 \tag{4.2j}$$

$$n_\alpha\Big\{-\nabla^2u_{0,\alpha}+\nu_1A_{(1),\alpha}-\frac{1}{40}h^2\nabla^2A_{(1),\alpha}-2(1-\nu_1)S_{(2)\alpha}$$
$$+\frac{2}{5}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(2)\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)S_{(2)\beta,\alpha\beta}\Big]+\frac{1}{D_1}\int_h\bar{f}_\alpha\rho x_0dx_0\Big\}=0 \tag{4.2k}$$

根据(4.2f)，条件(3.6h)业已满足．

从(4.2a，b，c，d)和(4.2f，g，h，i)，我们可以证明

$$\begin{aligned}&u_\alpha=A_{(0)}=A_{(0),\alpha}=S_{(3)\alpha}=0\\&u_{0,\alpha}=A_{(1),\alpha}=S_{(2)\alpha}=0\\&u_0+\frac{1}{40}h^2A_{(1)}=0\end{aligned} \tag{4.3}$$

在利用了(4.3)后，(4.2e，j，k)可以写成

$$n_\alpha\Big\{\frac{1}{2}(1-\nu_1)u_{\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)u_{\beta,\alpha\beta}-\frac{3}{40}h^2\nabla^2A_{(0),\alpha}$$
$$+\frac{27}{1120}h^2\Big[\frac{1}{2}(1-\nu_1)S_{(3)\alpha,\beta\beta}+\frac{1}{2}(1+\nu_1)S_{(3)\beta,\alpha\beta}\Big]$$