应用数学和力学编委会编
重庆出版社出版

边缘转动受弹性限制矩形板横向振动固有频率的一个近似解法

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（潘立宙推荐，1994年6月30日收到）

摘　要

本文给出了边缘转动受弹性限制矩形板横向振动固有频率的一个近似解法，不但公式简单易程序化，而且有着较高的精度，本文最后给出了一些算例，并与已有结果作了比较。

关键词　横向振动　固有频率　李兹法

一、引　言

在实际工程中，建筑、桥梁及船泊的一些结构往往可简化为边缘转动受弹性限制（边缘支承介于简支与固支之间）的矩形板，因而研究其横向振动很有实际意义。近年来，已有许多文章报道了边缘转动受弹性限制矩形板横向振动特性的研究结果，但主要限于基频的近似估算[1]、[2]、[3]，对高阶固有频率的计算还不多见。Laura[4]、[5]曾取多项式为位移函数近似计算边缘转动受弹性限制矩形板横向振动的基频，Warburton[6]将基函数取为简支梁函数与固支梁函数的叠加，用Rayleigh-Ritz法近似计算低阶固有频率，Mukhopadhyay[7]用半解析法[8]求解了高阶固有频率，计算结果有较好的精度。

李兹法是广泛使用的近似计算固有频率的一种有效方法，其精度完全取决于基函数的选取，本文针对边缘转动受弹性限制矩形板的支承特点，恰当地将基函数选择为简支梁函数与多项式函数的叠加，用李兹法近似计算固有频率，公式简单，易程序化，工作量小，计算结果表明本文方法有着很高的精度。

二、数　学　模　型

由板的振动理论可知，矩形薄板横向自主振动的微分方程为

$$\nabla^4 w+\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2}=0 \tag{2.1}$$

其中　w为板的横向位移，ρ，h分别是材料密度和板的厚度，$D=\dfrac{Eh^3}{12(1-\mu^2)}$为板的弯曲刚度，$E$，$\mu$分别是材料的弹性模量和泊松比。

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当矩形板边缘转动受如图1所示的弹性限制时，其横向振动的边界条件为

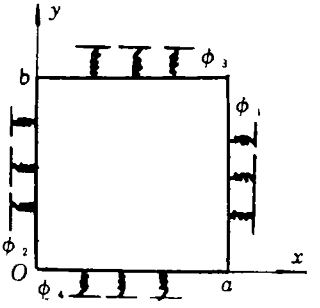

图1 边缘转动受弹性限制的矩形板

$$\left.\begin{aligned}
&w=0, \quad \frac{\partial w}{\partial x}=-\phi_1 D\left\{\frac{\partial^2 w}{\partial x^2}+\mu \frac{\partial^2 w}{\partial y^2}\right\}, \quad x=a \\
&w=0, \quad \frac{\partial w}{\partial x}=\phi_2 D\left\{\frac{\partial^2 w}{\partial x^2}+\mu \frac{\partial^2 w}{\partial y^2}\right\}, \quad x=0 \\
&w=0, \quad \frac{\partial w}{\partial y}=-\phi_3 D\left\{\frac{\partial^2 w}{\partial y^2}+\mu \frac{\partial^2 w}{\partial x^2}\right\}, \quad y=b \\
&w=0, \quad \frac{\partial w}{\partial y}=\phi_4 D\left\{\frac{\partial^2 w}{\partial y^2}+\mu \frac{\partial^2 w}{\partial x^2}\right\}, \quad y=0
\end{aligned}\right\} \tag{2.2}$$

其中 $\phi_i(i=1,2,3,4)$为矩形板四边转动的柔性系数，当$\phi_i(i=1,2,3,4)$取极限值(0或∞)则可得到固支边或简支边。

设$w=z(x,y)\sin(pt+\varphi)$，则图1所示矩形板的势能和动能分别为

$$\left.\begin{aligned}
U_{\max}=&\frac{D}{2}\iint\left\{(\nabla^2 z)^2-2(1-\mu)\left[\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2\right]\right\}dxdy \\
&+\frac{D}{2}\left\{\int_0^b \frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial x}\bigg|_{x=0}dy-\int_0^b \frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial x}\bigg|_{x=a}dy\right. \\
&\left.+\int_0^a \frac{\partial^2 z}{\partial y^2}\frac{\partial z}{\partial y}\bigg|_{y=0}dx-\int_0^a \frac{\partial^2 z}{\partial y^2}\frac{\partial z}{\partial y}\bigg|_{y=b}dx\right\} \\
T_{\max}=&\frac{1}{2}\rho h p^2\iint z^2 dxdy
\end{aligned}\right\} \tag{2.3}$$

其中 p为矩形板横向振动的固有圆频率，$z(x,y)$为横向位移函数。

由哈密顿原理可知有变分

$$\delta(U_{\max}-T_{\max})=0 \tag{2.4}$$

设横向位移函数$z(x,y)$可写成

$$z(x,y)=\sum_{m=1}^{q}\sum_{m=1}^{r}A_{mm}X_m(x)Y_n(y) \tag{2.5}$$

其中 $X_m(x)$称为x方向的基函数，$Y_n(y)$称为y方向的基函数，A_{mn}为待定函数，将(2.5)代入(2.4)，得到

$$\sum_{m=1}^{q}\sum_{n=1}^{r}\left[C_{mn}^{(ij)}-\lambda^2 m_{mn}^{(ij)}\right]A_{mn}=0 \quad \begin{pmatrix} i=1,2,\cdots,q \\ j=1,2,\cdots,r \end{pmatrix} \tag{2.6}$$

其中

$$\left.\begin{aligned}
&\lambda^2=\frac{\rho h a^3 b}{D}p^2\\
&C_{mn}^{(ij)}=a^2\iint\Big[(\nabla^2X_mY_n)(\nabla^2X_iY_i)-(1-\mu)\Big(\frac{d^2X_m}{dx^2}Y_nX_i\frac{d^2Y_i}{dy^2}\\
&\quad+X_m\frac{d^2Y_n}{dy^2}\frac{d^2X_i}{dx^2}Y_j-2\frac{dX_m}{dx}\frac{dY_n}{dy}\frac{dx_i}{dx}\frac{dY_j}{dy}\Big)\Big]dxdy\\
&\quad+\frac{a^2}{2}\Big\{\int_0^b Y_nY_j\Big(\frac{d^2X_m}{dx^2}\frac{dX_i}{dx}+\frac{d^2X_i}{dx^2}\frac{dX_m}{dx}\Big)\Big|_{x=0}dy-\int_0^b Y_nY_j\Big(\frac{d^2X_m}{dx^2}\frac{dX_i}{dx}\\
&\quad+\frac{d^2X_i}{dx^2}\frac{dX_m}{dx}\Big)\Big|_{x=a}dy+\int_0^a X_mX_i\Big(\frac{d^2Y_n}{dy^2}\frac{dY_j}{dy}+\frac{d^2Y_j}{dy^2}\frac{dY_n}{dy}\Big)\Big|_{y=0}dx\\
&\quad-\int_0^a X_mX_i\Big(\frac{d^2Y_n}{dy^2}\frac{dY_j}{dy}+\frac{d^2Y_j}{dy^2}\frac{dY_n}{dy}\Big)\Big|_{y=b}dx\Big\}\\
&m_{mn}^{(ij)}=\frac{1}{ab}\iint X_mY_nX_iY_jdxdy
\end{aligned}\right\}\tag{2.7}$$

考虑到边缘转动受弹性限制矩形板的支承特点，设

$$\left.\begin{aligned}
X_m(x)&=\sin\Big(\frac{m\pi}{a}x\Big)+\sum_{k=0}^{3}C_{mk}x^k\\
Y_n(y)&=\sin\Big(\frac{n\pi}{b}y\Big)+\sum_{k=0}^{3}D_{nk}y^k
\end{aligned}\right\}\tag{2.8}$$

其中 C_{mk}，$D_{nk}(k=0,1,2,3)$为待定常数。

令$X_m(x)$，$Y_n(y)$满足与矩形板相应的梁的边界条件，有

$$\left.\begin{aligned}
&X_m(x)=0,\quad \frac{dX_m}{dx}=-\phi_1D(1-\mu^2)\frac{d^2X_m}{dx^2}\quad(x=a)\\
&X_m(x)=0,\quad \frac{dX_m}{dx}=\phi_2D(1-\mu^2)\frac{d^2X_m}{dx^2}\quad(x=0)\\
&Y_n(y)=0,\quad \frac{dY_n}{dy}=-\phi_3D(1-\mu^2)\frac{d^2Y_n}{dy^2}\quad(y=b)\\
&Y_n(y)=0,\quad \frac{dY_n}{dy}=\phi_4D(1-\mu^2)\frac{d^2Y_n}{dy^2}\quad(y=0)
\end{aligned}\right\}\tag{2.9}$$

将(2.8)代入(2.9)可解得唯一的C_{mk}，$D_{nk}(k=0,1,2,3)$，此外，由于三角函数与多项式乘积的积分有解析式，因而很容易求得$C_{mn}^{(ij)}$及$m_{mn}^{(ij)}$的精确值。

三、算 例

考虑如图3所示三边简支一边转动受弹性限制的方板

令 $\xi=\frac{x}{a}$，$\eta=\frac{y}{a}$，(2.8)可写成

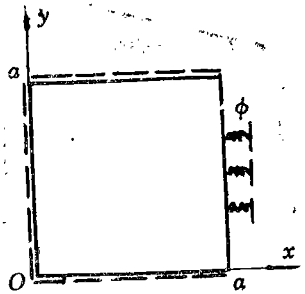

图2 三边简支一边转动受弹性限制的方板

$$\left.\begin{aligned} X_m(\xi)&=\sin m\pi\xi+\sum_{k=0}^{3}C_{mk}\xi^k \\ Y_n(\xi)&=\sin n\pi\eta+\sum_{k=0}^{3}D_{nk}\eta^k \end{aligned}\right\} \tag{3.1}$$

在(2.9)中，令$\phi_1=\phi$，$\alpha=\dfrac{\phi D}{a}$，$\phi_2=\phi_3=\phi_4=\infty$，有

$$\left.\begin{aligned} &X_m(\xi)=0, \quad \frac{dX_m}{d\xi}=-\alpha(1-\mu^2)\frac{d^2X_m}{d\xi^2} \quad (x=a) \\ &X_m(\xi)=0, \quad \frac{d^2X_m}{d\xi^2}=0 \quad (x=0) \\ &Y_n(\eta)=0, \quad \frac{d^2Y_n}{d\eta^2}=0 \quad (y=0,\ a) \end{aligned}\right\} \tag{3.2}$$

将(3.1)代入(3.2)中，可得到

$$\left.\begin{aligned} &C_{m0}=C_{m2}=0,\ C_{m1}=-C_{m3}=\frac{(-1)^m m\pi}{2(3\alpha(1-\mu^2)+1)} \\ &D_{nk}=0 \quad (k=0,1,2,3) \end{aligned}\right\} \tag{3.3}$$

容易求得

$$\left.\begin{aligned} &\int_0^1 X_i(\xi)X_j(\xi)d\xi=\frac{1}{2}\delta(i-j)-\frac{6}{\pi^3}\left(\frac{(-1)^i}{i^3}C_{j1}+\frac{(-1)^j}{j^3}C_{i1}\right)+\frac{8}{105}C_{i1}C_{j1} \\ &\int_0^1 X_i'(\xi)X_j'(\xi)d\xi=\frac{1}{2}ij\pi^2\delta(i-j)-\frac{6}{\pi}\left(\frac{(-1)^i}{i}C_{j1}+\frac{(-1)^j}{j}C_{i1}\right)+\frac{4}{5}C_{i1}C_{j1} \\ &\int_0^1 X_i''(\xi)X_j''(\xi)d\xi=\frac{1}{2}i^2j^2\pi^4\delta(i-j)-6\pi(i(-1)^iC_{j1}+j(-1)^jC_{i1})+12C_{i1}C_{i1} \\ &\int_0^1 X_i''(\xi)X_j(\xi)d\xi=-\frac{1}{2}i^2\pi^2\delta(i-j)+\frac{6}{\pi}\left(\frac{(-1)^i}{i}C_{j1}+\frac{(-1)^j}{j}C_{i1}\right)-\frac{4}{5}C_{i1}C_{j1} \\ &\int_0^1 Y_i(\eta)Y_j(\eta)d\eta=\frac{1}{2}\delta(i-j),\quad \int_0^1 Y_i'(\eta)Y_j'(\eta)d\eta=\frac{1}{2}ij\pi^2\delta(i-j) \\ &\int_0^1 Y_i''(\eta)Y_j''(\eta)d\eta=\frac{1}{2}i^2j^2\pi^4\delta(i-j),\quad \int_0^1 Y_i''(\eta)Y_j(\eta)d\eta=-\frac{1}{2}i^2\pi^2\delta(i-j) \end{aligned}\right\} \tag{3.4}$$

其中，$\delta(i-j)=\begin{cases}0 & (i\neq j)\\ 1 & (i=j)\end{cases}$，将上式代入(2.7)，求出$C_{mn}^{(ij)}$，$m_{mn}^{(ij)}$后，由(2.6)可求得各阶固有频率。

下面给出本文方法算出的前10阶固有频率，同时列出一些已有结果以供参考比较。

表 1　三边简支一边转动受弹性限制方板的前10阶固有频率 $\lambda=\left(\frac{\rho h a^4}{D}p^2\right)^{1/2}$

(($\mu=0.3$, $q=r=5$), [ ]取自文献[9], ( )取自文献[7])

α	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
0.0	23.647 [23.646]	51.678 [51.674]	58.647 [58.646]	86.145 [86.134]	100.284 [100.270]	113.230	133.835	140.851	168.990	187.544
0.01	23.393 (23.600)	51.459 (51.300)	57.815 (57.210)	85.353 (85.899)	100.089 (99.765)	111.531	133.092	139.196	168.810	184.684
0.05	22.636	50.879	55.543	83.366	99.626	107.323	131.392	135.345	168.419	178.261
0.1	22.037	50.486	53.960	82.118	99.351	104.757	130.432	133.149	168.211	174.800
0.5	20.600	49.719	50.843	79.917	98.890	100.460	128.911	129.675	167.901	169.680
1.0	20.222 (20.188)	49.549 (49.320)	50.156 (50.145)	79.469 (79.433)	98.799 (97.476)	99.629	128.624	129.026	167.845	168.773
5.0	19.846	49.391	49.521	79.065	98.718	98.892	128.372	128.455	167.796	167.988
10.0	19.793	49.370	49.435	79.011	98.707	98.794	128.338	128.381	167.790	167.886
100.0	19.745	49.350	49.357	78.962	98.697	98.706	128.308	128.312	167.784	167.794
1000.0	19.740 (19.460)	49.348 (49.210)	49.349 (49.348)	78.957 (79.877)	98.696 (97.214)	98.697	128.305	128.306	167.783	167.784
∞	19.739 [19.739]	49.348 [49.348]	49.348 (49.348)	78.957 [78.957]	98.696 [98.696]	98.696	128.305	128.305	167.783	167.783

四、讨　论

本文将边缘转动受弹性限制矩形板横向振动的基函数选择为简支梁函数和多项式函数的叠加，其本质是以简支梁函数作为基函数的主解，由于简支梁函数不满足边缘转动的弹性限制条件，因而选取多项式函数作为辅助解对其进行修正，使其满足全部边界条件。必须指出，基函数的主解并不仅限于简支梁函数，可取任意的梁函数，但为提高精度，应选取满足尽可能多边界条件的梁函数作基函数的主解，本文将基函数的主解选取为简支梁函数，不但能满足边界的横向位移条件，而且计算简单，工作量小，如选取其它梁函数作基函数的主解，工作量均会有较多的增加。此外，本文方法也可用来计算边缘有任意弹性支承矩形板横向振动的固有频率。

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An Approximate Solution of Eigen-Frequencies of Transverse Vibration of Rectangular Plates with Elastical Restraints

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Abstract

This paper presents an approximate solution for calculating eigen-frequencies of transverse vibration of rectangular plates with elastical restrained against rotation along edges. The formulae are not only very simple and easily programmed but also have high accuracy. Finally, some numerical results are given and compared with other results obtained.

Key words transverse vibration, eigen-frequencies, Ritz method