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圆柱型正交各向异性圆形薄板的非线性非对称弯曲问题(Ⅱ)

黄家寅　秦圣立　夏云杰　许晓平　孔祥和

（曲阜师范大学物理系，山东曲阜 273165）

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摘　要

本文利用“正则摄动法”[2]研究文献[1]中由“两变量法”[3]所得到的关于W_{nm}，φ_{nm}，ν_{nm}和$\psi_{nm}(n=0,1,2,\cdots,N;\ m=0,\ 1,\ 2,\ \cdots,M)$的递推方程和递推边界条件的求解问题。求得了本问题的一致有效渐近解。最后作为实例，我们利用“混合摄动法”[4]研究了轴对称线性弯曲问题，并将所得结果与文献[5]中给出的精确解相比较，两者基本上是一致的。

关键词　正交各向异性圆板　全板　摄动法

一、挠度函数W(r，θ)和应力函数Φ(r，θ)的渐近解

由文献[1]中所给出的方程(3.12)，(3.13)和边界条件(3.14)～(3.20)，今后有时把它们简记为(I，…，…)，且方程中的$\delta_2\ll 1$可视为小参数，我们利用正则摄动法[2]求其外部解W_{00}和φ_{00}，为此我们将W_{00}和φ_{00}展为δ_2的幂级数，即

$$W_{00}=\sum_{i=0}^{p}\delta_2^iW_{00i} \tag{1.1}$$

$$\varphi_{00}=\sum_{j=0}^{p}\delta_2^j\varphi_{00j} \tag{1.2}$$

将(1.1)式和(1.2)式分别代入文献[1]中的(I，3.12)和(I，3.13)式及边界条件(I，3.14)～(I，3.20)，则得：

$$\begin{aligned}&\frac{\partial^2}{\partial r^2}\Big(\sum_{i=0}^{p}\delta_2^iW_{00i}\Big)\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\Big)\\&+\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(\sum_{i=0}^{p}\delta_2^iW_{00i}\Big)\frac{\partial^2}{\partial r^2}\Big(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\Big)\\&-2\frac{\partial}{\partial r}\Big(\frac{1}{r^2}\frac{\partial}{\partial\theta}\Big)\Big(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\Big)\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial\theta}\Big)\Big(\sum_{i=0}^{p}\delta_2^iW_{00i}\Big)\\&=-q(r,\ \theta)\end{aligned} \tag{1.3}$$

$$\frac{\partial^4}{\partial r^4}\left(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\right)+\delta_1\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\right)$$

$$+\delta_2\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{r^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\right)\right.$$

$$+\frac{\partial^2}{\partial r^2}\left(\sum_{i=0}^{p}\delta_2^iW_{00i}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\sum_{i=0}^{p}\delta_2^iW_{00i}\right)$$

$$\left.-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\sum_{h=0}^{p}\delta_2^iW_{00i}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\sum_{h=0}^{p}\delta_2^iW_{00i}\right)\right]$$

$$=0 \tag{1.4}$$

$$\left(\sum_{i=0}^{p}\delta_2^iW_{00i}\right)\Big|_{r=1}=0 \tag{1.5}$$

$$\left[\frac{\partial}{\partial r}\left(\sum_{i=0}^{p}\delta_2^iW_{00i}\right)\right]\Big|_{r=1}=0 \tag{1.6}$$

$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\right)\right]\Big|_{r=1}=0 \tag{1.7}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\right)\right]\Big|_{r=1}=0 \tag{1.8}$$

和

$$\left[\left(\frac{1}{r}\frac{\partial_4}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\right)\right] \tag{1.9}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\sum_{j=0}^{p}\delta_2^j\varphi_{00j}\right)\right] \tag{1.10}$$

$$\left(\sum_{i=0}^{p}\delta_2^iW_{00i}\right);\quad \frac{\partial}{\partial r}\left(\sum_{i=0}^{p}\delta_2^iW_{00i}\right) \tag{1.11}$$

在$r=0$处，(1.9)～(1.11)均为有限值。

由方程(1.3)和(1.4)以及边界条件(1.5)～(1.11)，比较δ_2的同幂次项的系数，则得以下的递推方程和边界条件：

$$\frac{\partial^2}{\partial r^2}W_{000}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{000}+\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{000}\frac{\partial^2}{\partial r^2}\varphi_{000}$$

$$-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{000}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{000}=-q(r,\ \theta) \tag{1.12}$$

$$\frac{\partial^4}{\partial r^4}\varphi_{000}+\delta_1\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{000}=0 \tag{1.13}$$

$$W_{000}|_{r=1}=0 \tag{1.14}$$

$$\left[\frac{\partial}{\partial r}W_{000}\right]\bigg|_{r=1}=0 \tag{1.15}$$

$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{000}\right]\bigg|_{r=1}=0 \tag{1.16}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{000}\right]\bigg|_{r=1}=0 \tag{1.17}$$

和

$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{000}\right] \tag{1.18}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{000}\right] \tag{1.19}$$

$$W_{000};\ \frac{\partial}{\partial r}W_{000} \tag{1.20}$$

在$r=0$处，(1.18)～(1.20)均为有限值。

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$$\sum_{\substack{i=0\\j=n-i}}^{n}\left(\frac{\partial^2}{\partial r^2}W_{00i}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00j}+\sum_{\substack{i=0\\j=n-i}}^{n}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00i}\frac{\partial^2}{\partial r^2}\varphi_{00j}$$
$$-2\sum_{\substack{i=0\\j=n-i}}^{n}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{00j}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{00j}=0 \tag{1.21}$$

$$\frac{\partial^4}{\partial r^4}\varphi_{00n}+\delta_1\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{00n}+\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00\,n-1}+\sum_{\substack{i=0\\j=n-i-1}}^{n}\frac{\partial^2}{\partial r^2}W_{00i}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00j}-\sum_{\substack{i=0\\j=n-i-1}}^{n}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{00i}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{00j}=0 \tag{1.22}$$

$$W_{00n}|_{r=1}=0 \tag{1.23}$$

$$\left[\frac{\partial}{\partial r}W_{00n}\right]\bigg|_{r=1}=0 \tag{1.24}$$

$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00n}\right]_{r=1}=0 \tag{1.25}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{00n}\right]\bigg|_{r=1}=0 \tag{1.26}$$

和
$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00n}\right] \tag{1.27}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{00n}\right] \tag{1.28}$$

$$W_{00n};\ \frac{\partial}{\partial r}W_{00n}, \tag{1.29}$$

在$r=0$处，(1.27)～(1.29)式均为有限值。

由方程(1.13)和边界条件(1.16)，(1.17)，(1.18)，(1.19)和(1.20)，以及φ_{000}的自然周期条件$\varphi_{000}(r,\ \theta)=\varphi_{000}(r,\ \theta+2\pi)$，应用傅里叶级数法，可解出$\varphi_{000}$为

$$\varphi_{000}=D(2r^{\sqrt{1+\delta_1}+1}-\sqrt{1+\delta_1}\,r^3)\cos\theta \tag{1.30}$$

其中 D为待定的常数，$\delta_1=E_\theta\left(\frac{1}{G}-2\frac{\mu_r}{E_r}\right)$.

将φ_{000}代入方程(1.12)，得关于W_{000}的偏微分方程：

$$A(r)\cos\theta\frac{\partial^2W_{000}}{\partial r^2}+B(r)\cos\theta\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{000}$$
$$+2A(r)\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{000}=-q(r,\ \theta) \tag{1.31}$$

其中

$$A(r)=2D\sqrt{1+\delta_1}\,(r^{\sqrt{1+\delta_1}-1}-r) \tag{1.32}$$
$$B(r)=2D\sqrt{1+\delta_1}\,((\sqrt{1+\delta_1}+1)r^{\sqrt{1+\delta_1}-1}-3r) \tag{1.33}$$

因为方程(1.31)的左端包含$\cos\theta$和$\sin\theta$项，我们将W_{000}和$q(r,\ \theta)$展为傅里叶级数，即

$$W_{000}=\sum_{n=0}^{\infty}X_n(r)\cos n\theta+\sum_{n=1}^{\infty}Y_n(r)\sin n\theta \tag{1.34}$$

$$-q(r,\ \theta)=\sum_{n=0}^{\infty}C_n(r)\cos n\theta+\sum_{n=1}^{\infty}E_n(r)\sin n\theta \tag{1.35}$$

将(1.34)和(1.35)式代入方程(1.31)，然后比较$\cos n\theta$，$\sin n\theta$的系数，则得以下常微分方程：

$$A\frac{d^2X_0}{dr^2}+\frac{B}{r}\frac{dX_0}{dr}=2C_1(r) \tag{1.36}$$

$$A\frac{d^2Y_1}{dr^2}+\frac{B}{r}\frac{dY_1}{dr}-\frac{B}{r^2}Y_1+2A\frac{d}{dr}\left(\frac{Y_1}{r}\right)=2E_2(r) \tag{1.37}$$

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$$A\frac{d^2X_n}{dr^2}+\frac{B}{r}\frac{dX_n}{dr}-\frac{n^2B}{r^2}X_n=C_{n-1}+C_{n+1} \tag{1.38}$$

$$A\frac{d^2Y_n}{dr^2}+\frac{B}{r}\frac{dY_n}{dr}-\frac{n^2B}{r^2}Y_n=E_{n-1}+E_{n+1} \tag{1.39}$$

由边界条件(1.14)，(1.15)，$(1.20)_{1,2}$，可得关于X_n，Y_n的边界条件，即

$$X_0|_{r=1}=0 \tag{1.40}$$

$$\left.\frac{\partial X_0}{\partial r}\right|_{r=1}=0 \tag{1.41}$$

$$X_0;\ \frac{\partial X_0}{\partial r}\text{在}r=0\text{处取有限值} \tag{1.42}$$

$$Y_1|_{r=1}=0 \tag{1.43}$$

$$\left.\frac{\partial Y_1}{\partial r}\right|_{r=1}=0 \tag{1.44}$$

$$Y_1;\ \frac{\partial Y_1}{\partial r}\text{在}r=0\text{处取有限值} \tag{1.45}$$

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$$X_n|_{r=1}=0 \tag{1.46}$$

$$\left.\frac{\partial X_n}{\partial r}\right|_{r=1}=0 \tag{1.47}$$

$$X_n;\ \frac{\partial X_n}{\partial r}\text{在}r=0\text{处取有限值} \tag{1.48}$$

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$$Y_n|_{r=1}=0 \tag{1.49}$$

$$\left.\frac{\partial Y_n}{\partial r}\right|_{r=1}=0 \tag{1.50}$$

$$Y_n;\ \frac{\partial Y_n}{\partial r}\text{在}r=0\text{处取有限值} \tag{1.51}$$

由方程(1.36)～(1.39)和边界条件(1.45)～(1.51)，应用幂级数法[6]或偿试法，(或数值解法）可解X_n，Y_n($n=0$，1，2，…)。

将X_n，Y_n代入(1.34)式，可得W_{000}，然后把W_{000}和φ_{000}代入(1.21)和(1.22)式（取$n=1$）则得关于W_{001}和φ_{001}的微分方程．由(1.23)～(1.29)式(取$n=1$)，得关于W_{001}和φ_{001}的边界条件.用类似的方法可解W_{001}和φ_{001}，直到求出W_{00n}和φ_{00n}($n=0$，1，2……p).将W_{00n}和φ_{00n}分别代入(1，1)式和(1，2)式，可得W_{00}和φ_{00}．

得到W_{00}和φ_{00}之后，把它们分别代入(I，3.21)式和(I，3.22)式，则得关于W_{10}和φ_{10}的类似方程．将W_{00}和φ_{00}分别代入(I，3.30)式和(I，3.31)式，可得关于W_{01}和φ_{01}的微分方程．由(I，3.23)～(I，3.29)式和(I，3.32)～(I，3.38)式所对应的关于W_{10}，φ_{10}和W_{01}，φ_{01}的边界条件．应用上述方法，可逐次求得W_{10}，φ_{10}和W_{01}和φ_{01}，直到W_{nm}和φ_{nm}($n=0$，1，2，…，N；$m=0$，1，2，…，M)．

将φ_{00}代入方程(I，3.49)，得到关于ν_{00}的偏微分方程：

$$\left(\left(\frac{\partial u}{\partial r}\right)^4+\frac{\delta_2}{\eta^4}\left(\frac{\partial u}{\partial \theta}\right)^4\right)\frac{\partial^4\nu_{00}}{\partial\xi^4}-\left[\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}+\frac{1}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)^2\frac{\partial^2}{\partial\eta^2}\varphi_{00}-2\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\frac{1}{\eta}\left(\frac{1}{\eta}\frac{\partial}{\partial\theta}\right)\varphi_{00}\right]\frac{\partial^2\nu_{00}}{\partial\xi^2}=0 \tag{1.52}$$

我们令(I，2.1)中的待定函数$u(r,\theta)$满足以下的方程：

$$\left(\frac{\partial u}{\partial r}\right)^4+\frac{\delta_2}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^4=\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}+\frac{1}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)^2\frac{\partial^2}{\partial\eta^2}\varphi_{00}-2\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial}{\partial\theta}\right)\varphi_{00} \tag{1.53}$$

方程(1.53)是关于$u(r,\theta)$的一阶非线性偏微分方程，满足自然周期条件$u(r,\theta)=u(r,\theta+2\pi)$，以及以下的边界条件：

$$u(r,\theta)|_{r=1}=0,\ u>0\qquad(0<r<1)\tag{1.54}$$

易知边值问题(1.53)～(1.54)的解存在，同时方程(1.52)化为：

$$\frac{\partial^4\nu_{00}}{\partial\xi^4}-\frac{\partial^2\nu_{00}}{\partial\xi^2}=0\tag{1.55}$$

由方程(1.55)和边界条件(I, 3.24)，可解得ν_{00},

$$\nu_{00}=C_0(\eta,\theta)\exp(-\xi)\tag{1.56}$$

其中　$c_0(\eta,\theta)$是待定的任意函数。

将ν_{00}，W_{00}代入方程(I, 3.50)得ψ_{00}的微分方程：

$$\left(\left(\frac{\partial u}{\partial r}\right)^4+\delta_1\frac{1}{\eta^2}\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial u}{\partial\theta}\right)^2+\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^4\right)\frac{\partial^4\psi_{00}}{\partial\xi^4}$$
$$=-\delta_2\left[\frac{1}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)^2\frac{\partial^2W_{00}}{\partial\eta^2}\frac{\partial^2\nu_{00}}{\partial\xi^2}-2\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial}{\partial\theta}W_{00}\frac{\partial^2\nu_{00}}{\partial\xi^2}\right.$$
$$\left.+\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\frac{\partial^2\nu_{00}}{\partial\xi^2}\right]\tag{1.57}$$

由方程(1.57)和边界条件(I, 3.43)（取$n=2$，$m=0$，且带有负下标的量均取零），可确定ψ_{00}，即

$$\psi_{00}=-\frac{1}{H}\left[\left(\frac{1}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)^2\frac{\partial^2W_{00}}{\partial\eta^2}\right)C_0(\eta,\theta)\right.$$
$$-\left(\frac{2}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial W_{00}}{\partial\theta}\right)C_0(\eta,\theta)$$
$$\left.+\left(\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\right)C_0(\eta,\theta)\right]\exp(-\xi)\tag{1.58}$$

其中，$H=\left(\frac{\partial u}{\partial r}\right)^4+\delta_1\frac{1}{\eta^2}\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial u}{\partial\theta}\right)^2+\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^4$.

将ν_{00}和φ_{00}，φ_{10}代入方程(I, 3.51)（取$n=1$，$m=0$，且带有负下标的量均取零），可得ν_{10}的微分方程，即

$$\left(\left(\frac{\partial u}{\partial r}\right)^4+\frac{\delta_2}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^4\right)\frac{\partial^4\nu_{10}}{\partial\xi^4}-\left[\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}\right.$$
$$\left.+\frac{1}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)^2\frac{\partial^2}{\partial\eta^2}\varphi_{00}-2\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial}{\partial\theta}\varphi_{00}\right)\right]\frac{\partial^2\nu_{10}}{\partial\xi^2}$$
$$=\left\{\left[2\frac{1}{\eta}\left(\frac{\partial u}{\partial\theta}\right)\left(\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial}{\partial\theta}\varphi_{00}\right)-2\left(\frac{\partial u}{\partial r}\right)\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right.\right.\right.$$
$$\left.\left.+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}\right]\frac{\partial C_0}{\partial\eta}+\left[\frac{4\delta_2}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^3-\frac{2}{\eta}\left(\frac{\partial u}{\partial\theta}\right)\frac{\partial^2\varphi_{00}}{\partial\eta^2}\right.$$
$$\left.+\frac{2}{\eta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial\varphi_{00}}{\partial\theta}\right)\right]\frac{\partial C_0}{\partial\theta}+\left[\frac{2\delta_2}{\eta^3}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)^2\right.$$
$$+\frac{2\delta_2}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^2\left(\frac{\partial^2u}{\partial\theta^2}\right)+\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{10}$$

$$+\frac{1}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)^2\frac{\partial^2\varphi_{10}}{\partial\eta^2}-\left(\frac{\partial^2 u}{\partial r^2}\right)\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}$$

$$-\left(\frac{1}{\eta}\frac{\partial u}{\partial r}+\frac{1}{\eta^2}\left(\frac{\partial^2 u}{\partial\theta^2}\right)\right)\frac{\partial^2\varphi_{00}}{\partial\eta^2}$$

$$-2\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\left(\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial\varphi_{10}}{\partial\theta}\right)\Bigg]C_0\Bigg\}\exp(-\xi) \tag{1.59}$$

令方程(1.59)的右端关于 $\exp(-\xi)$ 的系数等于零，则得关于 $C_0(\eta,\ \theta)$ 的一阶线性偏微分方程：

$$\left[\frac{4\delta_2}{\eta^4}\left(\frac{\partial u}{\partial r}\right)^3-\frac{2}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)\frac{\partial^2\varphi_{00}}{\partial\eta^2}+\frac{2}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial\varphi_{00}}{\partial\theta}\right)\right]\frac{\partial C_0}{\partial\theta}$$

$$+\left[2\frac{1}{\eta}\left(\frac{\partial u}{\partial\theta}\right)\left(\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial\varphi_{00}}{\partial\theta}\right)-2\left(\frac{\partial u}{\partial r}\right)\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right.\right.$$

$$\left.\left.+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}\right]\frac{\partial C_0}{\partial\eta}+\left[\frac{2\delta_2}{\eta^8}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)^2+\frac{2\delta_2}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^2\left(\frac{\partial^2 u}{\partial\theta^2}\right)\right.$$

$$+\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{10}+\frac{1}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)^2\frac{\partial^2\varphi_{10}}{\partial\eta^2}$$

$$-\left(\frac{\partial^2 u}{\partial r^2}\right)\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}-\left(\frac{1}{\eta}\frac{\partial u}{\partial r}+\frac{1}{\eta^2}\left(\frac{\partial^2 u}{\partial\theta^2}\right)\right)\frac{\partial^2\varphi_{00}}{\partial\eta^2}$$

$$\left.-2\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\left(\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial\varphi_{10}}{\partial\theta}\right)\right]C_0$$

$$=0 \tag{1.60}$$

将 W_{10} 和 ν_{00} 代入边界条件(Ⅰ, 3.24)，则得 $C_0(\eta,\ \theta)$ 的边界条件

$$A_0\nu_{00}\Big|_{\substack{\eta=1\\ \xi=0}}=-\frac{\partial W_{10}}{\partial r}\Big|_{r=1} \tag{1.61}$$

或者

$$C_0(\eta,\ \theta)\Big|_{\eta=1}=\frac{1}{u_{,r}}\frac{\partial W_{10}}{\partial r}\Big|_{r=1} \tag{1.62}$$

且 $C_0(\eta,\ \theta)$ 满足自然周期条件：$C_0(\eta,\ \theta)=C_0(\eta,\ \theta+2\pi)$，由此可确定 $C_0(\eta,\ \theta)$。同时方程(1.59)简化为：

$$\left[\left(\frac{\partial u}{\partial r}\right)^4+\frac{\delta_2}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^4\right]\frac{\partial^4\nu_{10}}{\partial\xi^4}$$

$$-\left[\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}+\frac{1}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)^2\frac{\partial^2}{\partial\eta^2}\varphi_{00}\right.$$

$$\left.-2\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial}{\partial\theta}\right)\varphi_{00}\right]\frac{\partial^2\nu_{10}}{\partial\xi^2}=0 \tag{1.63}$$

利用(1.53)式，由方程(1.63)式可得边界层型函数 ν_{10}，即

$$\nu_{10}=C_1(\eta,\ \theta)\exp[-\xi] \tag{1.64}$$

其中 $C_1(\eta,\ \theta)$ 是待定的任意函数。

将 ψ_{00}，ν_{00}，ν_{10}，W_{00}，W_{10} 代入方程（Ⅰ, 3.52）（取 $n=1$，$m=0$，且带有负下标的量均为零），则得关于 ψ_{10} 的微分方程，然后积分可得关于 ψ_{10} 的边界层型的函数。利用求 $C_0(\eta,\ \theta)$ 相类似的方法，可求出 $C_1(\eta,\ \theta)$。这样可逐次得 ν_{nm}，$\psi_{nm}(n=0,\ 1,\ 2,\cdots,\ N;\ m=0,\ 1,$

$2, \cdots, M$).

把以上所求得的 W_{nm}, φ_{nm}, ν_{nm}, ψ_{nm} $(n=0, 1, 2, \cdots, N; m=0, 1, 2, \cdots, M)$ 分别代入(I,3.1)和(I, 3.2)式，便得到边值问题(I, 1.3)～(I, 1.4)的关于ε_1为N阶，而关于ε_2为M阶的一致有效渐近解.

二、算　例

以上我们得到了圆柱型正交各向异性圆板的非线性非对称弯曲问题的一致有效渐近解．以下我们将讨论一种特殊情况，并把所得结果与文献[5]中给出的精确解进行比较.

考察折算刚度按半径线性变化的非均匀圆柱正交各向异性圆板在均匀载荷作用下的弯曲问题．在周边固定夹支的条件下，其挠度函数W的方程和边界条件[5]为：

$$r^2(1+Ar)W'''+r(1+2Ar)W''+(Br-1)W'=Qr^3 \tag{2.1}$$

其中，

$$A=\frac{\Delta Q_{11}}{Q_{11}^0\delta},\ B=\frac{\Delta Q_{12}-\Delta Q_{22}}{Q_{11}^0\delta},$$

$$Q=\frac{q}{2HQ_{11}^0},\ \Delta Q_{11}=Q_{11}-Q_{11}^0,$$

$$\Delta Q_{22}=Q_{22}-Q_{22}^0,\ \Delta Q_{12}=Q_{12}-Q_{12}^0,$$

$$Q_{11}=\frac{E_r}{1-\nu_{r\theta}\nu_{\theta r}},\ Q_{11}^0=\frac{E}{1-\nu^2},$$

$$Q_{22}=\frac{E_\theta}{1-\nu_{r\theta}\nu_{\theta r}},\ Q_{22}^0=\frac{E}{1-\nu^2},$$

$$Q_{12}=\frac{\nu_{r\theta}E_\theta}{1-\nu_{r\theta}\nu_{\theta r}}=\frac{\nu_{\theta r}E_r}{1-\nu_{r\theta}\nu_{\theta r}},$$

$$Q_{12}^0=\frac{E}{1-\nu^2},\ D_r=Q_{11}\frac{h^3}{12}=Q_{11}H.$$

D_r是径向抗弯刚度；Q_{11}，Q_{22}，Q_{12}分别是直线正交各向异性区域的折算刚度；Q_{11}^0，Q_{22}^0，Q_{12}^0分别是中心处各向同性区的折算刚度；E_r，E_θ分别是直线正交各向异性区的径向和环向杨氏模量；$\nu_{r\theta}$，$\nu_{\theta r}$分别是直线正交异性区的径向和环向的泊松比；E，ν分别是中心处各向同性材料的杨氏模量和泊松比；h是板的厚度；δ是过渡区的核半径.

对于周边固定夹支的情况，其边界条件可写为：

$$\left.\begin{aligned}&W|_{r=C}=0\\&\left.\frac{dW}{dr}\right|_{r=C}=0\\&\left.\frac{dW}{dr}\right|_{r=0}=0\end{aligned}\right\} \tag{2.2}$$

其中　C是圆板半径.

若取$C=100\text{cm}$，$h=0.1\text{cm}$，

$E_r=0.78\times10^5\times9.8\text{N/cm}^2$，$E_\theta=3.8\times10^5\times9.8\text{N/cm}^2$，

$\nu_{\theta r}=0.31$，　　　　$\nu_{r\theta}=0.064$，

$$E=7.455\times10^5\times9.8\mathrm{N/cm^2},\quad \nu=0.2,$$

按以上所给数据，知

$$A=\frac{\Delta Q_{11}}{Q^0_{11}\delta}=-0.009\mathrm{cm^{-1}}$$

$$B=\frac{\Delta Q_{12}-\Delta Q_{22}}{Q^0_{11}\delta}=-0.00467\mathrm{cm^{-1}}.$$

为简便起见，我们只讨论全核的情况，即其中的$\delta=100$cm. 在此情况下，

$$|A|=0.009=\varepsilon_1\ll1,$$

$$|B|=0.00467=\varepsilon_2\ll1,$$

此例属于$A<0$，$-1<Ar<0$的情况，因此方程(2.1)可改写为：

$$r^2(1-\varepsilon_1r)W'''+r(1-2\varepsilon_1r)W'' -(\varepsilon_2r+1)W'=Qr^3 \tag{2.3}$$

或者，

$$r^2W'''+rW''-W'-\varepsilon_1r^3W'''-2\varepsilon_1r^2W'' -\varepsilon_2rW'=Qr^3 \tag{2.4}$$

为了同文献[5]所给出的精确解作比较，现在我们利用混合摄动法[4]解同一问题.为此，我们将W展开成关于ε_1为N阶，关于ε_2为M阶的幂级数：

$$W(r,\ \varepsilon_1,\ \varepsilon_2)=\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij} \tag{2.5}$$

将(2.5)式分别代入方程(2.4)和边界条件(2.2)，则得：

$$\begin{aligned}
&r\frac{d^3}{dr^3}\Big(\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij}\Big)\\
&+r\frac{d^2}{dr^2}\Big(\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij}\Big)\\
&-\frac{d}{dr}\Big(\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij}\Big)\\
&-\varepsilon_1r^3\frac{d^3}{dr^3}\Big(\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij}\Big)\\
&-2\varepsilon_1r^2\frac{d^2}{dr^2}\Big(\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij}\Big)\\
&-\varepsilon_2r\frac{d}{dr}\Big(\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij}\Big)\\
&=Qr^3
\end{aligned} \tag{2.6}$$

$$\left.\left\{\begin{aligned}&\left[\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij}\right]\Bigg|_{r=C}=0\\&\left[\frac{d}{dr}\left(\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij}\right)\right]\Bigg|_{r=C}=0\\&\left[\frac{d}{dr}\left(\sum_{i=0}^{N}\sum_{j=0}^{M}\varepsilon_1^i\varepsilon_2^jW_{ij}\right)\right]\Bigg|_{r=0}=0\end{aligned}\right.\right\}\tag{2.7}$$

由方程(2.6)和边界条件(2.7)，分别比较 $\varepsilon_1\varepsilon_2$ 的同幂次的系数，使其分别相等，则得递推方程和递推边界条件.

$$r^2\frac{d^3W_{00}}{dr^3}+r\frac{d^2W_{00}}{dr^2}-\frac{dW_{00}}{dr}=Qr^3\tag{2.8}$$

…………

$$\begin{aligned}&r^2\frac{d^3W_{ij}}{dr^3}+r\frac{d^2W_{ij}}{dr^2}-\frac{dW_{ij}}{dr}\\&\quad=r^3\frac{d^3W_{(i-1)j}}{dr^3}+2r^2\frac{d^2W_{(i-1)j}}{dr^2}\\&\qquad+r\frac{d}{dr}W_{i(j-1)}\end{aligned}\tag{2.9}$$

$$\left.\begin{aligned}&W_{00}|_{r=C}=0\\&\frac{dW_{00}}{dr}\bigg|_{r=C}=0\\&\frac{dW_{00}}{dr}\bigg|_{r=0}=0\end{aligned}\right\}\tag{2.10}$$

…………

$$\left.\begin{aligned}&W_{ij}|_{r=C}=0\\&\frac{dW_{ij}}{dr}\bigg|_{r=C}=0\\&\frac{dW_{ij}}{dr}\bigg|_{r=0}=0\end{aligned}\right\}\tag{2.11}$$

由方程(2.8)可得W_{00}的解为：

$$W_{00}=\frac{1}{32}Qr^4+\frac{C_{100}}{4}r^2+C_{200}\ln r+C_{300}\tag{2.12}$$

利用边界条件(2.10)可确定积分常数，

$$C_{200}=0,\quad C_{100}=-\frac{Q}{4}C^2,\quad C_{300}=\frac{Q}{32}C^4,$$

由此得，

$$W_{00}=\frac{Q}{32}r^4-\frac{Q}{16}C^2r^2+\frac{Q}{32}C^4\tag{2.13}$$

得到W_{00}之后，把W_{00}代入方程(2.9)（取$i=1$；$j=0$），得关于W_{10}的微分方程：

$$r^2\frac{d^3}{dr^3}W_{10}+r\frac{d^2}{dr^2}W_{10}-\frac{dW_{10}}{dr}$$

$$=r^3\frac{d^3}{dr^3}W_{00}+2r^2\frac{d^2}{dr^2}W_{00} \tag{2.14}$$

由(2.11)式（取$i=1$；$j=0$），得关于W_{10}的边界条件：

$$\left.\begin{aligned}&W_{10}|_{r=C}=0\\&\left.\frac{dW_{10}}{dr}\right|_{r=C}=0\\&\left.\frac{dW_{10}}{dr}\right|_{r=0}=0\end{aligned}\right\} \tag{2.15}$$

由方程(2.14)，并把W_{00}代入其中，并积分，得：

$$W_{10}=\frac{Q}{50}r^5+\frac{1}{9}C_{100}r^3+\frac{1}{4}C_{110}r^2$$
$$+C_{210}\ln r+C_{310}. \tag{2.16}$$

由边界条件(2.15)可确定积分常数：

$$C_{210}=0,\ C_{110}=-\frac{Q}{30}C^3,\ C_{310}=\frac{29}{1800}QC^5.$$

由此得：

$$W_{10}=\frac{Q}{50}r^5-\frac{Q}{36}C^2r^3-\frac{Q}{120}C^3r^2+\frac{29}{1800}QC^5 \tag{2.17}$$

将W_{00}代入(2.9)式（取$i=0$，$j=1$），并注意具有负下标的量取为零．则得：

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dW_{01}}{dr}\right)\right]=\frac{1}{8}Qr^2+\frac{1}{2}C_{10} \tag{2.18}$$

由(2.11)（取$i=0$，$j=1$），得关于W_{01}的边界条件：

$$\left.\begin{aligned}&W_{01}|_{r=C}=0\\&\left.\frac{dW_{01}}{dr}\right|_{r=C}=0\\&\left.\frac{dW_{01}}{dr}\right|_{r=0}=0\end{aligned}\right\} \tag{2.19}$$

积分(2.18)式，并利用边界条件(2.19)，则得：

$$W_{01}=\frac{1}{600}Qr^5-\frac{1}{72}QC^2r^3+\frac{1}{60}QC^3r^2$$

$$-\frac{1}{225}QC^5 \tag{2.20}$$

利用上述类似的方法可逐次求出 $W_{ij}(i=0,\ 1,\ 2,\cdots,N;\ j=0,\ 1,\ 2,\cdots,M)$．我们限于求到$\varepsilon_1^0\varepsilon_2^4$阶的项，则得$W(r,\ \varepsilon_1,\ \varepsilon_2)$的渐近解为：

$$\begin{aligned}W(r,\ \varepsilon_1,\ \varepsilon_2)=&W_{00}+\varepsilon_1W_{10}+\varepsilon_2W_{01}\\&+\varepsilon_1^2W_{20}+\varepsilon_1\varepsilon_2W_{11}+\varepsilon_2^2W_{02}+\varepsilon_1^3W_{30}\\&+\varepsilon_1^2\varepsilon_2W_{21}+\varepsilon_1\varepsilon_2^2W_{12}+\varepsilon_2^3W_{03}\\&+\varepsilon_1^4W_{40}+\varepsilon_1^3\varepsilon_2W_{31}+\varepsilon_1^2\varepsilon_2^2W_{22}\\&+\varepsilon_1\varepsilon_2^3W_{13}+\varepsilon_2^4W_{04}+o(\varepsilon_i^4)\end{aligned}$$

$$
\begin{aligned}
&=\frac{Q}{32}r^4-\frac{Q}{16}C^2r^2+\frac{Q}{32}C^4 \\
&\quad+\varepsilon_1\left(\frac{Q}{50}r^5-\frac{Q}{36}C^2r^3-\frac{Q}{120}C^3r^2+\frac{29}{1800}QC^5\right) \\
&\quad+\varepsilon_2\left(\frac{Q}{600}r^5-\frac{Q}{72}C^2r^3+\frac{Q}{60}C^3r^2-\frac{Q}{225}C^5\right) \\
&\quad+\varepsilon_1^2\left(\frac{Q}{72}r^6-\frac{Q}{64}C^2r^4-\frac{Q}{270}C^3r^3\right. \\
&\quad\left.-\frac{7}{1440}QC^4r^2+\frac{89}{8640}QC^6\right) \\
&\quad+\varepsilon_1\varepsilon_2\left(\frac{Q}{540}r^6-\frac{Q}{96}C^2r^4+\frac{Q}{540}C^3r^3\right. \\
&\quad\left.+\frac{Q}{80}C^4r^2-\frac{5}{864}QC^6\right) \\
&\quad+\varepsilon_2^2\left(\frac{Q}{17280}r^6-\frac{Q}{768}C^2r^2+\frac{Q}{270}C^3r^3\right. \\
&\quad\left.-\frac{3}{760}QC^4r^2+\frac{977}{656640}QC^6\right) \\
&\quad+\varepsilon_1^3\left(\frac{Q}{98}r^7-\frac{Q}{100}C^2r^5-\frac{Q}{480}C^4r^3\right. \\
&\quad\left.-\frac{5}{1512}QC^5r^2+\frac{6401}{151200}QC^7\right) \\
&\quad+\varepsilon_1^2\varepsilon_2\left(\frac{Q}{588}r^7-\frac{3}{400}QC^2r^5+\frac{Q}{1440}C^3r^4\right. \\
&\quad\left.+\frac{29}{6480}C^4r^3+\frac{71}{15120}QC^5r^2-\frac{12911}{3175200}QC^7\right) \\
&\quad+\varepsilon_1\varepsilon_2^2\left(\frac{31}{352800}Qr^7-\frac{Q}{720}C^2r^5+\frac{13}{5760}QC^3r^4\right. \\
&\quad\left.+\frac{13}{6840}QC^4r^3-\frac{2011}{478800}QC^5r^2+\frac{1441}{1072512}QC^7\right) \\
&\quad+\varepsilon_2^3\left(\frac{Q}{705600}r^7-\frac{Q}{29400}C^2r^5+\frac{Q}{2880}C^3r^4-\frac{Q}{1140}C^4r^3\right. \\
&\quad\left.+\frac{6269}{8937600}QC^5r^2-\frac{75701}{545193600}QC^7\right) \\
&\quad+\varepsilon_1^4\left(\frac{Q}{128}r^8-\frac{Q}{240}C^2r^6-\frac{Q}{750}C^3r^5-\frac{7Q}{5760}C^4r^4\right. \\
&\quad\left.-\frac{5Q}{3402}C^5r^3-\frac{9781Q}{907200}C^6r^2+\frac{18973}{1701000}QC^8\right) \\
&\quad+\varepsilon_1^3\varepsilon_2\left(\frac{Q}{672}r^8-\frac{Q}{180}C^2r^6+\frac{Q}{3000}C^3r^5\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{79Q}{34560}C^4r^4+\frac{23Q}{17010}C^5r^3+\frac{4477Q}{2154600}C^6r^2 \\
&-\frac{18446139}{9307872000}QC^8\Big) \\
&+\varepsilon_1^2\varepsilon_2^2\Big(\frac{6157}{51609600}Qr^8-\frac{73Q}{103680}C^2r^6+\frac{121Q}{108000}C^3r^5 \\
&+\frac{2657Q}{1969920}C^4r^4+\frac{71Q}{86040}C^5r^3 \\
&-\frac{3992453Q}{735436800}C^6r^2+0.0025QC^8\Big) \\
&+\varepsilon_1\varepsilon_2^3\Big(\frac{13Q}{4838400}r^8-\frac{73Q}{1016064}C^2r^6+\frac{37Q}{108000}C^3r^5 \\
&-\frac{23Q}{72960}C^4r^4+\frac{1782371Q}{2224687500}C^5r^3 \\
&-0.0122298QC^6r^2+0.01147035QC^8\Big) \\
&+\varepsilon_2^4\Big(\frac{Q}{38707200}r^8-\frac{Q}{846720}C^2r^6+\frac{Q}{43200}C^3r^5 \\
&-\frac{Q}{12160}C^4r^4+\frac{6269Q}{4021920}C^5r^3-0.02216456QC^6r^2 \\
&+0.020666099QC^8\Big)+o(\varepsilon_1^4)
\end{aligned}
\qquad (2.21)
$$

利用(2.21)式，我们算得无量纲挠度$K(\bar{r})$值如下表：

表 1 **无量纲挠度**$K(\bar{r})=W(\bar{r})/(qc^4/Q_{22}h^3)$**的值**($\bar{r}=r/C$)

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\bar{c}$	0.1899012	0.180510	0.179590	0.167179	0.150072	0.124475	0.103264	0.067473	0.0036889	0.0011279	0
c	0.245589	0.242635	0.233467	0.217759	0.195407	0.166632	0.132143	0.093445	0.0534430	0.0178964	0

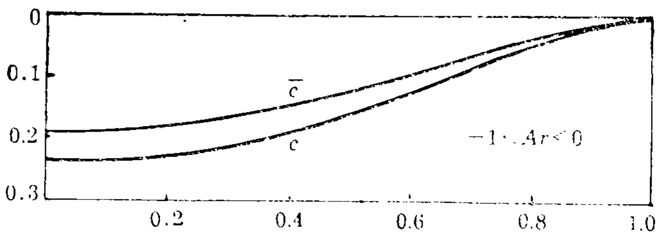

图1 无量纲挠度曲线（c是精确解曲线）

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The Problems of the Nonlinear Unsymmetrical Bending for Cylindrically Orthotropic Circular Plate (Ⅱ)

Huang Jiayin, Qin Shengli, Xia Yunjie, Xu Xiaoping, Kong Xianghe.

(*Qufu Normal University, Qufu, Shandong* 273165)

Abstract

In this paper, we study the recursive equations under the recursive boundary conditions for W_{nm}, φ_{nm}, ν_{nm} and ψ_{nm}($n=0,1,2,\cdots\cdots N$; $m=0, 1, 2, \cdots\cdots M$), which were derived by the "two-variable method"[3] in the preceding paper[1]. We solve these problems by using the methcd of regula pertubation[2], and the uniformly valid asymptotic solution is obtained. Lastly we consider a particular example, i. e., the bending problems of the axisymmetrical circular plate by using "the mixed perturbation method"[4], and compare our results with the exact solution found in Ref. [5]. They are similarly coincided.

Key words orthotropic circular plate, whole core, perturbation