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四边固定正交各向异性矩形板的非线性弯曲问题

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摘　　要

本文利用“两变量法”[1]和“混和摄动法”[2]研究了正交各向异性矩形板在非均布横向载荷作用下的非线性弯曲问题，在四边固定条件下，得到了一致有效渐近解。

关键词　正交各向异性矩形板　非线性弯曲　两变量法　混合摄动法　一致有效渐近解

一、引　　言

在我国，钱伟长教授对奇异摄动理论的发展有着开创性的贡献。他在1948年解决圆板大挠度问题时，即开创了“合成展开法”[3]．江福汝教授为研究非线性非对称弯曲问题提出了“两变量法”[1]．在文献[4]～[6]中对正交各向异性圆板的非线性弯曲问题进行了研究．文献[7]在一个小参数的情况下，对正交各向异性矩形板的非线性弯曲进行了研究．在本文中，我们利用“两变量法”[1]和“混合摄动法”[2]，对四边固定正交各向异性板在非均布横向载荷作用下的非线性弯曲问题进行了研究，求得一致有效渐近解．

二、基本微分方程和边界条件

在非均布横向载荷作用下，正交各向异性矩形板的挠度函数$W(x,y)$和应力函数$\Phi(x,y)$满足以下的微分方程

$$\frac{D_1}{h}\frac{\partial^4 W}{\partial x^4}+2\frac{D_3}{h}\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{D_2}{h}\frac{\partial^4 W}{\partial y^4}=\frac{q(x,y)}{h}+\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2\Phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}-2\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}\tag{2.1a}$$

$$\frac{1}{E_1}\frac{\partial^4\Phi}{\partial y^4}+\left(\frac{1}{G}-\frac{\nu_{12}}{E_1}-\frac{\nu_{21}}{E_2}\right)\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{1}{E_2}\frac{\partial^4\Phi}{\partial x^4}=\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\tag{2.1b}$$

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其中

$$D_1=\frac{E_1h^3}{12(1-\nu_{12}\nu_{21})},\quad D_2=\frac{E_2h^3}{12(1-\nu_{12}\nu_{21})}$$

$$D_k=\frac{Gh^3}{12},\quad D_3=\nu_{12}D_2+2D_k=\nu_{12}D_1+2D_k$$

h是矩形板的厚度；E_1，E_2，ν_{12}，ν_{21}分别是两个主方向的杨氏模量和泊松比；G 是板的剪切模量；D_1，D_2分别是x方向和y方向的抗弯刚度，D_3是折合刚度，D_k是抗扭刚度矩形板如图1 所示。

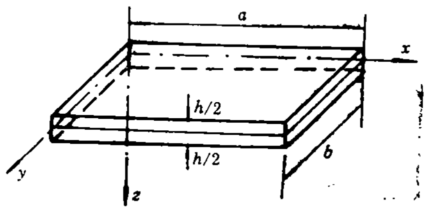

图 1

假设板的四边是固支的，则边界条件和角点条件为

$$\left.\begin{aligned}
&W|_{x=0}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{x=0}=0\\
&W|_{y=0}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{y=0}=0\\
&W|_{x=a}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{x=a}=0\\
&W|_{y=b}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{y=b}=0\\
&W|_{(0,0)}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{(0,0)}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{(0,0)}=0\\
&W|_{(a,0)}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{(a,0)}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{(a,0)}=0\\
&W|_{(0,b)}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{(0,b)}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{(0,b)}=0\\
&W|_{(a,b)}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{(a,b)}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{(a,b)}=0\\
&\left.\left(\frac{\partial^2\Phi}{\partial x^2}-\nu_{21}\frac{\partial^2\Phi}{\partial y^2}\right)\right|_{x=0}=0\\
&\left.\left(\frac{\partial^2\Phi}{\partial x^2}-\nu_{21}\frac{\partial^2\Phi}{\partial y^2}\right)\right|_{x=a}=0\\
&\left.\left(\frac{\partial^2\Phi}{\partial y^2}-\nu_{12}\frac{\partial^2\Phi}{\partial x^2}\right)\right|_{y=0}=0\\
&\left.\left(\frac{\partial^2\Phi}{\partial y^2}-\nu_{12}\frac{\partial^2\Phi}{\partial x^2}\right)\right|_{y=b}=0
\end{aligned}\right\}\tag{2.2}$$

引入无量纲变量[6]：

$$\tilde{W}=\frac{W}{a},\quad \tilde{x}=\frac{x}{a},\quad \tilde{y}=\frac{y}{a},\quad \tilde{\Phi}=\frac{\Phi}{E_1a^2},\quad \tilde{q}=\frac{aq}{hE_1}$$

并将方程(2.1)和边界条件(2.2)无量纲化（略去符号“～”），则得

$$\varepsilon_1^2\frac{\partial^4 W}{\partial x^4}+\varepsilon_2^2\frac{\partial^4 W}{\partial x^2\partial y^2}+\varepsilon_3^2\frac{\partial^4 W}{\partial y^4}-\left[\frac{\partial^2\Phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}-2\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}\right]=q(x,y) \tag{2.3a}$$

$$\delta_2\frac{\partial^4\Phi}{\partial y^4}+\delta_1\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial x^4}-\delta_2\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right]=0 \tag{2.3b}$$

若假定$G<E_2<E_1$，则其中

$$\varepsilon_1^2=\frac{h^2}{12(1-\nu_{12}\nu_{21})a^2}\ll 1$$

$$\varepsilon_3^2=\frac{E_2}{E_1}\varepsilon_1^2=\delta_2\varepsilon_1^2\ll 1$$

$$\varepsilon_2^2=2\left(\varepsilon_1^2\nu_{21}+\frac{1}{6}\frac{G}{E_1}\frac{h^2}{a^2}\right)\ll 1$$

$$\delta_1=E_2\left(\frac{1}{G}-\frac{\nu_{12}}{E_1}-\frac{\nu_{21}}{E_2}\right),\quad \delta_2=\frac{E_2}{E_1}<1$$

又

$$\left.\begin{aligned}
&W|_{x=0}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{x=0}=0\\
&W|_{x=1}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{x=1}=0\\
&W|_{y=0}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{y=0}=0\\
&W|_{y=b/a}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{y=b/a}=0\\
&W|_{(0,0)}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{(0,0)}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{(0,0)}=0\\
&W|_{(1,0)}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{(1,0)}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{(1,0)}=0\\
&W|_{(0,b/a)}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{\left(0,\frac{b}{a}\right)}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{\left(0,\frac{b}{a}\right)}=1\\
&W|_{(1,b/a)}=0,\quad \left.\frac{\partial W}{\partial x}\right|_{(1,b/a)}=0,\quad \left.\frac{\partial W}{\partial y}\right|_{(1,b/a)}=0\\
&\left.\left(\frac{\partial^2\Phi}{\partial x^2}-\nu_{21}\frac{\partial^2\Phi}{\partial y^2}\right)\right|_{x=0}=0\\
&\left.\left(\frac{\partial^2\Phi}{\partial x^2}-\nu_{21}\frac{\partial^2\Phi}{\partial y^2}\right)\right|_{x=1}=0\\
&\left.\left(\frac{\partial^2\Phi}{\partial y^2}-\nu_{12}\frac{\partial^2\Phi}{\partial x^2}\right)\right|_{y=0}=0\\
&\left.\left(\frac{\partial^2\Phi}{\partial y^2}-\nu_{12}\frac{\partial^2\Phi}{\partial x^2}\right)\right|_{y=b/a}=0
\end{aligned}\right\} \tag{2.4}$$

三、微分算子展开

为了得到递推方程和递推边界件条，我们在$x=0$的邻域引入变量ξ和η[7]

$$\xi=\frac{u(x,y)}{\varepsilon_2},\quad \eta=x \tag{3.1}$$

这里$u(x,y)$是一个待定的函数，正的，在$x=0$时，$u(0,y)=0$。把对x的偏导数换为对ξ、η的偏导数，并把ξ，η视为独立变量，即

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial}{\partial x^i}=\varepsilon_1^{-i}(A_{i,0}+\varepsilon_1 A_{i,1}+\cdots+\varepsilon_1^i A_{i,i}) \qquad (i=1,2,3,4) \tag{3.2}$$

式中：

$$A_{1,0}=u_x\frac{\partial}{\partial \xi};\quad A_{1,1}=\frac{\partial}{\partial \eta}$$

$$A_{2,0}=u_x^2\frac{\partial^2}{\partial \xi^2};\quad A_{2,1}=2u_x\frac{\partial^2}{\partial \xi\partial \eta}+u_{xx}\frac{\partial}{\partial \xi},\quad A_{2,2}=\frac{\partial^2}{\partial \eta^2}$$

$$A_{3,0}=u_x^3\frac{\partial^3}{\partial \xi^3};\quad A_{3,1}=3u_x^2\frac{\partial^3}{\partial \xi^2\partial \eta}+3u_xu_{xx}\frac{\partial^2}{\partial \xi^2}$$

$$A_{3,2}=3u_x\frac{\partial^3}{\partial \xi\partial \eta^2}+3u_{xx}\frac{\partial^2}{\partial \xi\partial \eta}+u_{xxx}\frac{\partial}{\partial \xi};\quad A_{3,3}=\frac{\partial^3}{\partial \eta^3};$$

$$A_{4,0}=u_x^4\frac{\partial^4}{\partial \xi^4},\quad A_{4,1}=4u_x^3\frac{\partial^4}{\partial \xi^3\partial \eta}+6u_x^2u_{xx}\frac{\partial^3}{\partial \xi^3};$$

$$A_{4,2}=6u_x^2\frac{\partial^4}{\partial \xi^2\partial \eta^2}+12u_xu_{xx}\frac{\partial^3}{\partial \xi^2\partial \eta}+4u_xu_{xxx}\frac{\partial^2}{\partial \xi^2}+3u_{xx}^2\frac{\partial^2}{\partial \xi^2}$$

$$A_{4,3}=4u_x\frac{\partial^4}{\partial \xi\partial \eta^3}+6u_{xx}\frac{\partial^3}{\partial \xi\partial \eta^2}+4u_{xxx}\frac{\partial^2}{\partial \xi\partial \eta}+u_{xxxx}\frac{\partial}{\partial \xi}$$

$$A_{4,4}=\frac{\partial^4}{\partial \eta^4}$$

同样在$x=1$的邻域引入变量$\tilde{\xi}$和$\tilde{\eta}$，

$$\tilde{\xi}=\frac{\tilde{u}(x,y)}{\varepsilon_1};\qquad \tilde{\eta}=x \tag{3.3}$$

把对x的偏导数换为对$\tilde{\xi}$和$\tilde{\eta}$的偏导数，即

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \tilde{\xi}}\frac{\partial \tilde{\xi}}{\partial x}+\frac{\partial}{\partial \tilde{\xi}}\frac{\partial \tilde{\eta}}{\partial x}$$

$$\frac{\partial}{\partial x^i}=\varepsilon_1^{-i}(\tilde{A}_{i,0}+\varepsilon_1\tilde{A}_{i,1}+\cdots+\varepsilon_1^i\tilde{A}_{i,i}) \qquad (i=1,2,3,4) \tag{3.4}$$

$\tilde{A}_{i,0}$，$\tilde{A}_{i,1},\cdots,\tilde{A}_{i,i}$与前面的$A_{i,0}$，$A_{i,1}$，$\cdots$，$A_{i,i}$相似。在$y=0$的邻域引进两变量$\alpha$和$\beta$；

$$\alpha=\frac{P(x,y)}{\varepsilon_1};\qquad \beta=y \tag{3.5}$$

将对y的偏导数换为对α和β的偏导数，即

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial \alpha}\ \frac{\partial \alpha}{\partial y}+\frac{\partial}{\partial \beta}\ \frac{\partial \beta}{\partial y}$$

$$\frac{\partial}{\partial y^i}=\varepsilon_1^{-i}(B_{i,0}+\varepsilon_1 B_{i,1}+\cdots+\varepsilon_1^i B_{i,i}) \qquad (i=1,2,3,4) \tag{3.6}$$

其中 $B_{1,0}=p_y\frac{\partial}{\partial \alpha}$, $B_{1,1}=\frac{\partial}{\partial \beta}$

$$B_{2,0}=p_y^2\frac{\partial^2}{\partial \alpha^2};\ B_{2,1}=2p_y\frac{\partial^2}{\partial \alpha\partial \beta}+p_{yy}\frac{\partial}{\partial \alpha},\ B_{2,2}=\frac{\partial^2}{\partial \beta^2}$$

$$B_{3,0}=p_y^3\frac{\partial^3}{\partial \alpha^3};\ B_{3,1}=3p_y^2\frac{\partial^3}{\partial \alpha^2\partial \beta}+3p_y p_{yy}\frac{\partial^2}{\partial \alpha^2};$$

$$B_{3,2}=3p_y\frac{\partial^3}{\partial \alpha\partial \beta^2}+3p_{yy}\frac{\partial^2}{\partial \alpha\partial \beta}+p_{yyy}\frac{\partial}{\partial \alpha};\ B_{3,3}=\frac{\partial^3}{\partial \beta^3};$$

$$B_{4,0}=p_y^4\frac{\partial^4}{\partial \alpha^4};\ B_{4,1}=4p_y^3\frac{\partial^4}{\partial \alpha^3\partial \beta}+6p_y^2 p_{yy}\frac{\partial^3}{\partial \alpha^3};$$

$$B_{4,2}=6p_y^2\frac{\partial^4}{\partial \alpha^2\partial \beta^2}+12p_y p_{yy}\frac{\partial^3}{\partial \alpha^2\partial \beta}+4p_y p_{yyy}\frac{\partial^2}{\partial \alpha^2}+3p_{yy}^2\frac{\partial^2}{\partial \alpha^2};$$

$$B_{4,3}=4p_y\frac{\partial^4}{\partial \alpha\partial \beta^3}+6p_{yy}\frac{\partial^3}{\partial \alpha\partial \beta^2}+4p_{yyy}\frac{\partial^2}{\partial \alpha\partial \beta}+p_{yyyy}\frac{\partial}{\partial \alpha}$$

$$B_{4,4}=\frac{\partial^4}{\partial \beta^4}$$

同样在$y=b/a$的领域引进两变量$\tilde{\alpha}$和$\tilde{\beta}$:

$$\tilde{\alpha}=\frac{\tilde{p}(x,y)}{\varepsilon_1};\ \tilde{\beta}=y \tag{3.7}$$

将对y的偏导数换为对$\tilde{\alpha}$和$\tilde{\beta}$的偏导数，即

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial \tilde{\alpha}}\ \frac{\partial \tilde{\alpha}}{\partial y}+\frac{\partial}{\partial \tilde{\beta}}\ \frac{\partial \tilde{\beta}}{\partial y}$$

$$\frac{\partial}{\partial y^i}=\varepsilon_1^{-i}(\tilde{B}_{i,0}+\varepsilon_1\tilde{B}_{i,1}+\cdots+\varepsilon_1^i \tilde{B}_{i,i}) \qquad (i=1,2,3,4) \tag{3.8}$$

$\tilde{B}_{i,0}$, $\tilde{B}_{i,1},\cdots$, $\tilde{B}_{i,i}$与$B_{i,0}$, $B_{i,1}$, $\cdots$, $B_{i,i}$相似.

四、递推方程及递推边界条件

设挠度函数$W(x,y)$和应力函数$\Phi(x,y)$对ε_1为N阶和对ε_2为M阶的展开式为:

$$W(x,y,\varepsilon_1,\varepsilon_2)=\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{mn}(x,y)+\sum_{n=0}^{M}\sum_{m=0}^{N}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}\ \nu_{mn}^{(1)}(\xi,\eta,y)$$

$$+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}\nu_{mn}^{(2)}(\tilde{\xi},\tilde{\eta},y)+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}(x,\alpha,\beta)$$

$$+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}(x,\tilde{\alpha},\tilde{\beta}) \tag{4.1}$$

$$\Phi(x,y,\varepsilon_1,\varepsilon_2)=\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_2^n\varepsilon_2^m\varphi_{nm}(x,y)$$

$$+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\ \varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}(\xi,\eta,y)+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\ \varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}(\tilde{\xi},\tilde{\eta},y)$$

$$+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\ \varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}(x,\alpha,\beta)+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\ \varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}(x,\tilde{\alpha},\tilde{\beta})\qquad(4.2)$$

式中 $\nu_{nm}^{(1)}$, $\psi_{nm}^{(1)}$ 和 $\nu_{nm}^{(2)}$, $\psi_{nm}^{(2)}$分别是$x=0$和$x=1$邻域的边界层型函数; $\nu_{nm}^{(3)}$, $\psi_{nm}^{(3)}$ 和 $\nu_{nm}^{(4)}$, $\psi_{nm}^{(4)}$分别是$y=0$和$y=b/a$邻域的边界层型函数. α_1, …, α_8; β_1, …, β_8则为待定常数.

将微分算子(3.1)～(3.8)，挠度函数和应力函数代入方程(2.3)得:

$$\Big\{\varepsilon_1^2\Big[\frac{\partial^4}{\partial x^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\Big)+\delta_2\frac{\partial^4}{\partial y^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\Big)\Big]$$

$$+\varepsilon_2^2\frac{\partial^4}{\partial x^2\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\Big)$$

$$-\Big[\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\Big)\frac{\partial^2}{\partial x^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\Big)$$

$$+\frac{\partial^2}{\partial x^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\Big)\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_2^n\varepsilon_2^m W_{nm}\Big)$$

$$-2\frac{\partial^2}{\partial x\partial y}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\Big)\frac{\partial^2}{\partial x\partial y}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\Big)\Big]\Big\}$$

$$+\Big\{\varepsilon_1^2\Big[\varepsilon_1^{-4}\Big(\sum_{i=0}^{4}\varepsilon_1^i A_{4,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\Big)$$

$$+\varepsilon_1^{-4}\Big(\sum_{i=0}^{4}\varepsilon_1^i\tilde{A}_{4,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\Big)$$

$$+\frac{\partial^4}{\partial x^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\Big)+\frac{\partial^4}{\partial x^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\Big)$$

$$+\delta_2\Big(\frac{\partial^4}{\partial y^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\Big)+\frac{\partial^4}{\partial y^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\Big)$$

$$+\varepsilon_1^{-4}\Big(\sum_{i=0}^{4}\varepsilon_1^i B_{4,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha}\ \nu_{nm}^{(3)}\Big)$$

$$+\varepsilon_1^{-4}\Big(\sum_{i=0}^{4}\varepsilon_1^i\tilde{B}_{4,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\Big)\Big)\Big]$$

$$+\varepsilon_2^2\Big[\varepsilon_1^{-4}\Big(\sum_{i=0}^{2}\varepsilon_1^i A_{2,i}\Big)\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\Big)$$

$$+\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^i\tilde{A}_{2,i}\Big)\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\Big)$$

$$+\frac{\partial^2}{\partial x^2}\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^i B_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\Big)$$

$$+\frac{\partial^2}{\partial x^2}\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^i\tilde{B}_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\Big)\Big]$$

$$-\Big[\Big(\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\Big)\Big)(L_0)+\Big(\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\ \varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\Big)$$

$$+\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\ \varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\Big)$$

$$+\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^i B_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\ \varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\Big)$$

$$+\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^i\tilde{B}_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\ \varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\Big)\Big)(L_1)$$

$$+\Big(\frac{\partial^2}{\partial x^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\Big)\Big)(K_0)$$

$$+\Big(\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^i A_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\ \varepsilon_2^{m+\beta_2}\psi_{nm}^{(3)}\Big)$$

$$+\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^i\tilde{A}_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\ \varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\Big)$$

$$+\frac{\partial^2}{\partial x^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\ \varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\Big)+\frac{\partial^2}{\partial x^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\ \varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\Big)\Big)(K_1)$$

$$-2\Big(\frac{\partial^2}{\partial x\partial y}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\Big)\Big)(J_0)$$

$$-2\Big(\varepsilon_1^{-}\Big(\sum_{i=0}^{1}\varepsilon_1^i A_{i,0}\Big)\frac{\partial}{\partial y}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\ \varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\Big)$$

$$+\varepsilon_1^{-1}\Big(\sum_{i=0}^{1}\varepsilon_1^i\tilde{A}_{1,i}\Big)\frac{\partial}{\partial y}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\ \varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\Big)$$

$$+\frac{\partial}{\partial x}\varepsilon_1^{-1}\Big(\sum_{i=0}^{1}\varepsilon_1^iB_{1,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\ \varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\Big)$$

$$+\frac{\partial}{\partial x}\varepsilon_1^{-1}\Big(\sum_{i=0}^{1}\varepsilon_1^i\tilde{B}_{1,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\ \varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\Big)\Big)(J_1)\ \Big]\Big\}$$

$$=q \tag{4.3}$$

$$\Big\{\frac{\partial^4}{\partial x^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\Big)+\delta_1\frac{\partial^4}{\partial x^2\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\Big)$$

$$+\delta_2\Big[\frac{\partial^4}{\partial y^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\Big)-\Big(\frac{\partial^2}{\partial x\partial y}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^mW_{nm}\Big)\Big)^2$$

$$+\Big(\frac{\partial^2}{\partial x^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^mW_{nm}\Big)\Big)\Big(\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^mW_{nm}\Big)\Big)\Big]\Big\}$$

$$+\Big\{\Big[\varepsilon_1^{-4}\Big(\sum_{i=0}^{4}\varepsilon_1^iA_{4,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\ \varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\Big)$$

$$+\varepsilon_1^{-4}\Big(\sum_{i=0}^{4}\varepsilon_1^i\tilde{A}_{4,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\ \varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\Big)$$

$$+\frac{\partial^4}{\partial x^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\ \varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\Big)+\frac{\partial^4}{\partial x^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\ \varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\Big)\Big]$$

$$+\delta_1\Big[\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iA_{2,i}\Big)\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\ \varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\Big)$$

$$+\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^i\tilde{A}_{2,i}\Big)\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\ \varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\Big)$$

$$+\frac{\partial^2}{\partial x^2}\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iB_{3,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\ \varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\Big)$$

$$+\frac{\partial^2}{\partial x^2}\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^i\tilde{B}_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\ \varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\Big)\Big]$$

$$+\delta_2\Big[\frac{\partial^4}{\partial y^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\ \varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\Big)+\frac{\partial^4}{\partial y^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\ \varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\Big)$$

$$+\varepsilon_1^{-4}\Big(\sum_{i=0}^{4}\varepsilon_1^iB_{4,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\ \varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\Big)$$

$$+\varepsilon_1^{-4}\left(\sum_{i=0}^{4}\varepsilon_1^i\tilde{B}_{4,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\ \varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\right)$$

$$-\left(\frac{\partial^2}{\partial x\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right)\right)(J_0)$$

$$-\left(\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i A_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right)\right.$$

$$+\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i\tilde{A}_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right)$$

$$+\frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right)$$

$$\left.+\frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i\tilde{B}_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right)\right)(J_1)$$

$$+\left(\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right)\right)(K_0)$$

$$+\left(\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i A_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right)\right.$$

$$+\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i\tilde{A}_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right)$$

$$+\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right)$$

$$\left.\left.\left.+\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right)\right)(K_1)\right]\right\}=0 \tag{4.4}$$

其中

$$L_0=\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i A_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right)$$

$$+\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i\tilde{A}_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right)$$

$$+\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right)+\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right)$$

$$L_1=\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right)+L_0$$

$$K_0=\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right)+\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right)$$

$$+\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i B_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right)$$

$$+\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i \tilde{B}_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right)$$

$$K_1=\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right)+K_0$$

$$J_0=\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i A_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right)$$

$$+\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i \tilde{A}_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right)$$

$$+\frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right)$$

$$+\frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i \tilde{B}_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(2)}\right)$$

$$J_1=\frac{\partial^2}{\partial x\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right)+J_0$$

将(4.1)～(4.2)，(3.1)～(3.8)代入边界条件(2.4)得：

$$\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_1^m W_{nm}\right]\Bigg|_{x=0}+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right]\Bigg|_{\eta=0}=0 \tag{4.5}$$

$$\left[\frac{\partial}{\partial x}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{x=0}+\left[\varepsilon^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i A_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right)\right]\Bigg|_{\eta=0}$$
$$=0 \tag{4.6}$$

$$\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{x=1}+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right]\Bigg|_{\tilde{\eta}=1}=0 \tag{4.7}$$

$$\left[\frac{\partial}{\partial x}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{x=1}+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i \tilde{A}_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right)\right]\Bigg|_{\tilde{\eta}=1}$$
$$=0 \tag{4.8}$$

$$\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{y=0}+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right]\Bigg|_{\beta=0}=0 \tag{4.9}$$

$$\left[\frac{\partial}{\partial y}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{y=0}+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right)\right]\Bigg|_{\beta=0}$$

$$=0 \tag{4.10}$$

$$\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{y=b/a}+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right]\Bigg|_{\tilde{\beta}=\frac{b}{a}}=0 \tag{4.11}$$

$$\left[\frac{\partial}{\partial y}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{y=\frac{b}{a}}+\left[\varepsilon_1^{-i}\left(\sum_{i=0}^{1}\varepsilon_1^i\tilde{B}_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha^8}\nu_{nm}^{(4)}\right)\right]\Bigg|_{\tilde{\beta}=b/a}$$

$$=0 \tag{4.12}$$

$$\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=0\\y=0}}+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right]\Bigg|_{\substack{\eta=0\\y=0}}$$

$$+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right]\Bigg|_{\substack{x=0\\\beta=0}}=0 \tag{4.13}$$

$$\left[\frac{\partial}{\partial x}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right)\right]\Bigg|_{\substack{x=0\\y=0}}+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i A_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right)\right]\Bigg|_{\substack{\eta=0\\y=0}}$$

$$+\left[\frac{\partial}{\partial x}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right)\right]\Bigg|_{\substack{x=0\\\beta=0}}=0 \tag{4.14}$$

$$\left[\frac{\partial}{\partial y}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=0\\y=0}}+\left[\frac{\partial}{\partial y}\sum_{x=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right]\Bigg|_{\substack{\eta=0\\y=0}}$$

$$+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right)\right]\Bigg|_{\substack{x=0\\\beta=0}}=0 \tag{4.15}$$

$$\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=0\\y=0}}+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right]\Bigg|_{\substack{\tilde{\eta}=1\\y=0}}$$

$$+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right]\Bigg|_{\substack{x=0\\\beta=0}}=0 \tag{4.16}$$

$$\left[\frac{\partial}{\partial x}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=1\\y=0}}+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i\tilde{A}_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right)\right]\Bigg|_{\substack{\tilde{\eta}=1\\y=0}}$$

$$+\left[\frac{\partial}{\partial x}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right]\Bigg|_{\substack{x=1\\\beta=0}}=0 \tag{4.17}$$

$$\left[\frac{\partial}{\partial y}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=1\\y=0}}+\left[\frac{\partial}{\partial y}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right]\Bigg|_{\substack{\tilde{\eta}=1\\y=0}}$$

$$+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\ \varepsilon_2^{m+\alpha_6}\nu_{nm}^{(3)}\right)\right]\Bigg|_{\substack{x=1\\\beta=0}}=0 \tag{4.18}$$

$$\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=0\\y=b/a}}+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right]\Bigg|_{\substack{\eta=0\\y=b/a}}$$

$$+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right]\Bigg|_{\substack{x=0\\\tilde{\beta}=b/a}}=0 \tag{4.19}$$

$$\left[\frac{\partial}{\partial x}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=0\\y=b/a}}+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i A_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right]\Bigg|_{\substack{\eta=0\\y=b/a}}$$

$$+\left[\frac{\partial}{\partial x}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right]\Bigg|_{\substack{x=0\\\tilde{\beta}=b/a}}=0 \tag{4.20}$$

$$\left[\frac{\partial}{\partial y}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=0\\y=b/a}}+\left[\frac{\partial}{\partial y}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\ \varepsilon_2^{m+\alpha_2}\nu_{nm}^{(1)}\right]\Bigg|_{\substack{\eta=0\\y=a/b}}$$

$$+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i \tilde{B}_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right]\Bigg|_{\substack{x=0\\\tilde{\beta}=b/a}}=0 \tag{4.21}$$

$$\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=1\\y=b/a}}+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right]\Bigg|_{\substack{\tilde{\eta}=1\\y=b/a}}$$

$$+\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right]\Bigg|_{\substack{x=1\\\tilde{\beta}=b/a}}=0 \tag{4.22}$$

$$\left[\frac{\partial}{\partial x}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=1\\y=b/a}}+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i \tilde{A}_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{nm}^{(2)}\right]\Bigg|_{\substack{\tilde{\eta}=1\\y=b/a}}$$

$$+\left[\frac{\partial}{\partial x}\left[\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right)\right]\Bigg|_{\substack{x=1\\\tilde{\beta}=b/a}}=0 \tag{4.23}$$

$$\left[\frac{\partial}{\partial y}\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}\right]\Bigg|_{\substack{x=1\\y=b/a}}+\left[\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\ \varepsilon_2^{m+\alpha_4}\nu_{n8}^{(2)}\right]\Bigg|_{\substack{\tilde{\eta}=1\\y=b/a}}$$

$$+\left[\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^i \tilde{B}_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\ \varepsilon_2^{m+\alpha_8}\nu_{nm}^{(4)}\right)\right]\Bigg|_{\substack{x=1\\\tilde{\beta}=b/a}}=0 \tag{4.24}$$

$$\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)-\nu_{21}\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)\right]\Bigg|_{x=0}$$

$$+\left[\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i A_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\ \varepsilon_2^{m+\beta_2}\ \psi_{nm}^{(1)}\right)\right.$$

$$\left.-\nu_{21}\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\ \varepsilon_2^{m+\beta_2}\ \psi_{nm}^{(1)}\right)\right]\Bigg|_{\eta=0}=0 \tag{4.25}$$

$$\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)-\nu_{21}\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)\right]\Bigg|_{x=1}$$

$$+\left[\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i \tilde{A}_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\ \varepsilon_2^{m+\beta_4}\ \psi_{nm}^{(2)}\right)\right.$$

$$\left.-\nu_{21}\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\ \varepsilon_2^{m+\beta_4}\ \psi_{nm}^{(2)}\right)\right]\Bigg|_{\tilde{\eta}=1}=0 \tag{4.26}$$

$$\left[\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)-\nu_{12}\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)\right]\Bigg|_{y=0}$$

$$+\left[\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i B_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\ \varepsilon_2^{m+\beta_6}\ \psi_{nm}^{(2)}\right)\right.$$

$$\left.-\nu_{12}\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\ \varepsilon_2^{m+\beta_6}\ \psi_{nm}^{(3)}\right)\right]\Bigg|_{\beta=0}=0 \tag{4.27}$$

$$\left[\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)-\nu_{12}\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_1^m\varphi_{nm}\right)\right]\Bigg|_{y=b/a}$$

$$+\left[\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^i \tilde{B}_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\varepsilon_2^{m+\beta_7}\psi_{nm}^{(4)}\right)\right.$$

$$\left.-\nu_{12}\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\right)\right]\Bigg|_{\tilde{\beta}=b/a}=0 \tag{4.28}$$

首先我们确定α_1, α_2, $\cdots$,α_8和 β_1, β_2, $\cdots$,β_8的值．由(4.6),(4.8),(4.10), (4.12), (4.25)～(4.28)式比较ε_1和ε_2的最低次幂系数可知 $\alpha_1=\alpha_3=\alpha_5=\alpha_7=1$, $\beta_1=\beta_3=\beta_5=\beta_7=2$, $\alpha_2=\alpha_4=\alpha_6=\alpha_8=\beta_2=\beta_4=\beta_6=\beta_8=0$．由(4.4)式的第二个大括号可知,$\beta_1=\beta_3=\beta_5=\beta_7=4$．而由(4.3)式的第二个大括号可知,$\alpha_1=\alpha_3=\alpha_5=\alpha_7=2$．我们取 $\alpha_1=\alpha_3=\alpha_5=\alpha_7=2$, $\beta_1=\beta_3=\beta_5=\beta_7=4$, $\alpha_2=\alpha_4=\alpha_6=\alpha_8=\beta_2=\beta_4=\beta_6=\beta_8=0$, 将它们代入(4.3)和(4.4)式，然后比较第一个大括号中$\varepsilon_1\varepsilon_2$同次幂系数可得递推方程及递推边界条件

$$\frac{\partial^2\varphi_{00}}{\partial y}\ \frac{\partial^2 W_{00}}{\partial x^2}+\frac{\partial^2\varphi_{00}}{\partial x^2}\ \frac{\partial^2 W_{00}}{\partial y^2}-2\frac{\partial^2\varphi_{00}}{\partial x\partial y}\ \frac{\partial^2 W_{00}}{\partial x\partial y}=-q(x,\ y) \tag{4.29}$$

$$\frac{\partial^4\varphi_{00}}{\partial x^4}+\delta_1\frac{\partial^4\varphi_{00}}{\partial x^2\partial y^2}+\delta_2\left(\frac{\partial^4\varphi_{00}}{\partial y^4}-\frac{\partial^2 W_{00}}{\partial x\partial y}\ \frac{\partial^2 W_{00}}{\partial x\partial y}+\frac{\partial^2 W_{00}}{\partial x^2}\ \frac{\partial^2 W_{00}}{\partial y^2}\right)$$

$$=0 \tag{4.30}$$

$$
\left.\begin{array}{l}
W_{00}\Big|_{x=0}=0,\ \dfrac{\partial W_{00}}{\partial x}\Big|_{x=0}=0 \\
W_{00}\Big|_{x=1}=0,\ \dfrac{\partial W_{00}}{\partial x}\Big|_{x=1}=0 \\
W_{00}\Big|_{y=0}=0,\ \dfrac{\partial W_{00}}{\partial y}\Big|_{y=0}=0 \\
W_{00}\Big|_{y=\frac{b}{a}}=0,\dfrac{\partial W_{00}}{\partial y}\Big|_{y=b/a}=0 \\
W_{00}\Big|_{\substack{x==0\\ y==0}}=0,\dfrac{\partial W_{00}}{\partial x}\Big|_{\substack{x=0\\ y=0}}=0,\dfrac{\partial W_{00}}{\partial y}\Big|_{\substack{x=0\\ y=0}}=0 \\
W_{00}\Big|_{\substack{x=1\\ y=0}}=0,\dfrac{\partial W_{00}}{\partial x}\Big|_{\substack{x=1\\ y=0}}=0,\dfrac{\partial W_{00}}{\partial y}\Big|_{\substack{x=1\\ y=0}}=0 \\
W_{00}\Big|_{\substack{x=0\\ y=\frac{b}{a}}}=0,\dfrac{\partial W_{00}}{\partial x}\Big|_{\substack{x=0\\ y=b/a}}=0,\dfrac{\partial W_{00}}{\partial y}\Big|_{\substack{x=0\\ y=b/a}}=0 \\
W_{00}\Big|_{\substack{x=1\\ y=b/a}}=0,\dfrac{\partial W_{00}}{\partial x}\Big|_{\substack{x=1\\ y=b/a}}=0,\dfrac{\partial W_{00}}{\partial y}\Big|_{\substack{x=1\\ y=b/a}}=0 \\
\left[\dfrac{\partial^2\varphi_{00}}{\partial x^2}-\nu_{21}\dfrac{\partial^2\varphi_{00}}{\partial y^2}\right]\Big|_{x=0}=0 \\
\left[\dfrac{\partial^2\varphi_{00}}{\partial x^2}-\nu_{21}\dfrac{\partial^2\varphi_{00}}{\partial y^2}\right]\Big|_{x=1}=0 \\
\left[\dfrac{\partial^2\varphi_{00}}{\partial y^2}-\nu_{12}\dfrac{\partial^2\varphi_{00}}{\partial x^2}\right]\Big|_{y=0}=0 \\
\left[\dfrac{\partial^2\varphi_{00}}{\partial y^2}-\nu_{12}\dfrac{\partial^2\varphi_{00}}{\partial x^2}\right]\Big|_{y=\frac{b}{a}}=0 \\
\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots
\end{array}\right\} \tag{4.31}
$$

$$
\begin{aligned}
&\frac{\partial^4}{\partial x^4}W_{(n-2)m}+\delta_2\frac{\partial^4}{\partial y^4}W_{(n-2)m}+\frac{\partial^4}{\partial x^2\partial y^2}W_{n(m-2)} \\
&\quad-\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial^2\varphi_{ij}}{\partial y^2}\frac{\partial^2}{\partial x^2}W_{(n-i)(m-j)}-\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial^2\varphi_{ij}}{\partial x^2}\frac{\partial^2}{\partial y^2}W_{(n-i)(m-j)} \\
&\quad+2\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial^2\varphi_{ij}}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y}W_{(n-i)(m-j)}=0
\end{aligned} \tag{4.32}
$$

$$
\begin{aligned}
&\frac{\partial^4}{\partial x^4}\varphi_{nm}+\delta_1\frac{\partial^4}{\partial x^2\partial y^2}\varphi_{nm}+\delta_2\Big[\frac{\partial^4}{\partial y^4}\varphi_{nm} \\
&\quad-\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial^2 W_{ij}}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y}W_{(n-i)(m-j)}+\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial^2 W_{ij}}{\partial x^2}\frac{\partial^2}{\partial y^2}W_{(n-i)(m-j)}=0
\end{aligned} \tag{4.33}
$$

$$
W_{nm}\Big|_{x=0}+\nu^{(1)}_{(n-2)m}\Big|_{\eta=0}=0 \tag{4.34}
$$

$$\frac{\partial W_{nm}}{\partial x}\bigg|_{x=0}+[A_{1,0}\nu^{(1)}_{(n-1)m}+A_{1,1}\nu^{(1)}_{(n-2)m}]|_{\eta=0}=0 \tag{4.35}$$

$$W_{nm}|_{x=1}+\nu^{(2)}_{(n-2)m}|_{\tilde{\eta}=1}=0 \tag{4.36}$$

$$\frac{\partial W_{nm}}{\partial x}\bigg|_{x=1}+[\tilde{A}_{1,0}\nu^{(2)}_{(n-1)m}+\tilde{A}_{1,1}\nu^{(3)}_{(n-2)m}]|_{\tilde{\eta}=1}=0 \tag{4.37}$$

$$W_{nm}|_{y=0}+\nu^{(3)}_{(n-2)m}|_{\beta=0}=0 \tag{4.38}$$

$$\frac{\partial W_{nm}}{\partial y}\bigg|_{y=0}+[B_{1,0}\nu^{(3)}_{(n-1)m}+B_{1,1}\nu^{(3)}_{(n-2)m}]\bigg|_{\beta=0}=0 \tag{4.39}$$

$$W_{nm}|_{y=b/a}+\nu^{(4)}_{(n-2)m}|_{\tilde{\beta}=b/a}=0 \tag{4.40}$$

$$\frac{\partial W_{nm}}{\partial y}\bigg|_{y=b/a}+[\tilde{B}_{1,0}\nu^{(4)}_{(n-1)m}+\tilde{B}_{1,1}\nu^{(4)}_{(n-2)m}]\bigg|_{\tilde{\beta}=b/a}=0 \tag{4.41}$$

$$W_{nm}\bigg|_{\substack{x=0\\y=0}}+\nu^{(1)}_{(n-2)m}\bigg|_{\substack{\eta=0\\x=0}}+\nu^{(3)}_{(n-2)m}\bigg|_{\substack{x=0\\\beta=0}}=0 \tag{4.42}$$

$$\frac{\partial W_{nm}}{\partial x}\bigg|_{\substack{x=0\\y=0}}+[A_{1,0}\nu^{(1)}_{(n-1)m}+A_{1,1}\nu^{(1)}_{(n-2)m}]\bigg|_{\substack{\eta=0\\y=0}}+\frac{\partial}{\partial x}\nu^{(3)}_{(n=2)m}\bigg|_{\substack{x=0\\\beta=0}}=0 \tag{4.43}$$

$$\frac{\partial W_{nm}}{\partial y}\bigg|_{\substack{x=0\\y=0}}+\frac{\partial}{\partial y}\nu^{(1)}_{(n-2)m}\bigg|_{\substack{\eta=0\\y=0}}+[B_{1,0}\nu^{(3)}_{(n-1)m}+B_{1,1}\nu^{(3)}_{(n-2)m}]\bigg|_{\substack{x=0\\\beta=0}}=0 \tag{4.44}$$

$$W_{nm}\bigg|_{\substack{x=1\\y=0}}+\nu^{(2)}_{(n-2)m}\bigg|_{\substack{\tilde{\eta}=1\\y=0}}+\nu^{(3)}_{(n-2)m}\bigg|_{\substack{x=1\\\beta=0}}=0 \tag{4.45}$$

$$\frac{\partial W_{nm}}{\partial x}\bigg|_{\substack{x=1\\y=0}}+[\tilde{A}_{1,0}\nu^{(2)}_{(n-1)m}+\tilde{A}_{1,1}\nu^{(2)}_{(n-2)m}]\bigg|_{\substack{\tilde{\eta}=1\\y=0}}+\frac{\partial}{\partial x}\nu^{(3)}_{(n-2)m}\bigg|_{\substack{x=1\\\beta=0}}=0 \tag{4.46}$$

$$\frac{\partial W_{nm}}{\partial y}\bigg|_{\substack{x=1\\y=0}}+\frac{\partial}{\partial y}\nu^{(2)}_{(n-2)m}\bigg|_{\substack{\tilde{\eta}=1\\y=0}}+[B_{1,0}\nu^{(3)}_{(n-1)m}+B_{1,1}\nu^{(3)}_{(n-2)m}]\bigg|_{\substack{x=1\\\beta=0}}=0 \tag{4.47}$$

$$W_{nm}\bigg|_{\substack{x=0\\y=b/a}}+\nu^{(1)}_{(n-2)m}\bigg|_{\substack{\eta=0\\y=b/a}}+\nu^{(4)}_{(n-2)m}\bigg|_{\substack{x=0\\\tilde{\beta}=b/a}}=0 \tag{4.48}$$

$$\frac{\partial W_{nm}}{\partial x}\bigg|_{\substack{x=0\\y=b/a}}+[A_{1,0}\nu^{(1)}_{(n-1)m}+A_{1,1}\nu^{(1)}_{(n-2)m}]\bigg|_{\substack{\eta=0\\y=b/a}}$$

$$+\frac{\partial}{\partial x}\nu^{(4)}_{(n-2)m}\bigg|_{\substack{x=0\\\tilde{\beta}=b/a}}=0 \tag{4.49}$$

$$\frac{\partial W_{nm}}{\partial y}\bigg|_{\substack{x=0\\y=b/a}}+\frac{\partial}{\partial y}\nu^{(1)}_{(n-2)m}\bigg|_{\substack{\eta=0\\y=b/a}}+[\tilde{B}_{1,0}\nu^{(4)}_{(n-1)m}$$

$$+\tilde{B}_{1,1}\nu^{(4)}_{(n-2)m}]\bigg|_{\substack{x=0\\\tilde{\beta}=b/a}}=0 \tag{4.50}$$

$$W_{nm}\bigg|_{\substack{x=1\\y=b/a}}+\nu^{(2)}_{(n-2)m}\bigg|_{\substack{\tilde{\eta}=1\\y=b/a}}+\nu^{(4)}_{(n-2)m}\bigg|_{\substack{x=1\\\tilde{\beta}=b/a}}=0 \tag{4.51}$$

$$\frac{\partial W_{nm}}{\partial x}\bigg|_{\substack{x=1\\y=b/a}}+[\tilde{A}_{1,0}\nu^{(2)}_{(n-1)m}+\tilde{A}_{1,1}\nu^{(2)}_{(n-2)m}]\bigg|_{\substack{\tilde{\eta}=1\\y=b/a}}$$

$$+\frac{\partial}{\partial x}\nu^{(4)}_{(n-2)m}\bigg|_{\substack{x=1\\\tilde{\beta}=b/a}}=0 \tag{4.52}$$

$$\frac{\partial W_{nm}}{\partial y}\bigg|_{\substack{x=1\\y=b/a}}+\frac{\partial}{\partial y}\nu^{(2)}_{(n-2)m}\bigg|_{\substack{\tilde{\eta}=1\\y=b/a}}+[\tilde{B}_{1,0}\nu^{(4)}_{(n-1)m}$$

$$+\tilde{B}_{1,1}\nu^{(4)}_{(n-2)m}]\Big|_{\substack{x=1\\ \tilde{\beta}=b/a}}=0 \tag{4.53}$$

$$\left[\frac{\partial^2}{\partial x^2}\varphi_{nm}-\nu_{21}\frac{\partial^2}{\partial y^2}\varphi_{nm}\right]\Big|_{x=0}+[A_{2,0}\psi^{(1)}_{(n-2)m}+A_{2,1}\psi^{(1)}_{(n-3)m}$$

$$+A_{2,2}\psi^{(1)}_{(n-4)m}-\nu_{21}\frac{\partial^2}{\partial y^2}\psi^{(1)}_{(n-4)m}]\Big|_{\eta=0}=0 \tag{4.54}$$

$$\left[\frac{\partial^2\varphi_{nm}}{\partial x^2}-\nu_{21}\frac{\partial^2}{\partial y^2}\varphi_{nm}\right]\Big|_{x=1}+\Big[\tilde{A}_{2,0}\psi^{(2)}_{(n-2)m}+\tilde{A}_{2,1}\psi^{(3)}_{(n-3)m}$$

$$+\tilde{A}_{2,2}\psi^{(2)}_{(n-4)m}-\nu_{21}\frac{\partial^2}{\partial y^2}\psi^{(2)}_{(n-4)m}\Big]\Big|_{\tilde{\eta}=1}=0 \tag{4.55}$$

$$\left[\frac{\partial^2\varphi_{nm}}{\partial y^2}-\nu_{12}\frac{\partial^2}{\partial x^2}\varphi_{nm}\right]\Big|_{y=0}+\Big[B_{2,0}\psi^{(3)}_{(n-2)m}+B_{2,1}\psi^{(3)}_{(n-3)m}$$

$$+B_{2,2}\psi^{(3)}_{(n-4)m}-\nu_{12}\frac{\partial^2}{\partial x^2}\psi^{(3)}_{(n-4)m}\Big]\Big|_{\beta=0}=0 \tag{4.56}$$

$$\left[\frac{\partial^2\varphi_{nm}}{\partial y^2}-\nu_{12}\frac{\partial^2}{\partial x^2}\varphi_{nm}\right]\Big|_{y=b/a}+\Big[\tilde{B}_{2,0}\psi^{(4)}_{(n-2)m}+\tilde{B}_{2,1}\psi^{(4)}_{(n-3)m}$$

$$+\tilde{B}_{2,2}\psi^{(4)}_{(n-4)m}-\nu_{12}\frac{\partial^2}{\partial x^2}\psi^{(4)}_{(n-4)m}\Big]\Big|_{\tilde{\beta}=b/a}=0 \tag{4.57}$$

由(4.3)和(4.4)式的第二个大括号，分别比较$\nu^{(1)}_{nm}$和 $\psi^{(1)}_{nm}$，$\nu^{(2)}_{nm}$和$\psi^{(2)}_{nm}$，$\nu^{(3)}_{nm}$和$\psi^{(3)}_{nm}$，$\nu^{(4)}_{nm}$和$\psi^{(4)}_{nm}$的$\varepsilon_1\varepsilon_2$同次幂的系数，我们获得边界层正项的递推方程

$$A_{4,0}\nu^{(1)}_{00}-\frac{\partial^2\varphi_{00}}{\partial y^2}A_{2,0}\nu^{(1)}_{00}=0 \tag{4.58a}$$

$$\tilde{A}_{4,0}\nu^{(2)}_{00}-\frac{\partial^2\varphi_{00}}{\partial y^2}\tilde{A}_{2,0}\nu^{(2)}_{00}=0 \tag{4.58b}$$

$$\delta_2B_{4,0}\nu^{(3)}_{00}-\frac{\partial^2\varphi_{00}}{\partial x^2}B_{2,0}\nu^{(3)}_{00}=0 \tag{4.58c}$$

$$\delta_2\tilde{B}_{4,0}\nu^{(4)}_{00}-\frac{\partial^2\varphi_{00}}{\partial x^2}B_{2,0}\nu^{(4)}_{00}=0 \tag{4.58d}$$

$$A_{4,0}\psi^{(1)}_{00}+\delta_2A_{2,0}\nu^{(1)}_{00}\cdot\left(\frac{\partial^2W_{00}}{\partial y^2}+\frac{1}{2}B_{2,0}\nu^{(3)}_{00}+\frac{1}{2}\tilde{B}_{2,0}\nu^{(4)}_{00}\right)=0 \tag{4.59a}$$

$$\tilde{A}_{4,0}\psi^{(2)}_{00}+\delta_2\tilde{A}_{2,0}\nu^{(2)}_{00}\cdot\left(\frac{\partial^2W_{00}}{\partial y^2}+\frac{1}{2}B_{2,0}\nu^{(3)}_{00}+\frac{1}{2}\tilde{B}_{2,0}\nu^{(4)}_{00}\right)=0 \tag{4.59b}$$

$$B_{4,0}\psi^{(3)}_{00}+B_{2,0}\nu^{(3)}_{00}\left(\frac{\partial^2W_{00}}{\partial x^2}+\frac{1}{2}A_{2,0}\nu^{(1)}_{00}+\frac{1}{2}\tilde{A}_{2,0}\nu^{(2)}_{00}\right)=0 \tag{4.59c}$$

$$\tilde{B}_{4,0}\psi^{(4)}_{00}+\tilde{B}_{2,0}\nu^{(4)}_{00}\left(\frac{\partial^2W_{00}}{\partial x^2}+\frac{1}{2}A_{2,0}\nu^{(1)}_{00}+\frac{1}{2}\tilde{A}_{2,0}\nu^{(2)}_{00}\right)=0 \tag{4.59d}$$

…… …… ……

由于篇幅所限，下面我们只写出$\nu^{(1)}_{nm}$和$\psi^{(1)}_{nm}$的一般递推式。

$$\sum_{i=0}^{4} A_{4,i}\nu^{(1)}_{(n-i)m}+\delta_2\sum_{i=0}^{2} A_{2,i}\frac{\partial^2}{\partial y^2}\nu^{(1)}_{(n-2-i)m}+\frac{\partial^4}{\partial y^4}\nu^{(1)}_{(n-2)m}$$

$$-\Big[\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\frac{\partial^2\varphi_{ij}}{\partial y^2}A_{2,l}\nu^{(1)}_{(n-i-l)(m-j)}$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial^2 W_{ij}}{\partial x^2}\frac{\partial^2}{\partial y^2}\psi^{(1)}_{(n-j-4)(m-j)}$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\Big(\frac{\partial^2}{\partial y^2}\psi^{(1)}_{(n-i-l-4)(m-j)}\Big)A_{2,l}\nu^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\Big(\frac{\partial^2}{\partial y^2}\psi^{(1)}_{(n-i-l-4)(m-j)}\Big)\tilde{A}_{2,l}\nu^{(2)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(\frac{\partial^2}{\partial y^2}\psi^{(1)}_{(n-i-6)(m-j)}\Big)\frac{\partial^2}{\partial x^2}\nu^{(3)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(\frac{\partial^2}{\partial y^2}\psi^{(1)}_{(n-i-6)(m-j)}\Big)\frac{\partial^2}{\partial x^2}\nu^{(4)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\Big(\frac{\partial^2}{\partial y^2}\psi^{(2)}_{(n-i-l-4)(m-j)}\Big)A_{2,l}\nu^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\sum_{r=0}^{2}B_{2,r}\psi^{(3)}_{(n-i-l-r-2)(m-j)}A_{2,l}\nu^{(1)}_{ij}$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\sum_{r=0}^{2}\tilde{B}_{2,r}\psi^{(4)}_{(n-i-r-l-2)(m-j)}A_{2,l}\nu^{(1)}_{ij}$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(\frac{\partial^2}{\partial x^2}\phi(n-i-2)(m-j)\Big)\frac{\partial^2}{\partial y^2}\nu^{(1)}_{ij}$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\Big(\frac{\partial^2}{\partial y^2}W_{(n-i-l-2)(m-j)}\Big)A_{2,l}\psi^{(1)}_{ij}$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\Big(\frac{\partial^2}{\partial y^2}\nu^{(1)}_{(n-i-l-4)(m-j)}\Big)A_{2,l}\psi^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\Big(\frac{\partial^2}{\partial y^2}\nu^{(2)}_{(n-i-l-4)(m-j)}\Big)A_{2,l}\psi^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\sum_{r=0}^{2}B_{2,r}\nu^{(3)}_{(n-i-l-r-2)(m-j)}A_{2,l}\psi^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\sum_{r=0}^{2}\tilde{B}_{2,r}\nu^{(4)}_{(n-i-l-r-2)(m-j)}A_{2,l}\psi^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\left(\frac{\partial^2}{\partial y^2}\nu^{(1)}_{(n-i-l-4)(m-j)}\right)\tilde{A}_{2,l}\psi^{(2)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\left(\frac{\partial^2}{\partial y^2}\nu^{(1)}_{(n-i-6)(m-j)}\right)\frac{\partial^2}{\partial x^2}\psi^{(3)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\left(\frac{\partial^2}{\partial y^2}\nu^{(1)}_{(n-i-6)(m-j)}\right)\frac{\partial^2}{\partial x^2}\psi^{(4)}_{ij}$$

$$-2\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\left(\frac{\partial^2}{\partial x\partial y}\varphi_{(n-i-l-1)(m-j)}\right)A_{1,l}\frac{\partial}{\partial y}\nu^{(1)}_{ij}$$

$$-2\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\left(\frac{\partial^2}{\partial x\partial y}W_{(n-i-l-3)(m-j)}\right)A_{1,l}\frac{\partial}{\partial y}\psi^{(1)}_{ij}$$

$$-2\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\left(A_{1,r}\frac{\partial}{\partial y}\nu^{(1)}_{(n-i-l-r-4)(m-j)}\right)A_{1,l}\frac{\partial}{\partial y}\psi^{(1)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\left(\tilde{A}_{1,r}\frac{\partial}{\partial y}\nu^{(2)}_{(n-i-l-r-4)(m-j)}\right)A_{1,l}\frac{\partial}{\partial y}\psi^{(1)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}\left(\frac{\partial}{\partial x}B_{1,r}\nu^{(3)}_{(n-i-l-r-4)(m-j)}\right)A_{1,l}\frac{\partial}{\partial y}\psi^{(3)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\left(\frac{\partial}{\partial x}\tilde{B}_{1,r}\nu^{(4)}_{(n-i-l-r-4)(m-j)}\right)A_{1,l}\frac{\partial}{\partial y}\psi^{(1)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\left(\tilde{A}_{1,r}\frac{\partial}{\partial y}\psi^{(2)}_{(n-i-l-r-4)(m-j)}\right)A_{1,l}\frac{\partial}{\partial y}\nu^{(1)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\left(\frac{\partial}{\partial x}B_{1,r}\psi^{(3)}_{(n-i-l-r-4)(m-j)}\right)A_{1,l}\frac{\partial}{\partial y}\nu^{(1)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\left(\frac{\partial}{\partial x}\tilde{B}_{1,r}\psi^{(4)}_{(n-i-l-r-4)(m-j)}\right)A_{1,l}\frac{\partial}{\partial y}\nu^{(1)}_{ij}$$

$$=0 \tag{4.60}$$

$$\sum_{i=0}^{4}A_{4,i}\psi^{(1)}_{(n-i)m}+\delta_1\sum_{l=0}^{2}A_{2,l}\frac{\partial^2}{\partial y^2}\psi^{(1)}_{(n-l-2)m}$$

$$+\delta_2\Big[\frac{\partial^4}{\partial y^4}\psi^{(1)}_{(n-4)m}-2\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\Big(\frac{\partial^2}{\partial x\partial y}W_{(n-i-l-1)(m-j)}\Big)A_{1,l}\frac{\partial}{\partial y}\nu^{(1)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\Big(A_{1,r}\frac{\partial}{\partial y}\nu^{(1)}_{(n-i-l-r-2)(m-j)}\Big)A_{1,l}\frac{\partial}{\partial y}\nu^{(1)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\Big(\tilde{A}_{1,r}\frac{\partial}{\partial y}\nu^{(2)}_{(n-i-l-r-2)(m-j)}\Big)A_{1,l}\frac{\partial}{\partial y}\nu^{(1)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\Big(\frac{\partial}{\partial x}B_{1,r}\nu^{(3)}_{(n-i-l-r-2)(m-j)}\Big)A_{1,l}\frac{\partial}{\partial y}\nu^{(1)}_{ij}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{1}\sum_{r=0}^{1}\Big(\frac{\partial}{\partial x}\tilde{B}_{1,r}\nu^{(4)}_{(n-i-l-r-2)(m-j)}\Big)A_{1,l}\frac{\partial}{\partial y}\nu^{(1)}_{ij}$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(\frac{\partial^2}{\partial x^2}W_{(n-i-2)(m-j)}\Big)\frac{\partial^2}{\partial y^2}\nu^{(1)}_{ij}$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\Big(\frac{\partial^2}{\partial y^2}W_{(n-i-l)(m-j)}\Big)A_{2,l}\nu^{(1)}_{ij}$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\Big(\frac{\partial^2}{\partial y^2}\nu^{(1)}_{(n-i-l-2)(m-j)}\Big)A_{2,l}\nu^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\Big(\frac{\partial^2}{\partial y^2}\nu^{(2)}_{(n-i-l-2)(m-j)}\Big)A_{2,l}\nu^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\sum_{r=0}^{2}B_{2,r}\nu^{(3)}_{(n-i-l-r)(m-j)}A_{2,l}\nu^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\sum_{r=0}^{2}\tilde{B}_{2,r}\nu^{(4)}_{(n-i-l-r)(m-j)}A_{2,l}\nu^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{2}\tilde{A}_{2,l}\nu^{(2)}_{(n-i-l-2)(m-j)}\frac{\partial^2}{\partial y^2}\nu^{(l)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(\frac{\partial^2}{\partial x^2}\nu^{(3)}_{(n-i)(m-j)}\Big)\frac{\partial^2}{\partial y^2}\nu^{(1)}_{ij}$$

$$+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(\frac{\partial^2}{\partial x^2}\nu^{(4)}_{(n-i)(m-j)}\Big)\frac{\partial^2}{\partial y^2}\nu^{(1)}_{ij}\Big]=0 \quad (4.61)$$

$\nu^{(2)}_{nm}$，$\nu^{(3)}_{nm}$，$\nu^{(4)}_{nm}$和$\psi^{(2)}_{nm}$，$\psi^{(3)}_{nm}$，$\psi^{(4)}_{nm}$的递推式与上面的递推式有相同的形式。

五、挠度函数和应力函数的渐近解

在(4.30)式中的$\delta_2=\frac{E_2}{E_1}<1$，将其视为小参数，由正则摄动法[8]求解$W_{00}$和$\varphi_{00}$，令

$$W_{00}=\sum_{i=0}^{P}\delta_2^i W_{00i}(x,\ y) \tag{5.1}$$

$$\varphi_{00}=\sum_{i=0}^{P}\delta_2^i \varphi_{00i}(x,\ y) \tag{5.2}$$

将(5.1)和(5.2)代入(4.29)～(4.31)式得：

$$\frac{\partial^2}{\partial y^2}\left(\sum_{i=0}^{P}\delta_2^i\varphi_{00i}\right)\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)+\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{P}\delta_2^i\varphi_{00i}\right)\frac{\partial^2}{\partial y^2}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)$$

$$-2\frac{\partial^2}{\partial x\partial y}\left(\sum_{i=0}^{P}\delta_2^i\varphi_{00i}\right)\frac{\partial^2}{\partial x\partial y}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)=-q(x,\ y) \tag{5.3}$$

$$\frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{P}\delta_2^i\varphi_{00i}\right)+\delta_1\frac{\partial^4}{\partial x^2\partial y^2}\left(\sum_{i=0}^{P}\delta_2^i\varphi_{00i}\right)$$

$$+\delta_2\left[\frac{\partial^4}{\partial y^4}\left(\sum_{i=0}^{P}\delta_2^i\varphi_{00i}\right)-\frac{\partial^2}{\partial x\partial y}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\frac{\partial^2}{\partial x\partial y}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right.$$

$$\left.+\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\frac{\partial^2}{\partial y^2}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]=0 \tag{5.4}$$

$$\left.\begin{array}{l}
\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\Big|_{x=0}=0,\quad \left[\frac{\partial}{\partial x}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\Big|_{x=0}=0\\
\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\Big|_{x=1}=0,\quad \left[\frac{\partial}{\partial x}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\Big|_{x=1}=0\\
\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\Big|_{y=0}=0,\quad \left[\frac{\partial}{\partial y}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\Big|_{y=0}=0\\
\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\Big|_{y=b/a}=0,\quad \left[\frac{\partial}{\partial y}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\Big|_{y=b/a}=0\\
\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\Big|_{\substack{x=0\\y=0}}=0,\quad \left[\frac{\partial}{\partial x}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\Big|_{\substack{x=0\\y=0}}=0
\end{array}\right\}$$

$$\left.\left[\frac{\partial}{\partial y}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\right|_{\substack{x=0\\y=0}}=0$$

$$\left.\left[\sum_{i=0}^{P}\delta_2^i W_{00i}\right]\right|_{\substack{x=1\\y=0}}=0,\quad \left.\left[\frac{\partial}{\partial x}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\right|_{\substack{x=1\\y=0}}=0$$

$$\left.\left[\frac{\partial}{\partial y}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\right|_{\substack{x=1\\y=0}}=0$$

$$\left.\left[\sum_{i=0}^{P}\delta_2^i W_{00i}\right]\right|_{\substack{x=0\\y=b/a}}=0,\quad \left.\left[\frac{\partial}{\partial x}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\right|_{\substack{x=0\\y=b/a}}=0$$

$$\left.\left[\frac{\partial}{\partial y}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\right|_{\substack{x=0\\y=b/a}}=0$$

$$\left.\left[\sum_{i=0}^{P}\delta_2^i W_{00i}\right]\right|_{\substack{x=1\\y=b/a}}=0,\quad \left.\left[\frac{\partial}{\partial x}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\right|_{\substack{x=1\\y=b/a}}=0$$

$$\left.\left[\frac{\partial}{\partial y}\left(\sum_{i=0}^{P}\delta_2^i W_{00i}\right)\right]\right|_{\substack{x=1\\y=b/a}}=0$$

$$\left.\left[\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{P}\delta_2^i \varphi_{00i}\right)-\nu_{21}\frac{\partial^2}{\partial y^2}\left(\sum_{i=0}^{P}\delta_2^i \varphi_{00i}\right)\right]\right|_{x=0}=0$$

$$\left.\left[\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{P}\delta_2^i \varphi_{00i}\right)-\nu_{21}\frac{\partial^2}{\partial y^2}\left(\sum_{i=0}^{P}\delta_2^i \varphi_{00i}\right)\right]\right|_{x=1}=0$$

$$\left.\left[\frac{\partial^2}{\partial y^2}\left(\sum_{i=0}^{P}\delta_2^i \varphi_{00i}\right)-\nu_{12}\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{P}\delta_2^i \varphi_{00i}\right)\right]\right|_{y=0}=0$$

$$\left.\left[\frac{\partial^2}{\partial y^2}\left(\sum_{i=0}^{P}\delta_2^i \varphi_{00i}\right)-\nu_{12}\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{P}\delta_2^i \varphi_{00i}\right)\right]\right|_{y=b/a}=0 \tag{5.5}$$

比较(5.3)～(5.5)式的δ_2的同次幂系数，我们获得W_{00k}和φ_{00k}的递推方程和递推边界条件：

$$\frac{\partial^2\varphi_{000}}{\partial y^2}\frac{\partial^2 W_{000}}{\partial x^2}+\frac{\partial^2\varphi_{000}}{\partial x^2}\frac{\partial^2 W_{000}}{\partial y^2}-2\frac{\partial^2\varphi_{000}}{\partial x\partial y}\frac{\partial^2 W_{000}}{\partial x\partial y}=-q(x,\ y) \tag{5.6}$$

$$\frac{\partial^4\varphi_{000}}{\partial x^4}+\delta_1\frac{\partial^4\varphi_{000}}{\partial x^2\partial y^2}=0 \tag{5.7}$$

$$W_{000}\Big|_{x=0}=0;\quad \frac{\partial W_{000}}{\partial x}\Big|_{x=0}=0;\quad W_{00}\Big|_{x=1}=0$$

$$\frac{\partial W_{000}}{\partial x}\Big|_{x=1}=0;\quad W_{000}\Big|_{y=0}=0;\quad \frac{\partial W_{000}}{\partial y}\Big|_{y=0}=0$$

$$\left.\begin{aligned}
&W_{000}\Big|_{y=b/a}=0;\quad \frac{\partial W_{000}}{\partial y}\Big|_{y=b/a}=0\\
&W_{000}\Big|_{\substack{x=0\\y=0}}=0;\quad \frac{\partial W_{000}}{\partial x}\Big|_{\substack{x=0\\y=0}}=0,\quad \frac{\partial W_{000}}{\partial y}\Big|_{\substack{x=0\\y=0}}=0\\
&W_{000}\Big|_{\substack{x=1\\y=0}}=0;\quad \frac{\partial W_{000}}{\partial x}\Big|_{\substack{x=1\\y=0}}=0;\quad \frac{\partial W_{000}}{\partial y}\Big|_{\substack{x=1\\y=0}}=0\\
&W_{000}\Big|_{\substack{x=0\\y=b/a}}=0;\quad \frac{\partial W_{000}}{\partial x}\Big|_{\substack{x=0\\y=b/a}}=0;\quad \frac{\partial W_{000}}{\partial y}\Big|_{\substack{x=0\\y=b/a}}=0\\
&W_{000}\Big|_{\substack{x=1\\y=b/a}}=0;\quad \frac{\partial W_{000}}{\partial x}\Big|_{\substack{x=1\\y=b/a}}=0;\quad \frac{\partial W_{000}}{\partial y}\Big|_{\substack{x=1\\y=b/a}}=0\\
&\left[\frac{\partial^2\varphi_{000}}{\partial x^2}-\nu_{21}\frac{\partial^2\varphi_{000}}{\partial y^2}\right]\Big|_{x=0}=0\\
&\left[\frac{\partial^2\varphi_{000}}{\partial x^2}-\nu_{21}\frac{\partial^2\varphi_{000}}{\partial y^2}\right]\Big|_{x=1}=0\\
&\left[\frac{\partial^2\varphi_{000}}{\partial y^2}-\nu_{21}\frac{\partial^2\varphi_{000}}{\partial x^2}\right]\Big|_{y=0}=0,\left[\frac{\partial^2\varphi_{000}}{\partial y^2}-\nu_{12}\frac{\partial^2\varphi_{000}}{\partial x^2}\right]\Big|_{y=b/a}=0
\end{aligned}\right\}\tag{5.8}$$

$$\sum_{i=0}^{n}\frac{\partial^2\varphi_{00(n-i)}}{\partial y^2}\frac{\partial^2 W_{00i}}{\partial x^2}+\sum_{i=0}^{n}\frac{\partial^2\varphi_{00(n-i)}}{\partial x^2}\frac{\partial^2 W_{00i}}{\partial y^2}$$

$$-2\sum_{i=0}^{n}\frac{\partial^2\varphi_{00(n-i)}}{\partial x\partial y}\frac{\partial^2 W_{00i}}{\partial x\partial y}=0\tag{5.9}$$

$$\frac{\partial^4}{\partial x^4}\varphi_{00n}+\delta_1\frac{\partial^4}{\partial x^2\partial y^2}\varphi_{00n}+\delta_2\left[\frac{\partial^4}{\partial y^4}\varphi_{00n}-\sum_{i=0}^{n}\frac{\partial^2 W_{00(n-i)}}{\partial x\partial y}\frac{\partial^2 W_{00i}}{\partial x\partial y}\right.$$

$$\left.+\sum_{i=0}^{n}\frac{\partial^2 W_{00(n-i)}}{\partial x^2}\frac{\partial^2 W_{00i}}{\partial y^2}\right]=0\tag{5.10}$$

$$\left.\begin{aligned}
&[W_{00n}]\Big|_{x=0}=0;\quad \left[\frac{\partial W_{00n}}{\partial x}\right]\Big|_{x=0}=0;\quad [W_{00n}]\Big|_{x=1}=0\\
&\left[\frac{\partial W_{00n}}{\partial x}\right]\Big|_{x=1}=0;\quad [W_{00n}]\Big|_{y=0}=0;\quad \left[\frac{\partial W_{00n}}{\partial y}\right]\Big|_{y=0}=0\\
&[W_{00n}]\Big|_{y=b/a}=0;\quad \left[\frac{\partial W_{00n}}{\partial y}\right]\Big|_{y=b/a}=0\\
&[W_{00n}]\Big|_{\substack{x=0\\y=0}}=0;\quad \left[\frac{\partial W_{00n}}{\partial x}\right]\Big|_{\substack{x=0\\y=0}}=0;\quad \left[\frac{\partial W_{003}}{\partial y}\right]\Big|_{\substack{x=0\\y=0}}=0\\
&[W_{00n}]\Big|_{\substack{x=1\\y=0}}=0;\quad \left[\frac{\partial W_{00n}}{\partial x}\right]\Big|_{\substack{x=1\\y=0}}=0;\quad \left[\frac{\partial W_{00n}}{\partial y}\right]\Big|_{\substack{x=1\\y=0}}=0\\
&[W_{00n}]\Big|_{\substack{x=0\\y=b/a}}=0;\quad \left[\frac{\partial W_{00n}}{\partial x}\right]\Big|_{\substack{x=0\\y=b/a}}=0;\quad \left[\frac{\partial W_{00n}}{\partial y}\right]\Big|_{\substack{x=0\\y=b/a}}=0
\end{aligned}\right\}\tag{5.11}$$

$$\left.\begin{array}{l}
[W_{00n}]\Big|_{\substack{x=1\\y=b/a}}=0;\quad \left[\dfrac{\partial W_{00n}}{\partial x}\right]\Big|_{\substack{x=1\\y=b/a}}=0;\quad \left[\dfrac{\partial W_{00n}}{\partial y}\right]\Big|_{\substack{x=1\\y=b/a}}=0\\
\left[\dfrac{\partial^2\varphi_{00n}}{\partial x^2}-\nu_{21}\dfrac{\partial^2\varphi_{00n}}{\partial y^2}\right]\Big|_{x=0}=0\\
\left[\dfrac{\partial^2\varphi_{00n}}{\partial x^2}-\nu_{21}\dfrac{\partial^2\varphi_{00n}}{\partial y^2}\right]\Big|_{x=1}=0\\
\left[\dfrac{\partial^2\varphi_{00n}}{\partial y^2}-\nu_{21}\dfrac{\partial^2\varphi_{00n}}{\partial x^2}\right]_{y=0}=0\\
\left[\dfrac{\partial^2\varphi_{00n}}{\partial y^2}-\nu_{12}\dfrac{\partial^2\varphi_{00n}}{\partial x^2}\right]_{y=b/a}=0
\end{array}\right\}$$

利用分离变量法，可以求得(5.7)式满足边界条件的解为：

$$\varphi_{000}=\left[\frac{\delta_1+\nu_{21}}{\nu_{21}}\left(\left(1-\exp\left[n\pi\sqrt{\delta_1}\frac{a}{b}\right]\right)c_1+\left(1-\exp\left[-n\pi\sqrt{\delta_1}\frac{a}{b}\right]\right)c_2\right)x\right.$$
$$-\frac{\delta_1+\nu_{21}}{\nu_{21}}c'+c_1\exp\left[n\pi\sqrt{\delta_1}\frac{ax}{b}\right]$$
$$\left.+c_2\exp\left[-n\pi\sqrt{\delta_1}\frac{ax}{b}\right]\right]\sin\frac{n\pi a}{b}y \tag{5.12}$$

其中 c_1，c_2是待定系数，$c'=c_1+c_2$。将φ_{000}代入(5.6)式，得：

$$\frac{n^2\pi^2a^2}{b^2}X(x)\sin\frac{n\pi a}{b}y\frac{\partial^2W_{000}}{\partial x^2}-X''(x)\sin\frac{n\pi a}{b}y\frac{\partial^2W_{000}}{\partial y^2}$$
$$+\frac{n\pi a}{b}X'(x)\cos\frac{n\pi a}{b}y\frac{\partial^2W_{000}}{\partial x\partial y}=q(x,\ y) \tag{5.13}$$

其中 $X(x)$是φ_{000}中的x部分。

如果$q(x,\ y)$给定，我们可以采用付里叶级数法求出W_{000}；再利用(5.9)～(5.11)可求得所有W_{00n}和φ_{00n}，即求得了W_{00}和φ_{00}，将W_{00}和φ_{00}代入(4.32)和(4.33)式(取$n=1$，$m=0$，且带有负下标的量均为0)，我们获得关于W_{10}和φ_{10}微分方程。解微分方程，并利用边界条件，可得W_{10}和φ_{10}。这样可逐次求得W_{nm}，φ_{nm}($n=0$，1，…，N；$m=0$，1，…，M)

将φ_{00}代入(4.58a)和(4.58b)得：

$$u_x^4\frac{\partial^4\nu_{00}^{(1)}}{\partial\xi^4}-\varphi_{00,yy}u_x^2\frac{\partial^2\nu_{00}^{(1)}}{\partial\xi^2}=0 \tag{5.14a}$$

$$\tilde{u}_x^4\frac{\partial^4\nu_{00}^{(2)}}{\partial\tilde{\xi}^4}-\varphi_{00,yy}\tilde{u}_x^2\frac{\partial^2\nu_{00}^{(2)}}{\partial\tilde{\xi}^2}=0 \tag{5.14b}$$

若取待定函数$u(x,\ y)$和$\tilde{u}(x,\ y)$为

$$u(x,\ y)=\int_0^x\sqrt{\varphi_{00,yy}(x,\ y)}\,dx,\quad \tilde{u}(x,\ y)=\int_x^1\sqrt{\varphi_{00,yy}(x,\ y)}\,dx \tag{5.15}$$

则可获得

$$\nu_{00}^{(1)}=c_0^{(1)}(\eta,\ y)\exp[-\xi]=c_0^{(1)}(\eta,\ y)\exp\left[-\frac{1}{\varepsilon_1}\int_0^x\sqrt{\varphi_{00,yy}(x,\ y)}\,dx\right] \tag{5.16}$$

$$\nu_{00}^{(2)}=c_0^{(2)}(\tilde{\eta},\ y)\exp[-\tilde{\xi}]=c_0^{(2)}(\tilde{\eta},\ y)\exp\left[-\frac{1}{\varepsilon_1}\int_x^1\sqrt{\varphi_{00,yy}(x,\ y)}\,dx\right] \tag{5.17}$$

由(4.58c)和(4.58d)得：

$$\delta_2 p_y^4 \frac{\partial^4 \nu_{00}^{(3)}}{\partial \alpha^4} - \varphi_{00,yy} p_y^2 \frac{\partial^2 \nu_{00}^{(3)}}{\partial \alpha^2} = 0 \tag{5.14c}$$

$$\delta_2 \tilde{p}_y^4 \frac{\partial^4 \nu_{00}^{(4)}}{\partial \tilde{\alpha}^4} - \varphi_{00,xx} \tilde{p}_y^2 \frac{\partial^2 \nu_{00}^{(4)}}{\partial \tilde{\alpha}^2} = 0 \tag{5.14d}$$

若取待定函数$p(x, y)$和$\tilde{p}(x, y)$为

$$p(x, y) = \frac{1}{\sqrt{\delta_2}} \int_0^y \sqrt{\varphi_{00,xx}(x, y)}\, dy, \quad \tilde{p}(x, y) = \frac{1}{\sqrt{\delta_2}} \int_y^{b/a} \sqrt{\varphi_{00,xx}(x, y)}\, dy \tag{5.18}$$

则可获得

$$\nu_{00}^{(3)} = c_0^{(3)}(x, \beta) \exp[-\alpha] = c_0^{(3)}(x, \beta) \exp\left[-\frac{1}{\varepsilon_1 \sqrt{\varepsilon_2}} \int_0^y \sqrt{\varphi_{00,xx}(x, y)}\, dy\right] \tag{5.19}$$

$$\nu_{00}^{(4)} = c_0^{(4)}(x, \tilde{\beta}) \exp[-\tilde{\alpha}] = c_0^{(4)}(x, \tilde{\beta}) \exp\left[-\frac{1}{\varepsilon_1 \sqrt{\delta_2}} \int_y^{b/a} \sqrt{\varphi_{00,xx}(x, y)}\, dy\right] \tag{5.20}$$

将$\nu_{00}^{(1)}$，$\nu_{00}^{(2)}$，$\nu_{00}^{(3)}$，$\nu_{00}^{(4)}$代入方程(4.59)，我们获得

$$\psi_{00}^{(1)} = -\frac{\delta_2 c_0^{(1)}(x, y)}{\varphi_{00,yy}} \Big[W_{00,yy} + \frac{1}{2\delta_2} \varphi_{00,xx} c_0^{(3)}(x, \beta) \exp[-\alpha] + \frac{1}{2\delta_2} \varphi_{00,xx} c_0^{(4)}(x, \tilde{\beta}) \exp[-\tilde{\alpha}] \Big] e^{-\xi} \tag{5.21}$$

$$\psi_{00}^{(2)} = -\frac{\delta_2 c_0^{(2)}(\tilde{\eta}, y)}{\varphi_{00,yy}} \Big[W_{00,yy} + \frac{1}{2\delta_2} \varphi_{00,xx} c_0^{(3)}(x, \beta) \exp[-\alpha] + \frac{1}{2\delta_2} \varphi_{00,xx} c_0^{(4)}(x, \beta) \exp[-\tilde{\alpha}] \Big] \exp[-\tilde{\xi}] \tag{5.22}$$

$$\psi_{00}^{(3)} = -\frac{1}{\delta_2} \frac{c_0^{(3)}(x, \beta)}{\varphi_{00,xx}} \Big[W_{00,xx} + \frac{1}{2} \varphi_{00,yy} c_0^{(1)}(\eta, y) e^{-\xi} + \frac{1}{2} \varphi_{00,yy} c_0^{(2)}(\tilde{\eta}, y) \exp[-\tilde{\xi}] \Big] \exp[-\alpha] \tag{5.23}$$

$$\psi_{00}^{(4)} = -\frac{1}{\delta_2} \frac{c_0^{(4)}(x, \tilde{\beta})}{\varphi_{00,xx}} \Big[W_{00,xx} + \frac{1}{2} \varphi_{00,yy} c_0^{(1)}(\eta, y) e^{-\xi} + \frac{1}{2} \varphi_{00,yy} c_0^{(2)}(\tilde{\eta}, y) \exp[-\tilde{\xi}] \Big] \exp[-\tilde{\alpha}] \tag{5.24}$$

由(4.35)，(4.37)，(4.39)，(4.41)式，我们可获得$c_0^{(1)}(\eta, y)$，$c_0^{(2)}(\tilde{\eta}, y)$，$c_0^{(3)}(x, \beta)$和$c_0^{(4)}(x, \tilde{\beta})$的边界条件

$$c_0^{(1)}(\eta, y)\Big|_{\eta=0} = \frac{1}{\sqrt{\varphi_{00,yy}}} \frac{\partial W_{10}}{\partial x}\Big|_{x=0} \tag{5.25}$$

$$c_0^{(2)}(\tilde{\eta}, y)\Big|_{\tilde{\eta}=1} = \frac{1}{\sqrt{\varphi_{00,yy}}} \frac{\partial W_{10}}{\partial x}\Big|_{x=1} \tag{5.26}$$

$$c_0^{(3)}(x, \beta)\Big|_{\beta=0} = \sqrt{\frac{\delta_2}{\varphi_{00,xx}}} \frac{\partial W_{10}}{\partial y}\Big|_{y=0} \tag{5.27}$$

$$c_0^{(4)}(x,\tilde{\beta})\Big|_{\tilde{\beta}=b/a}=\sqrt{\frac{\delta_2}{\varphi_{00\cdot xx}}}\frac{\partial W_{10}}{\partial y}\Big|_{y=b/a} \tag{5.28}$$

将$\nu_{00}^{(1)}$，$\nu_{00}^{(2)}$，$\nu_{00}^{(3)}$，$\nu_{00}^{(4)}$，$\psi_{00}^{(1)}$，$\psi_{00}^{(2)}$，$\nu_{00}^{(3)}$，$\psi_{00}^{(4)}$，W_{00}，W_{10}，φ_{00}，φ_{10}代入方程(4.60)和(4.61)(取$n=1$，$m=0$，且带负下标的项均取零)，获得$\nu_{10}^{(1)}$和$\psi_{10}^{(1)}$的微分方程，再极分便可获得边界层型函数$\nu_{10}^{(1)}$和$\psi_{10}^{(1)}$．同确定$c_0^{(1)}(\eta,y)$和$c_0^{(2)}(\tilde{\eta},y)$一样，我们可以确定$c_1^{(1)}(\eta,y)$和$c_1^{(2)}(\tilde{\eta},y)$．按照以上步骤，我们可以逐渐获得$\nu_{nm}^{(1)}$，$\nu_{nm}^{(2)}$，$\nu_{nm}^{(3)}$，$\nu_{nm}^{(4)}$，$\psi_{nm}^{(1)}$，$\psi_{nm}^{(2)}$，$\psi_{nm}^{(3)}$，$\psi_{nm}^{(4)}(n=0,1,2,\cdots,N;\ m=0,1,2,\cdots,M)$．

获得W_{nm}，φ_{nm}，$\nu_{nm}^{(1)}$，$\nu_{nm}^{(2)}$，$\nu_{nm}^{(3)}$，$\nu_{nm}^{(4)}$，$\psi_{nm}^{(1)}$，$\psi_{nm}^{(2)}$，$\psi_{nm}^{(3)}$，$\psi_{nm}^{(4)}(n=0,1,\cdots,N;m=0,1,\cdots,M)$后，将它们代入(4.1)和(4.2)式，获得边值问题(2.4)的对ε_1为N阶而对ε_2为M阶的一致有效渐近解．

六、讨　论

本文利用“两变量法”，并在$G<E_2<E_1$的假设下，引进了三个小参数，对四边固定的正交各向异性矩形板的非线性弯曲问题进行了研究．应用这种方法所求得的结果较文献[7]有很大的进展．在本文中，得到了薄膜力的一个渐近解，这弥补了文献[7]没有薄膜力解的不足．又在$G<E_2<E_1$的假设下，我们引进三个小参数ε_1，ε_2和δ_2，获得了较文[7]更精确的边界层型函数$\psi_{00}^{(1)}$，$\psi_{00}^{(2)}$，$\psi_{00}^{(3)}$，$\psi_{00}^{(4)}$的表示式．

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The Proplems of Nonliear Bending for Orthotropic Rectangular Plate with Four Clamped Edges

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Abstract

In this paper, under the the non-uniform transverse load, the problems of nonliear bending for orthotropic rectangular plate are studied by using "the method of two-variable"[1] and "the method of mixing perturbation"[2]. The uniformly valid asymptotic solutions of Nth-order for ε_1 and Mth-order for ε_2 for orthotropic rectangular plate with four clamped edges are obtained.

Key words orthotropic rectangular plate, nonliear bending, method of two-variable, method of mixing perturbation, uniformly valid asymptotic solutions