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横观各向同性弹性层点力解

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摘　要

本文根据弹性层状结构的传递矩阵法思想，由横观各向同性弹性力学基本方程，导出了含应力和位移两类变量的混合方程，利用Fourier 变换和文献[7]的位移函数通解，以及计算机代数软件，得到了横观各向同性层的点力解，这个点力解可直接退化到各同性情形的解。

关键词　横观各向同性　弹性层　点力解　Fourier变换

一、引　言

横观各向同性材料在工程上的应用十分广泛，尤其是大多数纤维增强复合材料都表现出横观各向同性的弹性关系．弹性材料基本解的研究是开展力学解析分析的基础性研究，也是建立边界积分方程的必要条件．因此有很多作者在这方面做了大量工作；但对横观各向同性材料，大部分研究是针对无限体或半无限体[1~4]，文[5]研究了横观各向同性弹性层表面受载平衡问题，文[6]研究了各向同性弹性层的点力解．本文对横观各向同性弹性力学基本方程按传递矩阵法的思想，导出了混合方程．利用文[7]给出的横观各向同性弹性力学的通解，得到了经Fourier变换后的混合方程的通解．这样，可以利用Fourier反变换，得到弹性层问题的点力解，在推导过程中充分利用了计算机代数软件；全面考察了各种类型的横观各向同性材料，所得结果可直接退化到各向同性情况．

二、横观各向同性弹性层混合平衡方程及Fourier变换

厚度为H的弹性层，建立如下直角坐标系：上表面即为xy坐标平面，z轴垂直上表面向下．则弹性层上下表面只出现以下6个变量

$$\mathbf{a}=[\sigma_{zz},\tau_{xz},\tau_{yz},w,u,v]^T \tag{2.1}$$

同时，对横观各向同性材料，有如下的平衡方程、几何方程和本构方程：

$$\sigma_{ij,j}+f_i=0 \tag{2.2}$$

$$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2 \tag{2.3}$$

$$\sigma_{xx}=c_{11}\varepsilon_x+c_{12}\varepsilon_y+c_{13}\varepsilon_z,\quad \sigma_{yy}=c_{12}\varepsilon_x+c_{11}\varepsilon_y+c_{13}\varepsilon_z \tag{2.4a}$$

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$$\left.\begin{aligned}&\sigma_{zz}=c_{13}\varepsilon_x+c_{13}\varepsilon_y+c_{33}\varepsilon_z\\&\tau_{yz}=2c_{44}\varepsilon_{yz},\ \tau_{xz}=2c_{44}\varepsilon_{xz},\ \tau_{xy}=c_{66}\varepsilon_{xy}\end{aligned}\right\}\tag{2.4b}$$

式中：

$$c_{11}=c_{12}+2c_{66}\tag{2.5}$$

按传递矩阵法思想，可以将基本方程改写成如下形式：

$$\partial \mathbf{a}/\partial z=\mathbf{A}\mathbf{a}+\mathbf{c},\ \ \mathbf{b}=\mathbf{B}\mathbf{a}\tag{2.6}$$

式中：

$$\mathbf{c}=[-f_z,-f_x,-f_y,0,0,0]^T\tag{2.7a}$$

$$\mathbf{b}=[\sigma_{xx}+\sigma_{yy},\sigma_{xx}-\sigma_{yy},\tau_{xy}]^T\tag{2.7b}$$

而矩阵$\mathbf{A}$是6×6微分算子矩阵，$\mathbf{B}$是3×6微分算子矩阵。对式(2.6)进行Fourier变换：

$$\bar{f}(\alpha,\beta)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)e^{i(\alpha x+\beta y)}dxdy\tag{2.8}$$

得到常微分方程组：

$$\frac{d\bar{\mathbf{a}}}{dz}=\bar{\mathbf{A}}\bar{\mathbf{a}}+\bar{\mathbf{c}}\tag{2.9}$$

和线性代数方程组

$$\bar{\mathbf{b}}=\bar{\mathbf{B}}\bar{\mathbf{a}}\tag{2.10}$$

式(2.9)和(2.10)中

$$\bar{\mathbf{a}}=[\bar{\sigma}_{zz},\bar{\tau}_{xz},\bar{\tau}_{yz},\bar{w},\bar{u},\bar{v}]^T\tag{2.11a}$$

$$\bar{\mathbf{b}}=[\bar{\sigma}_{xx}+\bar{\sigma}_{yy},\bar{\sigma}_{xx}-\bar{\sigma}_{yy},\bar{\tau}_{xy}]^T\tag{2.11b}$$

$$\bar{\mathbf{c}}=[-\bar{f}_z,-\bar{f}_x,-\bar{f}_y,0,0,0]^T\tag{2.11c}$$

$$\bar{\mathbf{A}}=\begin{bmatrix}0 & i\alpha & i\beta & 0 & 0 & 0\\ i\alpha\dfrac{c_{13}}{c_{33}} & 0 & 0 & 0 & c_{66}\beta^2-\dfrac{c_{13}^2-c_{11}c_{33}}{c_{33}}\alpha^2 & -\dfrac{c_{13}^2+c_{33}c_{66}-c_{11}c_{33}}{c_{33}}\alpha\beta\\ i\beta\dfrac{c_{13}}{c_{33}} & 0 & 0 & 0 & -\dfrac{c_{13}^2+c_{33}c_{66}-c_{11}c_{33}}{c_{33}}\alpha\beta & c_{66}\alpha^2-\dfrac{c_{13}^2-c_{11}c_{33}}{c_{33}}\beta^2\\ \dfrac{1}{c_{33}} & 0 & 0 & 0 & i\alpha\dfrac{c_{13}}{c_{33}} & i\beta\dfrac{c_{13}}{c_{33}}\\ 0 & \dfrac{1}{c_{44}} & 0 & i\alpha & 0 & 0\\ 0 & 0 & \dfrac{1}{c_{44}} & i\beta & 0 & 0\end{bmatrix}\tag{2.11d}$$

$$\bar{\mathbf{B}}=\begin{bmatrix}\dfrac{2c_{13}}{c_{33}} & 0 & 0 & 0 & -2i\alpha\left(c_{11}-c_{66}-\dfrac{c_{13}^2}{c_{33}}\right) & -2i\beta\left(c_{11}-c_{66}-\dfrac{c_{13}^2}{c_{33}}\right)\\ 0 & 0 & 0 & 0 & -2i\alpha c_{66} & 2i\beta c_{66}\\ 0 & 0 & 0 & 0 & -i\beta c_{66} & -i\alpha c_{66}\end{bmatrix}\tag{2.11e}$$

三、Fourier变换下混合方程通解

式(2.9)是含6个变量的常系数微分方程组，其通解形式可以表示为

$$\bar{\mathbf{a}}=\mathbf{X}(z)\,\mathbf{X}^{-1}(0)\bar{\mathbf{a}}_0+\mathbf{X}(z)\int_0^z\mathbf{X}^{-1}(s)\bar{\mathbf{c}}(s)ds \tag{3.1}$$

式中$\bar{\mathbf{a}}_0$为$\bar{\mathbf{a}}(z)$在$z=0$处的值，$\mathbf{X}(z)$为式(2.9)齐次微分方程组的基解矩阵。

按文[7]的式(54)，横观各向同性材料的位移可用两个位移函数Ψ和F表示：

$$u=\frac{\partial\Psi}{\partial y}-\left(c_{44}\nabla_1^2+c_{33}\frac{\partial^2}{\partial z^2}\right)\frac{\partial F}{\partial x} \tag{3.2a}$$

$$v=-\frac{\partial\Psi}{\partial x}-\left(c_{44}\nabla_1^2+c_{33}\frac{\partial^2}{\partial z^2}\right)\frac{\partial F}{\partial y} \tag{3.2b}$$

$$w=(c_{13}+c_{44})\nabla_1^2\frac{\partial F}{\partial z} \tag{3.2c}$$

$$\left(c_{66}\nabla_1^2+c_{44}\frac{\partial^2}{\partial z^2}\right)\Psi=0 \tag{3.2d}$$

$$\left(s_1^2\nabla_1^2+\frac{\partial^2}{\partial z^2}\right)\left(s_2^2\nabla_1^2+\frac{\partial^2}{\partial z^2}\right)F=0 \tag{3.2e}$$

式中

$$\nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{3.3a}$$

$$s_1=\sqrt{\frac{(\bar{c}_{13}-c_{13})(\bar{c}_{13}+c_{13}+2c_{44})}{4c_{33}c_{44}}}+\sqrt{\frac{(\bar{c}_{13}+c_{13})(\bar{c}_{13}-c_{13}-2c_{44})}{4c_{33}c_{44}}} \tag{3.3b}$$

$$s_2=\sqrt{\frac{(\bar{c}_{13}-c_{13})(\bar{c}_{13}+c_{13}+2c_{44})}{4c_{33}c_{44}}}-\sqrt{\frac{(\bar{c}_{13}+c_{13})(\bar{c}_{13}-c_{13}-2c_{44})}{4c_{33}c_{44}}} \tag{3.3c}$$

以及

$$\bar{c}_{13}=\sqrt{c_{11}c_{33}} \tag{3.3d}$$

对式(3.2)作Fourier 变换，在假设$r^2=x^2+y^2\to\infty$时，Ψ和F的本身及有关偏导数趋于零的情况下，其中式(3.2d)和式(3.2e)有如下形式：

$$\left(-c_{66}\rho^2+c_{44}\frac{d^2}{dz^2}\right)\bar{\Psi}=0 \tag{3.4a}$$

$$\left(-\rho^2s_1^2+\frac{d^2}{dz^2}\right)\left(-\rho^2s_2^2+\frac{d^2}{dz^2}\right)\bar{F}=0 \tag{3.4b}$$

方程(3.4a)有解：

$$\bar{\Psi}=k_{01}\exp[-\rho s_0z]+k_{02}\exp[\rho s_0z] \tag{3.5}$$

式中$s_0=\sqrt{c_{66}/c_{44}}$，k_{01}和k_{02}是待定常数。对方程(3.4b)有如下形式的解：

当$s_1\neq s_2$时：

$$\bar{F}=\bar{F}_1+\bar{F}_2 \tag{3.6}$$

式中：

$$\bar{F}_i=k_{i1}\exp[-\rho s_iz]+k_{i2}\exp[\rho s_iz]\qquad(i=1,2) \tag{3.7}$$

当$s_1=s_2=s$时：

$$\bar{F}=k_{11}\exp[-\rho s_1z]+k_{12}\exp[\rho s_1z]+k_{21}z\exp[-\rho s_2z]+k_{22}z\exp[\rho s_2z] \tag{3.8}$$

式(3.7)和式(3.8)中，k_{i1}和k_{i2}是待定常数。

将式(3.5)，(3.7)和(3.8)代入式(3.2a,b,c) 的Fourier变换式，再利用式 (2.3) 和 (2.4)的Fourier变换式，就得到$s_1\neq s_2$和$s_1=s_2=s$两种情形的基解矩阵：

（1）$s_1 \neq s_2$时

$$\mathbf{X}=\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ \beta & \beta & \frac{ias_1}{\rho} & -\frac{ias_1}{\rho} & \frac{ias_2}{\rho} & -\frac{ias_2}{\rho} \\ -\alpha & -\alpha & \frac{i\beta s_1}{\rho} & -\frac{i\beta s_1}{\rho} & \frac{i\beta s_2}{\rho} & -\frac{i\beta s_2}{\rho} \\ 0 & 0 & -\frac{b_{21}s_1}{\rho} & \frac{b_{21}s_1}{\rho} & -\frac{b_{22}s_2}{\rho} & \frac{b_{22}s_2}{\rho} \\ -\frac{\beta}{\rho c_{44}s_0} & \frac{\beta}{\rho c_{44}s_0} & -\frac{iab_{31}}{\rho^2} & -\frac{iab_{31}}{\rho^2} & -\frac{iab_{32}}{\rho^2} & -\frac{iab_{32}}{\rho^2} \\ \frac{\alpha}{\rho c_{44}s_0} & -\frac{\alpha}{\rho c_{44}s_0} & -\frac{i\beta b_{31}}{\rho^2} & -\frac{i\beta b_{31}}{\rho^2} & -\frac{i\beta b_{32}}{\rho^2} & -\frac{i\beta b_{32}}{\rho^2} \end{bmatrix}$$

$$\cdot \mathrm{diag}\{\exp[-\rho s_0 z],\exp[\rho s_0 z],\exp[-\rho s_1 z],\exp[\rho s_1 z],\exp[-\rho s_2 z],\exp[\rho s_2 z]\} \tag{3.9a}$$

式中：

$$b_{21}=\frac{c_{13}+c_{44}}{c_{44}(c_{13}+c_{33}s_1^2)},\quad b_{22}=\frac{c_{13}+c_{44}}{c_{44}(c_{13}+c_{33}s_2^2)} \tag{3.9b}$$

$$b_{31}=\frac{c_{33}s_1^2-c_{44}}{c_{44}(c_{13}+c_{33}s_1^2)},\quad b_{32}=\frac{c_{33}s_2^2-c_{44}}{c_{44}(c_{13}+c_{33}s_2^2)} \tag{3.9c}$$

（2）$s_1=s_2=s$时

$$\mathbf{X}=\begin{bmatrix} 0, & 0, & 1, & 1, & z, & z \\ \beta, & \beta, & \frac{ias}{\rho}, & -\frac{ias}{\rho}, & -\frac{ia}{\rho^2}+\frac{iasz}{\rho}, & -\frac{ia}{\rho^2}-\frac{iasz}{\rho} \\ -\alpha, & -\alpha, & \frac{i\beta s}{\rho}, & -\frac{i\beta s}{\rho}, & -\frac{i\beta}{\rho^2}+\frac{i\beta sz}{\rho}, & -\frac{i\beta}{\rho^2}-\frac{i\beta sz}{\rho} \\ 0, & 0, & -\frac{s}{2\rho c_{44}}, & \frac{s}{2\rho c_{44}}, & -\frac{1}{2(c_{13}+c_{44})\rho^2}-\frac{sz}{2\rho c_{44}}, & -\frac{1}{2(c_{13}+c_{44})\rho^2}+\frac{sz}{2\rho c_{44}} \\ -\frac{\beta}{\rho c_{44}s_0}, & \frac{\beta}{\rho c_{44}s_0}, & -\frac{ai}{2\rho^2c_{44}}, & -\frac{ia}{2\rho^2c_{44}}, & \frac{iasc_{33}}{2c_{44}(c_{13}+c_{44})\rho^3}-\frac{iaz}{2\rho^2c_{44}}, & -\frac{iasc_{33}}{2c_{44}(c_{13}+c_{44})\rho^3}-\frac{iaz}{2\rho^2c_{44}} \\ \frac{\alpha}{\rho c_{44}s_0}, & -\frac{\alpha}{\rho c_{44}s_0}, & -\frac{i\beta}{2\rho^2c_{44}}, & -\frac{i\beta}{2\rho^2c_{44}}, & \frac{i\beta sc_{33}}{2c_{44}(c_{13}+c_{44})\rho^3}-\frac{i\beta z}{2\rho^2c_{44}}, & -\frac{i\beta sc_{33}}{2c_{44}(c_{13}+c_{44})\rho^3}-\frac{i\beta z}{2\rho^2c_{44}} \end{bmatrix}$$

$$\mathrm{diag}\{\exp[-\rho s_0 z],\exp[\rho s_0 z],\exp[-\rho sz],\exp[\rho sz],\exp[-\rho sz],\exp[\rho sz]\} \tag{3.10}$$

四、横观各向同性弹性层受集中体力作用时的通解

只需考虑两种受力情形：1．在点$(0,0,h)$作用z向单位集中力，即$f_x=f_y=0, f_z=\delta(x)\delta(y)\delta(z-h)$，则$\bar{\mathbf{c}}[-1/2\pi,0,0,0,0,0]^T\delta(z-h)$；2．在点$(0,0,h)$作用$x$向单位集中力，即$f_x=\delta(x)\delta(y)\delta(z-h), f_y=f_z=0$，则$\bar{\mathbf{c}}=[0,-1/2\pi,0,0,0,0]^T\delta(z-h)$。

对应式(3.1)的积分项：

$$\mathbf{X}(z)\int_0^z \mathbf{X}^{-1}(s)\bar{\mathbf{C}}(s)ds = N(z-h)\bar{\mathbf{R}} \tag{4.1}$$

式中

$$N(z-h)=\begin{cases}0 & \forall z<h\\ 1 & \forall z\geqslant h\end{cases} \tag{4.2}$$

$$\bar{\mathbf{R}} = \mathbf{X}(z)\overline{\mathbf{XV}} \tag{4.3}$$

其中$\overline{\mathbf{XV}}$的表达式分两种情形分别如下：

情况1 单位集中力沿z方向

$$\overline{\mathbf{XV}} = \mathbf{X}^{-1}(h)[-1/2\pi,0,0,0,0,0]^T$$

$$=\begin{cases}[0,0,b_{41}\exp[\rho hs_1],b_{41}\exp[-\rho hs_1],-b_{42}\exp[\rho hs_2],-b_{42}\exp[-\rho hs_2]]^T & s_1\neq s_2\\ \left[0,0,\dfrac{b_{33}\rho h-s}{4\pi s}e^{\rho hs},\ -\dfrac{b_{33}\rho h+s}{4\pi s}e^{-\rho hs},\ -\dfrac{b_{33}\rho}{4\pi s}e^{\rho hs},\ \dfrac{b_{33}\rho}{4\pi s}e^{-\rho hs}\right]^T & s_1=s_2\end{cases} \tag{4.4a}$$

$$b_{33}=\frac{c_{13}+c_{44}}{c_{33}},\ b_{41}=\frac{(c_{13}+c_{33}s_1^2)(c_{33}s_2^2-c_{44})}{4\pi c_{33}(c_{13}+c_{44})(s_1^2-s_2^2)},\ b_{42}=\frac{(c_{13}+c_{33}s_2^2)(c_{33}s_1^2-c_{44})}{4\pi c_{33}(c_{13}+c_{44})(s_1^2-s_2^2)} \tag{4.4b}$$

情况2 单位集中力沿x方向

$$\overline{\mathbf{XV}} = \mathbf{X}^{-1}(h)[0,-1/2\pi,0,0,0,0]^T$$

$$=\begin{cases}\Big[-\dfrac{\beta}{4\pi\rho^2}\exp[\rho hs_0],\ -\dfrac{\beta}{4\pi\rho^2}\exp[-\rho hs_0],\ -\dfrac{iab_{61}}{\rho s_1}\exp[\rho hs_1],\ \dfrac{iab_{61}}{\rho s_1}\exp[-\rho hs_1], \\ \quad \dfrac{iab_{62}}{\rho s_2}\exp[\rho hs_2],\ -\dfrac{iab_{62}}{\rho s_2}\exp[-\rho hs_2]\Big]^T, \quad s_1\neq s_2\\ \Big[-\dfrac{\beta}{4\pi\rho^2}\exp[\rho hs_0],-\dfrac{\beta}{4\pi\rho^2}\exp[-\rho hs_0],\dfrac{iasc_{44}}{4\pi\rho c_{11}}e^{\rho hs},\ -\dfrac{iasc_{44}}{4\pi\rho c_{11}}e^{-\rho hs}, \\ \quad -\dfrac{ia(c_{13}+c_{44})}{4\pi s^4 c_{33}}e^{\rho hs},\ -\dfrac{ia(c_{13}+c_{44})}{4\pi s^4 c_{33}}e^{-\rho hs}\Big]^T \\ \quad +h\left[0,0,\dfrac{ias^2(c_{13}+c_{44})}{4\pi c_{11}}e^{\rho hs},\ \dfrac{ias^2(c_{13}+c_{44})}{4\pi c_{11}}e^{-\rho hs},0,0\right]^T, \quad s_1=s_2=s\end{cases} \tag{4.5a}$$

$$b_{61}=-\frac{(c_{13}+c_{33}s_1^2)}{4\pi c_{33}(s_1^2-s_2^2)},\ b_{62}=-\frac{(c_{13}+c_{33}s_2^2)}{4\pi c_{33}(s_1^2-s_2^2)} \tag{4.5b}$$

五、自由边界弹性层点力解

由式(3.1)和(4.1)，得横观各向同性弹性层在Fourier变换下的通解：

$$\bar{\mathbf{a}} = \mathbf{T}\bar{\mathbf{a}}_0 + N(z-h)\bar{\mathbf{R}} \tag{5.1}$$

式中：

$$\mathbf{T}(z)=\mathbf{X}(z)\mathbf{X}^{-1}(0) \tag{5.2}$$

将式(5.1)进一步写成分块矩阵形式：

$$\begin{bmatrix}\bar{\mathbf{\Sigma}}\\ \bar{\mathbf{U}}\end{bmatrix}=\begin{bmatrix}\mathbf{T}_{11}(z) & \mathbf{T}_{12}(z)\\ \mathbf{T}_{21}(z) & \mathbf{T}_{22}(z)\end{bmatrix}\begin{bmatrix}\bar{\mathbf{\Sigma}}_0\\ \bar{\mathbf{U}}_0\end{bmatrix}+\begin{bmatrix}N(z-h)\bar{\mathbf{R}}_1\\ N(z-h)\bar{\mathbf{R}}_2\end{bmatrix} \tag{5.3}$$

式中$\bar{\boldsymbol{\Sigma}}=[\bar{\sigma}_{zz},\bar{\tau}_{xz},\bar{\tau}_{yz}]^T$，$\bar{\mathbf{U}}=[w,u,v]^T$等．由于$0\leqslant h\leqslant H$，在$z=H$处有

$$\begin{bmatrix}\bar{\boldsymbol{\Sigma}}_H\\ \bar{\mathbf{U}}_H\end{bmatrix}=\begin{bmatrix}\mathbf{T}_{11}(H) & \mathbf{T}_{12}(H)\\ \mathbf{T}_{21}(H) & \mathbf{T}_{22}(H)\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{\Sigma}}_0\\ \bar{\mathbf{U}}_0\end{bmatrix}+\begin{bmatrix}\bar{\mathbf{R}}_1(H)\\ \bar{\mathbf{R}}_2(H)\end{bmatrix} \tag{5.4}$$

可得：

$$\bar{\mathbf{U}}_0=\mathbf{T}_{12}^{-1}(H)[\bar{\boldsymbol{\Sigma}}_H-\mathbf{T}_{11}(H)\bar{\boldsymbol{\Sigma}}_0-\mathbf{R}_1(H)] \tag{5.5}$$

将式(5.5)代入式(5.3)，并命$\bar{\boldsymbol{\Sigma}}_0=\bar{\boldsymbol{\Sigma}}_H=0$，即得：

$$\begin{bmatrix}\bar{\boldsymbol{\Sigma}}\\ \bar{\mathbf{U}}\end{bmatrix}=N(z-h)\bar{\mathbf{R}}(z)+\bar{\mathbf{R}}_{\Delta}(z) \tag{5.6}$$

式中：

$$\bar{\mathbf{R}}_{\Delta}=\begin{bmatrix}-\mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(H)\bar{\mathbf{R}}_1(H)\\ -\mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(H)\bar{\mathbf{R}}_1(H)\end{bmatrix} \tag{5.7}$$

对式(5.6)进行Fourier逆变换，即能得到$\mathbf{a}$的表达式，可以写成如下的形式：

$$\mathbf{a}=\mathbf{a}_{\infty}+\mathbf{a}_h \tag{5.8}$$

式中$\mathbf{a}_{\infty}$是无限体受集中力时的基本解[1,2]，$\mathbf{a}_h$是有关弹性层的附加项，在附录中将分情形写出$\mathbf{a}_h$的表达式．篇幅有限，在附录中只列出位移表达式．

六、结　论

在各向同性情况下，文献[6]也采用混合方程及Fourier 变换法求得弹性层的点力解，但对横观各向同性材料，要利用直接求特征值及特征向量的方法求得基解矩阵有一定的难度；本文利用文献［7］的通解，成功地解决了这一问题，同时又利用计算机代数软件包MATHEMATICAL进行繁琐的推导工作，保证了结果的正确性．另外，本文还分别考虑了特征值不等及相等这两种情况，有利于各种横观各向同性材料的处理，同时，本文的解可直接退化到各同向性情形．

附　录

下文中的上标z，x标志单位集中力的方向．

（1） 当$s_1\neq s_2$时

$$u^z=\frac{x}{r}\int_0^{\infty} f_u^z J_1(\rho r)d\rho,\quad v^z=\frac{y}{x}u^z \tag{A·1}$$

式中：

$$f_u^z=\frac{1}{t}(m_{31}r_1^z+m_{32}r_2^z)-b_{61}\exp[\rho s_1(z-h)]+b_{62}\exp[\rho s_2(z-h)] \tag{A·2}$$

$$t=2-2\mathrm{ch}(H\rho s_1)\mathrm{ch}(H\rho s_2)+\left(\frac{s_1}{s_2}+\frac{s_2}{s_1}\right)\mathrm{sh}(H\rho s_1)\mathrm{sh}(H\rho s_2) \tag{A·3}$$

$$r_1^z=2b_{41}\mathrm{ch}[(H-h)\rho s_1]-2b_{42}\mathrm{ch}[(H-h)\rho s_2] \tag{A·4}$$

$$r_2^z=-2b_{41}\mathrm{sh}[(H-h)\rho s_1]+2b_{42}s_2\mathrm{sh}[(H-h)\rho s_2] \tag{A·5}$$

$$b_{61}=b_{62}=\frac{c_{13}+c_{44}}{4\pi c_{33}c_{44}(s_1^2-s_2^2)} \tag{A·6}$$

$$m_{31}=-b_{31}\{\mathrm{ch}(H\rho s_2)\mathrm{ch}(\rho s_1 z)-\mathrm{ch}[\rho s_1(H-z)]-\frac{s_2}{s_1}\mathrm{sh}(H\rho s_2)\mathrm{sh}(\rho s_1 z)\}$$
$$-b_{32}\{\mathrm{ch}(H\rho s_1)\mathrm{ch}(\rho s_2 z)-\mathrm{ch}[\rho s_2(H-z)]-\frac{s_1}{s_2}\mathrm{sh}(H\rho s_1)\mathrm{sh}(\rho s_2 z)\} \tag{A·7}$$

$$m_{32}=\frac{b_{31}}{s_1}\{\mathrm{sh}[\rho s_1(H-z)]+\mathrm{ch}(H\rho s_2)\mathrm{sh}(\rho s_1 z)\}-\frac{b_{31}}{s_2}\mathrm{ch}(\rho s_1 z)\mathrm{sh}(H\rho s_2)$$
$$+\frac{b_{32}}{s_2}\{\mathrm{sh}[\rho s_2(H-z)]+\mathrm{ch}(H\rho s_1)\mathrm{sh}(\rho s_2 z)\}-\frac{b_{32}}{s_1}\mathrm{ch}(\rho s_2 z)\mathrm{sh}(H\rho s_1) \quad (\text{A}\cdot 8)$$

$$w^z=\int_0^\infty f_w^z J_0(\rho r)d\rho \quad (\text{A}\cdot 9)$$

式中：

$$f_w^z=\frac{1}{t}(m_{41}r_1^z+m_{42}r_2^z)+b_{71}s_1\exp[\rho s_1(z-h)]-b_{72}s_2\exp[\rho s_2(z-h)] \quad (\text{A}\cdot 10)$$

$$b_{71}=\frac{c_{33}s_2^2-c_{44}}{4\pi c_{33}c_{44}(s_1^2-s_2^2)},\quad b_{72}=\frac{c_{33}s_1^2-c_{44}}{4\pi c_{33}c_{44}(s_1^2-s_4^2)} \quad (\text{A}\cdot 11)$$

$$m_{41}=b_{21}s_1\{\mathrm{sh}[\rho s_1(H-z)]+\mathrm{ch}(H\rho s_2)\mathrm{sh}(\rho s_1 z)\}-b_{21}s_2\mathrm{ch}(\rho s_1 z)\mathrm{sh}(H\rho s_2)$$
$$+b_{22}s_2\{\mathrm{sh}[\rho s_2(H-z)]+\mathrm{ch}(H\rho s_1)\mathrm{sh}(\rho s_2 z)\}-b_{22}s_1\mathrm{ch}(\rho s_2 z)\mathrm{sh}(H\rho s_1) \quad (\text{A}\cdot 12)$$

$$m_{42}=-b_{21}\{\mathrm{ch}(H\rho s_2)\mathrm{ch}(\rho s_1 z)-\mathrm{ch}[\rho s_1(H-z)]-\frac{s_1}{s_2}\mathrm{sh}(H\rho s_2)\mathrm{sh}(\rho s_1 z)\}$$
$$-b_{22}\{\mathrm{ch}(H\rho s_1)\mathrm{ch}(\rho s_2 z)-\mathrm{ch}[\rho s_2(H-z)]-\frac{s_2}{s_1}\mathrm{sh}(H\rho s_1)\mathrm{sh}(\rho s_2 z)\} \quad (\text{A}\cdot 13)$$

$$u^z=\frac{y^2}{r^2}\int_0^\infty f_{u0}^z J_0(\rho r)d\rho+\frac{x^2}{r^2}\int_0^\infty f_u^z J_0(\rho r)d\rho$$
$$-\frac{y^2-x^2}{r^3}\left[\int_0^\infty f_{u0}^z J_1(\rho r)/\rho\, d\rho-\int_0^\infty f_u^z J_1(\rho r)/\rho\, d\rho\right] \quad (\text{A}\cdot 14)$$

式中：

$$f_{u0}^z=\frac{\mathrm{ch}[(H-h)\rho s_0]\mathrm{sh}(\rho s_0 z)}{2\pi c_{44}s_0\mathrm{sh}(H\rho s_0)}-\frac{\exp[\rho s_0(z-h)]}{4\pi c_{44}s_0} \quad (\text{A}\cdot 15)$$

$$f_u^z=\frac{1}{t}(-m_{31}r_1^z+m_{32}r_2^z)-b_{72}/s_1\exp[\rho s_1(z-h)]+b_{71}/s_2\exp[\rho s_2(z-h)] \quad (\text{A}\cdot 16)$$

上式中t同式(A.3)，m_{ij}同式(A·7)和(A·8)，而

$$r_1^z=\frac{2b_{51}}{s_1}\mathrm{sh}[(H-h)\rho s_1]-\frac{2b_{52}}{s_2}\mathrm{sh}[(H-h)\rho s_2] \quad (\text{A}\cdot 17)$$

$$r_2^z=2b_{51}\mathrm{ch}[(H-h)\rho s_1]-2b_{52}\mathrm{ch}[(H-h)\rho s_2] \quad (\text{A}\cdot 18)$$

$$v^z=\frac{xy}{r^2}\left\{\int_0^\infty\left[f_u^z-f_{u0}^z\right]J_0(\rho r)d\rho-\frac{2}{r}\int_0^\infty\left[f_u^z-f_{u0}^z\right]J_1(\rho r)/\rho\, d\rho\right\} \quad (\text{A}\cdot 19)$$

$$w^x=\frac{x}{r}\int_0^\infty f_w^x J_1(\rho r)d\rho \quad (\text{A}\cdot 20)$$

式中：

$$f_w^x=\frac{1}{t}(m_{41}r_1^x-m_{42}r_2^x)-b_{61}\exp[\rho s_1(z-h)]+b_{62}\exp[\rho s_2(z-h)] \quad (\text{A}\cdot 21)$$

（2） 当$s_1=s_2=s$时

位移u^z，u^x等的表达式形式分别与式（A·1），（A·9），（A·14），（A·19）和（A·20）相同，而其中的f_u^z，f_u^x，m_{ij}，t等的表达式则分别表示如下：

$$f_u^z=\frac{1}{2c_{44}t}(m_{31}r_1^z+m_{32}r_2^z)-\frac{\rho(z-h)(c_{13}+c_{44})}{8\pi s c_{33}c_{44}}e^{\rho s(z-h)} \quad (\text{A}\cdot 22)$$

$$t=H^2\rho^2 s^2-\mathrm{sh}(H\rho s)^2 \quad (\text{A}\cdot 23)$$

$$r_1^z=-\frac{1}{2\pi\rho}\mathrm{ch}[(H-h)\rho s]+\frac{c_{13}+c_{44}}{2\pi c_{33}s}(H-h)\mathrm{sh}[(H-h)\rho s] \quad (\text{A}\cdot 24)$$

$$r_2^z=\frac{c_{44}}{2\pi\rho c_{33}s}\mathrm{sh}[(H-h)\rho s]-\frac{c_{13}+c_{44}}{2\pi c_{33}}(H-h)\mathrm{ch}[(H-h)\rho s] \quad (\text{A}\cdot 25)$$

$$m_{31}=H\rho^3s^2z\,\mathrm{ch}[\rho s(H-z)]+\frac{\rho c_{44}}{c_{13}+c_{44}}\mathrm{sh}(H\rho s)\mathrm{sh}(\rho sz)+\rho^2 sz\,\mathrm{sh}(H\rho s)\mathrm{ch}(\rho sz)$$

$$-\frac{H\rho^2s^3c_{33}}{c_{13}+c_{44}}\mathrm{sh}[\rho s(H-z)]-H\rho^2 s\,\mathrm{ch}(H\rho s)\mathrm{sh}(\rho sz) \qquad (\mathrm{A}\cdot 26)$$

$$m_{32}=\frac{\rho c_{33}s}{c_{13}+c_{44}}\mathrm{ch}(\rho sz)\mathrm{sh}(H\rho s)+H\rho^3sz\,\mathrm{sh}[\rho s(H-z)]$$

$$-(H-z)\rho^2\mathrm{sh}(H\rho s)\mathrm{sh}(\rho sz)-\frac{H\rho^2s^2c_{33}}{c_{13}+c_{44}}\mathrm{ch}[\rho s(H-z)] \qquad (\mathrm{A}\cdot 27)$$

$$f^z_w=\frac{1}{2c_{44}t}(m_{41}r_1^z+m_{42}r_2^z)-\left[\frac{c_{13}+3c_{44}}{8\pi sc_{33}c_{44}}-\frac{\rho(z-h)(c_{13}+c_{44})}{8\pi_{33}c_{44}}\right]e^{\rho s(z-h)} \qquad (\mathrm{A}\cdot 28)$$

$$m_{41}=\frac{\rho c_{33}s^3}{c_{13}+c_{44}}\mathrm{ch}(\rho sz)\mathrm{sh}(H\rho s)+H\rho^3s^3z\,\mathrm{sh}[\rho s(H-z)]+\frac{H\rho^2s^2c_{44}}{c_{13}+c_{44}}\mathrm{ch}[\rho s(H-z)]$$

$$+\rho^2s^2[H\mathrm{ch}(H\rho s)\mathrm{ch}(\rho sz)-z\,\mathrm{sh}(H\rho s)\mathrm{sh}(\rho sz)] \qquad (\mathrm{A}\cdot 29)$$

$$m_{42}=H\rho^3s^2z\,\mathrm{ch}[\rho s(H-z)]+\frac{\rho c_{44}}{c_{13}+c_{44}}\mathrm{sh}(H\rho s)\mathrm{sh}(\rho sz)$$

$$+(H-z)\rho^2s\,\mathrm{sh}(H\rho s)\mathrm{ch}(\rho sz)+\frac{H\rho^2sc_{44}}{c_{13}+c_{44}}\mathrm{sh}[\rho s(H-z)] \qquad (\mathrm{A}\cdot 30)$$

$$r_1^z=-\frac{sc_{44}}{2\pi\rho c_{11}}\mathrm{sh}[(H-h)\rho s]+\frac{c_{44}s^2-c_{11}}{2\pi c_{11}}(H-h)\mathrm{ch}[(H-h)\rho s] \qquad (\mathrm{A}\cdot 31)$$

$$r_2^z=-\frac{1}{2\pi\rho}\mathrm{ch}[(H-h)\rho s]-\frac{c_{13}+c_{44}}{2\pi sc_{33}}(H-h)\mathrm{sh}[(H-h)\rho s] \qquad (\mathrm{A}\cdot 32)$$

f^z_{u0}与式(A·15)相同。

$$f^z_u=\frac{1}{2c_{44}t}(-m_{31}r_1^z+m_{32}r_2^z)-\left[\frac{\rho(z-h)(c_{13}+c_{44})}{8\pi s^2c_{33}c_{44}}+\frac{(c_{11}+s^2c_{44})}{8\pi sc_{11}c_{44}}\right]e^{\rho s(z-h)} \qquad (\mathrm{A}\cdot 33)$$

$$f^z_w=\frac{1}{2c_{44}t}(m_{41}r_1^z-m_{42}r_2^z)-\frac{\rho(z-h)(c_{13}+c_{44})}{8\pi sc_{33}c_{44}}e^{\rho s(z-h)} \qquad (\mathrm{A}\cdot 34)$$

若令：

$$\left.\begin{aligned}&s_0=s_1=s_2=1,\quad c_{11}=c_{22}=c_{33}=\frac{(1-\gamma)E}{(1+\gamma)(1-2\gamma)}\\&c_{12}=c_{13}=\frac{\gamma E}{(1+\gamma)(1-2\gamma)},\quad c_{44}=c_{66}=\frac{E}{2(1+\gamma)}\end{aligned}\right\} \qquad (\mathrm{A}\cdot 35)$$

则可直接退化到各向同性解。

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Point Force Solution for a Transversely Isotropic Elastic Layer

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Abstract

By introduction of transmitting matrices' technique for layered structure, mixed equations with stresses and displacement are derived from the basic equations of transversely isotropic elasticity. Then, using Fourier transformation and the general solutions in Zhou et al.[7], the point force solution for transversely isotropic elastic layer is obtained and it can be degenerated to the corresponding solution of isotropic medium. In this paper, all equations are derived by the use of computer algebra software.

Key words transversely isotropy, elastic layer, point force solution, Fourier transformation