应用数学和力学，第17卷 第8期（1996年8月）
Applied Mathematics and Mechanics

关于Kähler流形上的Newton力学

张　荣　业[1]

（朱照宣推荐，1994年6月30日收到，1996年2月5日收到修改稿）

摘　要

本文讨论Kähler流形上的Newton力学．在此给出Newton定律、动能定律、动量定律、虚位移原理、D'Alembert-Lagrange原理、运动方程及“普遍运动方程”等的复的数学形式．

关键词　Kähler流形　连络　绝对微分　对偶对

一、引　论

古典力学或分析力学先后由S.I.Newton,J.Lagrange和W. Hamilton借助Euclid几何和微积分而建立(见[1])．随着现代微分几何之发展近人又把它建立在Riemann流形及其切丛和余切丛及辛流形上(见[2]、[3]、[4]、[5]、[6]),使之更精密更完美，并于力学、理论物理、相对论中广泛应用．这都是实的情形（即实力学系统）．复的情形如何？我们将在Kähler流形上建立复的力学系统——包括Newton 力学、Lagrange 力学、Hamilton力学．现先讨论Newton力学，余后续．

二、Kähler流形的表述

设M^n是具有连络D的n-维Kähler流形，在局部坐标系$(U,\ z^i)$下其度量为

$$h=h_{i\bar{k}}dz^j\otimes d\bar{z}^k \tag{2.1}$$

相应的Kähler形式为

$$\Omega=\frac{i}{2}h_{j\bar{k}}dz^j\wedge d\bar{z}^k \tag{2.2}$$

TM，T^*M为M^n的切丛及余切丛，它们在$U\subset M^n$上的标架场为$\{\partial/\partial z^j,\ \partial/\partial\bar{z}^j\}_{j=1}^n$，$\{dz^j,\ d\bar{z}^j\}_{j=1}^n$，且有分解

$$TM=TM^{1,0}\oplus TM^{0,1},\quad T^*M=T^*M^{1,0}\oplus T^*M^{0,1} \tag{2.3}$$

$\forall V\in\mathscr{X}(M)$在$U\subset M^n$上表为$V=v^i\partial/\partial z^j+v^{\bar{j}}\partial/\partial\bar{z}^j$．

$\forall\omega\in\mathscr{F}'(M)$在$U\subset M^n$上表为$\omega=b_jdz^j+b^{\bar{j}}d\bar{z}^j$．

$\mathscr{X}(M)$为TM的截面—M^n的向量场的空间；$\mathscr{F}'(M)=\mathscr{X}^*(M)$为$T^*M$的截面—$M^n$的1形

1 中国科学院数学研究所，北京 100080

式场的空间。$\forall p\in U$，$\{(\partial/\partial z^j)_p,\ (\partial/\partial \bar z^j)_p\}_{j=1}^n$，$\{(dz^j)_p,\ (d\bar z^j)_p\}_{j=1}^n$为纤维一切空间$T_p(M)$及余切空间$T_p^*(M)$的标架，$v^j, v^{\bar j}, b_j, b_{\bar j}$为相应的纤维坐标。为方便计$\partial/\partial z^j$，$d/dz^j$等的下标$p$均省略。有关微分几何及复流形的知识见陈省身[7]，Tanjiro Okubo[8]，R.O. Wells Jr.[9]及Kunihiko Kodaira[10]等的研究。

由于连络D与度量h相容：

$$dh_{j\bar k}=dh(\partial/\partial z^j,\ \partial/\partial \bar z^k)=h(D\partial/\partial z^j,\ \partial/\partial \bar z^k)+h(\partial/\partial z^j,\ D\partial/\partial \bar z^k) \tag{2.4}$$

$$Dh=0 \tag{2.5}$$

标架场$\partial/\partial z^j$，$\partial/\partial \bar z^j$，$dz^j$，$d\bar z^j$的绝对微分是：

$$D\partial/\partial z^j=\omega_j^l\partial/\partial z^l=\Gamma_{js}^l dz^s\otimes\partial/\partial z^l,\quad D\partial/\partial \bar z^j=\bar\omega_j^l\partial/\partial \bar z^l=\Gamma_{\bar j\bar s}^{\bar l}d\bar z^s\otimes d\bar z^l$$

$$Ddz^j=-\omega_l^j dz^l=-\Gamma_{ls}^j dz^s\otimes dz^l,\quad Dd\bar z^j=-\bar\omega_l^j d\bar z^l=-\Gamma_{\bar l\bar s}^{\bar j}d\bar z^s\otimes d\bar z^l$$

故(2.4) $\Rightarrow$ $dh_{j\bar k}=h(\omega_j^l\partial/\partial z^l,\ \partial/\partial \bar z^k)+h(\partial/\partial z^j,\ \bar\omega_k^s\partial/\partial \bar z^s)$

$$=\omega_j^l h_{l\bar k}+h_{j\bar s}\bar\omega_k^s$$

矩阵记法为

$$dH=\omega\cdot H+H\cdot\bar\omega^T \tag{2.6}$$

$\omega=(\omega_j^l)$为连络形式ω_j^l的矩阵，$H=(h_{j\bar k})$为度量张量$h_{j\bar k}$的矩阵，且$h_{j\bar k}=h_{\bar kj}=\overline{h_{k\bar j}}$，$\bar\omega^T$为$\omega$的共轭转置。

这样TM的截面—向量场$V=v^j\partial/\partial z^j+v^{\bar j}\partial/\partial \bar z^j$的绝对微分是

$$\begin{aligned}DV&=dv^j\partial/\partial z^j+v^jD\partial/\partial z^j+dv^{\bar j}\partial/\partial \bar z^j+v^{\bar j}D\partial/\partial \bar z^j\\&=(dv^j+v^l\omega_l^j)\partial/\partial z^j+(dv^{\bar j}+v^{\bar l}\bar\omega_l^j)\partial/\partial \bar z^j\\&=(dv^j+v^l\Gamma_{ls}^j dz^s)\partial/\partial z^j+(dv^{\bar j}+v^{\bar l}\Gamma_{\bar l\bar s}^{\bar j}d\bar z^s)\partial/\partial \bar z^j\\&=Dv^j\partial/\partial z^j+Dv^{\bar j}\partial/\partial \bar z^j\end{aligned} \tag{2.7}$$

$$\left[D\left(\frac{\partial}{\partial z^s}\right)\right]V=\left(\frac{\partial v^j}{\partial z^s}+v^l\Gamma_{ks}^j\right)\frac{\partial}{\partial z^j}+\frac{\partial v^{\bar j}}{\partial z^s}\frac{\partial}{\partial \bar z^j}=\frac{Dv^j}{\partial z^s}\frac{\partial}{\partial z^j}+\frac{Dv^{\bar j}}{\partial z^s}\frac{\partial}{\partial \bar z^j}$$

$$\left[D\left(\frac{\partial}{\partial \bar z^s}\right)\right]V=\left(\frac{\partial v^{\bar j}}{\partial \bar z^s}+v^{\bar l}\Gamma_{\bar l\bar s}^{\bar j}\right)\frac{\partial}{\partial \bar z^j}+\frac{\partial v^j}{\partial \bar z^s}\frac{\partial}{\partial z^j}=\frac{Dv^j}{\partial \bar z^s}\frac{\partial}{\partial z^j}+\frac{Dv^{\bar j}}{\partial \bar z^s}\frac{\partial}{\partial \bar z^j}$$

后者是V关于$\partial/\partial z^s$，$\partial/\partial \bar z^s$的绝对微商（协变微商）且

$$\begin{aligned}\frac{DV}{dt}&=\frac{Dv^j}{dt}\frac{\partial}{\partial z^j}+\frac{Dv^{\bar j}}{dt}\frac{\partial}{\partial \bar z^j}\\&=\frac{dz^s}{dt}\left[D\left(\frac{\partial}{\partial z^s}\right)\right]V+\frac{d\bar z^s}{dt}\left[D\left(\frac{\partial}{\partial \bar z^s}\right)\right]V\\&=\left[D\left(\frac{dz^s}{dt}\frac{\partial}{\partial z^s}+\frac{d\bar z^s}{dt}\frac{\partial}{\partial \bar z^s}\right)\right]V=D_XV\\&=\frac{dz^s}{dt}\left(\frac{\partial v^j}{\partial z^s}+v^l\Gamma_{ls}^j\right)\frac{\partial}{\partial z^j}+\frac{dz^j}{dt}\frac{\partial v^{\bar j}}{\partial z^s}\frac{\partial}{\partial \bar z^j}\\&\quad+\frac{d\bar z^s}{dt}\frac{\partial v^j}{\partial \bar z^s}\frac{\partial}{\partial z^j}+\frac{d\bar z^s}{dt}\left(\frac{\partial v^{\bar j}}{\partial \bar z^s}+v^{\bar l}\Gamma_{\bar l\bar s}^{\bar j}\right)\frac{\partial}{\partial \bar z^j}\\&=\left(\frac{dv^j}{dt}+v^l\Gamma_{ls}^j\frac{dz^s}{dt}\right)\frac{\partial}{\partial z^j}+\left(\frac{dv^{\bar j}}{dt}+v^{\bar l}\Gamma_{\bar l\bar s}^{\bar j}\frac{d\bar z^s}{dt}\right)\frac{\partial}{\partial \bar z^j}\end{aligned} \tag{2.8}$$

这是V沿曲线$C:(z^j(t),\ \bar z^j(t))$的切向量$X=\dot z^s\partial/\partial z^s+\dot{\bar z}^s\partial/\partial \bar z^s$关于参量$t$的绝对微商。

$V=X$时

$$\frac{DV}{dt}=\left(\frac{dv^j}{dt}+\Gamma^j_{ls}v^lv^s\right)\frac{\partial}{\partial z^j}+\left(\frac{dv^{\bar{j}}}{dt}+\Gamma^{\bar{j}}_{\bar{l}\bar{s}}v^{\bar{l}}v^{\bar{s}}\right)\frac{\partial}{\partial \bar{z}^j}=D_VV \quad (2.9)$$

(2.9)是V在V方向上关于t的绝对微商．本文主要用(2.9)．因

$$\partial H+\bar{\partial}H=dH=\omega\cdot H+H\cdot\bar{\omega}^T\Rightarrow\partial H=\omega\cdot H,\ \bar{\partial}H=H\cdot\bar{\omega}^T$$

故ω决定如次：

$$\left.\begin{aligned}&\omega=\partial H\cdot H^{-1},\ \omega^l_j=\partial h_{j\bar{k}}h^{\bar{k}l}\\ \text{且}\quad &\Gamma^l_{js}=h^{\bar{k}l}\frac{\partial h_{j\bar{k}}}{\partial z^s},\ \Gamma^{\bar{l}}_{\bar{j}\bar{s}}=\overline{\Gamma^l_{js}}=h^{\bar{l}k}\frac{\partial h_{k\bar{j}}}{\partial\bar{z}^s}\end{aligned}\right\}\quad(2.10)$$

利用度量(2.1)定义从同构：

$$\left.\begin{aligned}&b:\ TM\to T'M,\ \frac{\partial}{\partial z^j}\mapsto\frac{\partial}{\partial z^j}^b=\frac{1}{2}h_{j\bar{k}}d\bar{z}^k,\ \frac{\partial}{\partial\bar{z}^j}\mapsto\frac{\partial}{\partial\bar{z}^j}^b=\frac{1}{2}h_{k\bar{j}}dz^k\\ &b^{-1}\#:\ T^*M\to TM,\ dz^j\mapsto dz^{j\#}=2h^{\bar{k}j}\frac{\partial}{\partial\bar{z}^k},\ d\bar{z}^j\mapsto d\bar{z}^{j\#}=2h^{\bar{j}k}\frac{\partial}{\partial z^k}\end{aligned}\right\}\quad(2.11)$$

于是 $V^b=v^j\frac{\partial}{\partial z^j}^b+v^{\bar{j}}\frac{\partial}{\partial\bar{z}^j}^b=\frac{1}{2}v^jh_{j\bar{k}}d\bar{z}^k+\frac{1}{2}v^{\bar{j}}h_{k\bar{j}}dz^k\in\mathscr{F}'(M)$

$$\begin{aligned}DV^b&=Dv^j\frac{\partial}{\partial z^j}^b+Dv^{\bar{j}}\frac{\partial}{\partial\bar{z}^j}^b=\frac{1}{2}(Dv^jh_{j\bar{k}}d\bar{z}^k+Dv^{\bar{j}}h_{k\bar{j}}dz^k)\\ &=\frac{1}{2}h_{j\bar{k}}(dv^j+v^l\Gamma^j_{ls}dz^s)d\bar{z}^k+\frac{1}{2}h_{k\bar{j}}(dv^{\bar{j}}+v^{\bar{l}}\Gamma^{\bar{j}}_{\bar{l}\bar{s}}d\bar{z}^s)dz^k\\ &=\frac{1}{2}h_{j\bar{k}}\left(dv^j+v^lh^{\bar{k}j}\frac{\partial h_{l\bar{k}}}{\partial z^s}dz^s\right)d\bar{z}^k+\frac{1}{2}h_{k\bar{j}}\left(dv^{\bar{j}}+v^{\bar{l}}h^{\bar{j}k}\frac{\partial h_{k\bar{l}}}{\partial\bar{z}^s}d\bar{z}^s\right)dz^k\\ &=\frac{1}{2}\left[\left(h_{j\bar{k}}dv^j+v^l\frac{\partial h_{l\bar{k}}}{\partial z^s}dz^s\right)d\bar{z}^k+\left(h_{k\bar{j}}dv^{\bar{j}}+v^{\bar{l}}\frac{\partial h_{k\bar{l}}}{\partial\bar{z}^s}d\bar{z}^s\right)dz^k\right]\end{aligned}\quad(2.12)$$

$$\begin{aligned}\frac{DV^b}{dt}&=\frac{Dv^j}{dt}\frac{\partial}{\partial z^j}^b+\frac{Dv^{\bar{j}}}{dt}\frac{\partial}{\partial\bar{z}^j}^b=\frac{1}{2}\left(\frac{Dv^j}{dt}h_{j\bar{k}}d\bar{z}^k+\frac{Dv^{\bar{j}}}{dt}h_{k\bar{j}}dz^k\right)\\ &=\frac{1}{2}\left[h_{j\bar{k}}\left(\frac{dv^j}{dt}+v^l\Gamma^j_{ls}\frac{dz^s}{dt}\right)d\bar{z}^k+h_{k\bar{j}}\left(\frac{dv^{\bar{j}}}{dt}+v^{\bar{l}}\Gamma^{\bar{j}}_{\bar{l}\bar{s}}\frac{d\bar{z}^s}{dt}\right)dz^k\right]\\ &=\frac{1}{2}\left[\left(h_{j\bar{k}}\frac{dv^j}{dt}+v^l\frac{\partial h_{l\bar{k}}}{\partial z^s}\frac{dz^s}{dt}\right)d\bar{z}^k+\left(h_{k\bar{j}}\frac{dv^{\bar{j}}}{dt}+v^{\bar{l}}\frac{\partial h_{k\bar{l}}}{\partial\bar{z}^s}\frac{d\bar{z}^s}{dt}\right)dz^k\right]\end{aligned}\quad(2.13)$$

其中 $V=X=\dot{z}^j\frac{\partial}{\partial z^j}+\dot{\bar{z}}^j\frac{\partial}{\partial\bar{z}^j}$时，$\frac{dz^s}{dt}=\dot{z}^s=v^s$，$\frac{d\bar{z}^s}{dt}=\dot{\bar{z}}^s=v^{\bar{s}}$

$$(2.8)\Rightarrow\frac{DV}{dt}=0\Rightarrow\frac{Dv^j}{dt}=\frac{dv^j}{dt}+v^l\Gamma^j_{ls}\frac{dz^s}{dt}=0,\ \frac{Dv^{\bar{j}}}{dt}=\frac{dv^{\bar{j}}}{dt}+v^{\bar{l}}\Gamma^{\bar{j}}_{\bar{l}\bar{s}}\frac{d\bar{z}^s}{dt}=0\quad(2.14)$$

是向量场V沿M^n中曲线C:$(z^j(t),\ \bar{z}^j(t))$平移的条件．

$V=X$时，$V=v^j\partial/\partial z^j+v^{\bar{j}}\partial/\partial\bar{z}^j$是质点在$U\subset M^n$中沿曲线$C$运动的速度，加速度是

$$\frac{DV}{dt}=D_VV=\left(\frac{dv^j}{dt}+\Gamma^j_{ls}v^lv^s\right)\frac{\partial}{\partial z^j}+\left(\frac{dv^{\bar{j}}}{dt}+\Gamma^{\bar{j}}_{\bar{l}\bar{s}}v^{\bar{l}}v^{\bar{s}}\right)\frac{\partial}{\partial\bar{z}^j}\quad(2.15)$$

质点在Kähler流形M^n上运动的规律与质点在 Riemann 流形上运动的规律相象．事实上，n-维复解析流形可看作$2n$-维实解析流形．Kähler流形是特殊的复解析流形，其基底流形是

$2n$-维Riemann流形.

不失一般性,设运动的质点具有单位质量或把质量吸收到度量张量$h_{j\bar{k}}$中.在Kähler流形M^n中运动的质点加速度为0时：$DV/dt=0$，如同在Riemann流形上一样不是停止或作匀速直线运动，它不能作为自由质点飞离M^n直线运动．按此观点 Newton定律不成立．$DV/dt=0$时，质点的运动方程是

$$\ddot{z}^j+\Gamma^j_{ls}\dot{z}^l\dot{z}^s=0,\quad \ddot{\bar{z}}^{\bar{j}}+\Gamma^{\bar{j}}_{\bar{l}\bar{s}}\dot{\bar{z}}^{\bar{l}}\dot{\bar{z}}^{\bar{s}}=0 \tag{2.16}$$

这是M^n上的测地线方程，即质点在无别的外力作用下在M^n上作测地线运动而不飞离M^n．此现象可如同在Riemann流形上运动的质点用于相对论是受引力场的作用．改写(2.16)为

$$\ddot{z}^j=-\Gamma^j_{ls}\dot{z}^l\dot{z}^s,\quad \ddot{\bar{z}}^{\bar{j}}=-\Gamma^{\bar{j}}_{\bar{l}\bar{s}}\dot{\bar{z}}^{\bar{l}}\dot{\bar{z}}^{\bar{s}} \tag{2.17}$$

$F_j=-\Gamma^j_{ls}\dot{z}^l\dot{z}^s$，$F_{\bar{j}}=-\Gamma^{\bar{j}}_{\bar{l}\bar{s}}\dot{\bar{z}}^{\bar{l}}\dot{\bar{z}}^{\bar{s}}$ 视为引力场中作用于质点且与位置、速度及质量有关的引力．这样，在引力场中质点运动的空间是"弯曲"的．在N维复空间C^N的$n(n<N)$维Kähler 曲面（或子流形）上运动的质点无外力作用时的约束运动的方程也是这个形式．特别，$M^n=C^n$为n维复空间或M^n是局部平坦Kähler流形，则$\Gamma^j_{ls}=0$，$\Gamma^{\bar{j}}_{\bar{l}\bar{s}}=0$,质点为自由质点作直线运动．运动轨迹是$z^j=\dot{z}^j_0t+z^j_0$，$\bar{z}^j=\dot{\bar{z}}^j_0t+\bar{z}^j_0$，$j=1, 2, \cdots, n$，其中$z^j_0=z^j(0)$，$\dot{z}^j_0=\dot{z}^j(0)$等.

三、质点在Kähler流形上运动的Newton定律

我们概括在Kähler流形M^n上质点在外力作用下运动的Newton定律如下：

Newton定律 质点在Kähler流形M^n上受力场$F=F_kdz^k+F_{\bar{k}}d\bar{z}^k$作用而运动，其速度

$$V=v^j\frac{\partial}{\partial z^j}+v^{\bar{j}}\frac{\partial}{\partial \bar{z}^j}\qquad (U,\ z^j)\text{中} \tag{3.1}$$

加速度
$$\frac{DV}{dt}=\frac{Dv^j}{dt}\frac{\partial}{\partial z^j}+\frac{Dv^{\bar{j}}}{dt}\frac{\partial}{\partial \bar{z}^j}\qquad (U,\ z^j)\text{中} \tag{3.2}$$

运动方程
$$\frac{DV}{dt}=F^{\#}\Leftrightarrow\frac{DV^{\flat}}{dt}=F \tag{3.3}$$

即
$$\left.\begin{aligned}\frac{Dv^j}{dt}&=2h^{\bar{k}j}F_{\bar{k}} & & \frac{1}{2}h_{j\bar{k}}\frac{Dv^j}{dt}=F_{\bar{k}}\\ & & \Leftrightarrow & \\ \frac{Dv^{\bar{j}}}{dt}&=2h^{\bar{j}k}F_k & & \frac{1}{2}h_{k\bar{j}}\frac{Dv^{\bar{j}}}{dt}=F_k\end{aligned}\right\} \tag{3.4}$$

其中假定运动质点有不变的单位质量，以下同此．若运动质点有不变质量$m=\mathrm{const}\neq 1$，则运动方程为

$$m\frac{DV}{dt}=F^{\#}\ \text{或}\ m\frac{DV^{\flat}}{dt}=F$$

$$F^{\#}=F_kdz^{k\#}+F_{\bar{k}}d\bar{z}^{k\#}=2h^{\bar{j}k}F_k\frac{\partial}{\partial\bar{z}^j}+2h^{\bar{k}j}F_{\bar{k}}\frac{\partial}{\partial z^j},\quad \frac{DV}{dt}=D_VV$$

定义 在Kähler流形M^n上运动的质点的动能是

$$T=\frac{1}{2}h(V,\ V)=\frac{1}{2}h_{j\bar{k}}v^jv^{\bar{k}} \tag{3.5}$$

动能的微分

$$dT=\frac{1}{2}dh(V,\ V)=\frac{1}{2}(h(DV,\ V)+h(V,\ DV))$$

$$=\frac{1}{2}[h_{j\bar{k}}Dv^{j}v^{\bar{k}}+h_{k\bar{j}}Dv^{\bar{j}}v^{k}]$$

$$=\frac{1}{2}\left[h_{j\bar{k}}(dv^{j}+v^{l}\Gamma^{j}_{ls}dz^{s})v^{\bar{k}}+h_{k\bar{j}}(dv^{\bar{j}}+v^{\bar{l}}\Gamma^{\bar{j}}_{\bar{l}\bar{s}}d\bar{z}^{s})v^{k}\right]$$

$$=\frac{1}{2}\left[\left(h_{j\bar{k}}dv^{j}+v^{l}\frac{\partial h_{l\bar{k}}}{\partial z^{s}}dz^{s}\right)v^{\bar{k}}+\left(h_{k\bar{j}}dv^{\bar{j}}+v^{\bar{l}}\frac{\partial h_{k\bar{l}}}{\partial\bar{z}^{s}}d\bar{z}^{s}\right)v^{k}\right] \tag{3.6}$$

$$\frac{dT}{dt}=\frac{1}{2}\left[h\left(\frac{DV}{dt},V\right)+h\left(V,\ \frac{DV}{dt}\right)\right]$$

$$=\frac{1}{2}\left[h_{j\bar{k}}\frac{Dv^{j}}{dt}v^{\bar{k}}+h_{k\bar{j}}\frac{Dv^{\bar{j}}}{dt}v^{k}\right]$$

$$=\frac{1}{2}\left[\left(h_{j\bar{k}}\ddot{z}^{j}+\dot{z}^{l}\frac{\partial h_{l\bar{k}}}{\partial z^{s}}\dot{z}^{s}\right)\dot{\bar{z}}^{k}+\left(h_{k\bar{j}}\ddot{\bar{z}}^{j}+\dot{\bar{z}}^{l}\frac{\partial h_{k\bar{l}}}{\partial\bar{z}^{s}}\dot{\bar{z}}^{s}\right)\dot{z}^{k}\right] \tag{3.7}$$

其中，$V=v^{j}\partial/\partial z^{j}+v^{\bar{j}}\partial/\partial\bar{z}^{j}$是运动质点速度．$v^{k}=\dot{z}^{k}$，$v^{\bar{k}}=\dot{\bar{z}}^{k}$．

(2.12)、(3.6)⇒ $$dT=\langle DV^{b},\ V\rangle \tag{3.8}$$

其中⟨ , ⟩是$T^{*}M$与TM之间的对偶对．$V\in\mathscr{X}(M)$，$DV^{b}\in\mathscr{F}'(M)$．

定理1 $dT=\langle DV^{b},\ V\rangle=\langle V^{b},\ DV\rangle$ (3.9)

定理2 $\dfrac{dT}{dt}=\left\langle\dfrac{DV^{b}}{dt},\ V\right\rangle=\left\langle V^{b},\ \dfrac{DV}{dt}\right\rangle$ (3.10)

二定理之结论已隐含于前述推导．由Newton定律(3.3)及(3.10)，

$$\frac{dT}{dt}=\left\langle\frac{DV^{b}}{dt},\ V\right\rangle=\langle F,\ V\rangle \tag{3.11}$$

积分得

$$T_{2}-T_{1}=\int_{t_{1}}^{t_{2}}\frac{dT}{dt}dt=\int_{t_{1}}^{t_{2}}\langle F,\ V\rangle dt$$

$$=\int_{t_{1}}^{t_{2}}(F_{k}\dot{z}^{k}+F_{\bar{k}}\dot{\bar{z}}^{k})dt=\int_{\gamma}F_{k}dz^{k}+F_{\bar{k}}d\bar{z}^{k} \tag{3.12}$$

由(3.4)及(3.7)即知．注意dz^{k}，$d\bar{z}^{k}$是运动质点的曲线$\gamma\subset U$的坐标变量$z^{k}(t)$，$\bar{z}^{k}(t)$的微分．它们是(3.4)的解．积分线路是质点的运动轨线$\gamma\subset U$．而力场$F=F_{k}dz^{k}+F_{\bar{k}}d\bar{z}^{k}$中的$dz^{k}$，$d\bar{z}^{k}$是余切丛$T^{*}M$在$U\subset M^{n}$上的标架场．故有

动能定律 Kähler流形M^{n}上的质点，在力场F作用下运动而产生的动能T，在时间t_1到t_2间的变化等于该力场F在此期间沿运动轨线γ所作的功．

设$U=U(z,\ \bar{z})$是势场．若$F=-dU$，则势场是力场，这时

$$F=F_{k}dz^{k}+F_{\bar{k}}d\bar{z}^{k}=-\frac{\partial U}{\partial z^{k}}dz^{k}-\frac{\partial U}{\partial\bar{z}^{k}}d\bar{z}^{k}=-dU$$

(3.12)⇒ $$T_{2}-T_{1}=-\int_{\gamma}\frac{\partial U}{\partial z^{k}}dz^{k}+\frac{\partial U}{\partial\bar{z}^{k}}d\bar{z}^{k}=-\int_{\gamma}dU=U_{1}-U_{2} \tag{3.13}$$

$F=-dU$是恰当的微分1-形式，是函数U的全微分，其积分与线路无关，只与γ的端点有关．由(3.13)得

$$T_{2}+U_{2}=T_{1}+U_{1}$$

特别，
$$dT=\langle DV^b, V\rangle=\langle F, V\rangle dt=-dU$$
$$d(T+U)=0 \Rightarrow T+U=\text{const}$$

能量守恒定律 M^n上的质点在势场中运动时，动能与势能之和不变，其运动方程是

$$\frac{DV}{dt}=-dU^{\#} \Leftrightarrow \frac{DV^b}{dt}=-dU \tag{3.14}$$

即
$$\left.\begin{aligned} \frac{Dv^j}{dt}&=-2h^{\bar{k}j}\frac{\partial U}{\partial \bar{z}^k} & \Leftrightarrow \quad \frac{1}{2}h_{j\bar{k}}\frac{Dv^j}{dt}&=-\frac{\partial U}{\partial \bar{z}^k} \\ \frac{Dv^{\bar{j}}}{dt}&=-2h^{\bar{j}k}\frac{\partial U}{\partial z^k} & \frac{1}{2}h_{k\bar{j}}\frac{Dv^{\bar{j}}}{dt}&=-\frac{\partial U}{\partial z^k}\end{aligned}\right\} \tag{3.15}$$

定义 质量为m的质点，在Kähler流形上在力场F作用下运动的速度为V，则mV称为质点的动量，$\int_{t_1}^{t_2}F^{\#}dt$是力F的冲量。

设$m=1$由(3.3)，
$$\int_{t_1}^{t_2}\frac{DV}{dt}dt=\int_{t_1}^{t_2}F^{\#}dt \tag{3.16}$$

而
$$\begin{aligned}\int_{t_1}^{t_2}\frac{DV}{dt}dt&=\int_{t_1}^{t_2}[(dv^j+\Gamma^j_{ls}v^lv^sdt)\partial/\partial z^j+(dv^{\bar{j}}+\Gamma^{\bar{j}}_{\bar{l}\bar{s}}v^{\bar{l}}v^{\bar{s}}dt)\partial/\partial\bar{z}^j]\\ &=\int_{t_1}^{t_2}(dv^j\partial/\partial z^j+dv^{\bar{j}}\partial/\partial\bar{z}^j)\\ &\quad+\int_{t_1}^{t_2}(\Gamma^j_{ls}v^lv^s\partial/\partial z^j+\Gamma^{\bar{j}}_{\bar{l}\bar{s}}v^{\bar{l}}v^{\bar{s}}\partial/\partial\bar{z}^j)dt\end{aligned}$$

$$\int_{t_1}^{t_2}F^{\#}dt=\int_{t_1}^{t_2}2(h^{\bar{k}j}F_{\bar{k}}\partial/\partial z^j+h^{\bar{j}k}F_k\partial/\partial\bar{z}^j)dt$$

故
$$\begin{aligned}V_2-V_1+\int_{t_1}^{t_2}(\Gamma^j_{ls}v^lv^s\partial/\partial z^j+\Gamma^{\bar{j}}_{\bar{l}\bar{s}}v^{\bar{l}}v^{\bar{s}}\partial/\partial\bar{z}^j)dt\\ =2\int_{t_1}^{t_2}(h^{\bar{k}j}F_{\bar{k}}\partial/\partial z^j+h^{\bar{j}k}F_k\partial/\partial\bar{z}^j)dt\end{aligned}$$

即
$$\left.\begin{aligned} v_2^j-v_1^j+\int_{t_1}^{t_2}\Gamma^j_{ls}v^lv^sdt&=2\int_{t_1}^{t_2}h^{\bar{k}j}F_{\bar{k}}dt\\ v_2^{\bar{j}}-v_1^{\bar{j}}+\int_{t_1}^{t_2}\Gamma^{\bar{j}}_{\bar{l}\bar{s}}v^{\bar{l}}v^{\bar{s}}dt&=2\int_{t_1}^{t_2}h^{\bar{j}k}F_kdt\end{aligned}\right\} \tag{3.17}$$

动量定律 质点在M^n上受力场F作用而运动，从时间t_1到t_2间动量的变化，等于F在此期间的冲量。

这里我们假设运动质点有单位质量。$M^n=C^n$或M^n是局部平坦的，则$\Gamma^j_{ls}=0$，$\Gamma^{\bar{j}}_{\bar{l}\bar{s}}=0$。对一个质点的运动得到的所有结果，可以贯彻到在同一坐标系中的N个质点的质点系的运动（如同在Riemann流形中一样）。

改虑具有质量m_r及位置p_r的N个质点的质点系(m_r, p_r)，$r=1, 2, \cdots, N$。

虚位移原理 在Kähler流形M^n上，$p=\{p_r\}_{r=1}^N\in U\subset M^n$是质点系的平衡位置，若且仅若主动力$F_r$在此位置的任何虚位移上所作的功之和为零，可表示为

$$\sum_{r=1}^{N}\langle F_r,\ V_r\rangle=0,\qquad \forall V_r\in T_{p_r}M \tag{3.18}$$

$p_r\in U\subset M^n$，$\langle\ ,\ \rangle$是对偶对，即

$$\sum_{r=1}^{N}\sum_{j=1}^{n}(F_{rj}v_r^j+F_{r\bar{j}}v_r^{\bar{j}})=0 \tag{3.19}$$

设虚位移的时间是δt，则(3.19)变为

$$\sum_{r=1}^{N}\sum_{j=1}^{n}(F_{rj}\delta z_r^j+F_{r\bar{j}}\delta\bar{z}_r^j)=0 \tag{3.20}$$

其中$V_r=v_r^j\partial/\partial z^j+v_r^{\bar{j}}\partial/\partial\bar{z}^j$是虚速度向量，$F_r=F_{rk}dz^k+F_{r\bar{k}}d\bar{z}^k$是主作用力．

由此，我们得到

D'Alembert-Lagrange原理 若于Kähler流形M^n上运动的质点（系）的任何位置上补加假想的惯性力$-mDV^b/dt$(或$-m_rDV_r^b/dt$)于主作用力F_r，则此位置变为平衡位置，且可表示为

$$\left\langle F-m\frac{DV^b}{dt},\ \xi\right\rangle=0,\qquad \forall\xi\in T_pM,\ p\in U\subset M^n \tag{3.21}$$

$$\sum_{r=1}^{N}\left\langle F_r-m_r\frac{DV_r^b}{dt},\xi_r\right\rangle=0,\qquad \forall\xi_r\in T_{p_r}M,\ p_r\in U\subset M^n \tag{3.22}$$

即

$$\left(F_{\bar{k}}-\frac{1}{2}mh_{j\bar{k}}\frac{Dv^j}{dt}\right)\xi^{\bar{k}}+\left(F_k-\frac{1}{2}mh_{k\bar{j}}\frac{Dv^{\bar{j}}}{dt}\right)\xi^k=0 \tag{3.23}$$

$$\sum_{r=1}^{N}\left[\left(F_{r\bar{k}}-\frac{m_r}{2}h_{j\bar{k}}\frac{Dv_r^j}{dt}\right)\xi_r^{\bar{k}}+\left(F_{rk}-\frac{m_r}{2}h_{k\bar{j}}\frac{Dv_r^{\bar{j}}}{dt}\right)\xi_r^k\right]=0$$

若上述表达式乘以瞬时δt，$\xi_r^k\delta t=\delta z_r^k$，$\xi_r^{\bar{k}}\delta t=\delta\bar{z}_r^k$，那末我们得到类似(3.20)的表达式．(3.21)、(3.22)也称“动力学普遍方程”．在自由质点系中，ξ_r，ξ等完全任意，故(3.21)及(3.22)等价于Newton方程组．

四、坐 标 变 换

上述结论只在局部坐标系中得到，在别的坐标系如何变化转换？故必须考虑坐标变换．

设$(U,\ z^j)$及$(U',\ z'^j)$是Kähler流形M^n中的局部坐标系且$U\cap U'\neq\phi$,$\forall p\in U\cap U'$，由M^n的定义，变换$\varphi: z^j\to z'^k$全纯，φ诱导映射$\varphi_*=(\varphi^{-1})^*$，$\varphi^*=\varphi_*^{-1}$．在$\varphi$下标架场的变换为

$$\frac{\partial}{\partial z^j}=\frac{\partial z'^k}{\partial z^j}\frac{\partial}{\partial z'^k}=\varphi_*\frac{\partial}{\partial z^j},\quad \frac{\partial}{\partial\bar{z}^j}=\frac{\partial\bar{z}'^k}{\partial\bar{z}^j}\frac{\partial}{\partial\bar{z}'^k}=\varphi_*\frac{\partial}{\partial\bar{z}^j}$$

$$dz^j=\frac{\partial z^j}{\partial z'^k}dz'^k=\varphi_*dz^j=(\varphi^{-1})^*dz^j,\quad d\bar{z}^j=\frac{\partial\bar{z}^j}{\partial\bar{z}'^k}d\bar{z}'^k=\varphi_*d\bar{z}^j=(\varphi^{-1})^*d\bar{z}^j$$

$$\frac{\partial}{\partial z'^k}=\frac{\partial z^j}{\partial z'^k}\frac{\partial}{\partial z^j}=\varphi_*^{-1}\frac{\partial}{\partial z'^k},\quad \frac{\partial}{\partial\bar{z}'^k}=\frac{\partial\bar{z}^j}{\partial\bar{z}'^k}\frac{\partial}{\partial\bar{z}^j}=\varphi_*^{-1}\frac{\partial}{\partial\bar{z}'^k}$$

$$dz'^k=\frac{\partial z'^k}{\partial z^j}dz^j=\varphi^*dz'^k,\quad d\bar z'^k=\frac{\partial \bar z'^k}{\partial \bar z^j}d\bar z^j=\varphi^*d\bar z'^k$$

命 $a_j^{k'}=\frac{\partial z'^k}{\partial z^j}$，$\bar a_j^{k'}=\frac{\partial \bar z'^k}{\partial \bar z^j}$，$a_{k'}^j=\frac{\partial z^j}{\partial z'^k}$，$\bar a_{k'}^j=\frac{\partial \bar z^j}{\partial \bar z'^k}$

$A=(a_j^{k'})$是变换矩阵，上标为列，下标为行，且$A^{-1}=(a_{k'}^j)$，$\bar A=(\bar a_j^{k'})$

$$\begin{aligned}D\partial/\partial z'^j&=\omega_{j'}^{l'}\partial/\partial z'^l=\omega_{j'}^{l'}a_{l'}^s\partial/\partial z^s=\varphi_*^{-1}D\partial/\partial z'^j\\&=D(a_{j'}^s\partial/\partial z^s)=D(\varphi_*^{-1}\partial/\partial z'^j)\\&=da_{j'}^s\partial/\partial z^s+a_{j'}^sD\partial/\partial z^s=da_{j'}^s\partial/\partial z^s+a_{j'}^s\omega_s^r\partial/\partial z^r\\&=(da_{j'}^s+a_{j'}^r\omega_r^s)\partial/\partial z^s\end{aligned}\tag{4.1}$$

得连络的变换

$$\omega_{j'}^{l'}a_{l'}^s=da_{j'}^s+a_{j'}^r\omega_r^s,\quad \omega_{j'}^{l'}=(da_{j'}^s+a_{j'}^r\omega_r^s)a_s^{l'}$$

即
$$\omega'A^{-1}=dA^{-1}+A^{-1}\omega,\quad \omega'=(dA^{-1}+A^{-1}\omega)A\tag{4.2}$$

另方面

$$\begin{aligned}D\partial/\partial z^j&=\omega_j^l\partial/\partial z^l=\omega_j^la_l^{k'}\partial/\partial z'^k=\varphi_*D\partial/\partial z^j\\&=D(a_j^{k'}\partial/\partial z'^k)=D(\varphi_*\partial/\partial z^j)\\&=da_j^{k'}\partial/\partial z'^k+a_j^{k'}D\partial/\partial z'^k=da_j^{k'}\partial/\partial z'^k+a_j^{k'}\omega_{k'}^{l'}\partial/\partial z'^l\\&=(da_j^{k'}+a_j^{r'}\omega_{r'}^{k'})\partial/\partial z'^k\end{aligned}$$

得连络的变换

$$\omega_j^la_l^{k'}=da_j^{k'}+a_j^{r'}\omega_{r'}^{k'},\quad \omega_j^l=(da_j^{k'}+a_j^{r'}\omega_{r'}^{k'})a_{k'}^l$$

即
$$\omega A=dA+A\omega',\quad \omega=(dA+A\omega')A^{-1}\tag{4.3}$$

(4.2)，(4.3)表示ω'与ω的同样的关系。

连络D与变量h相容的条件是(2.6)，而

$$h=h_{j\bar k}dz^j\otimes d\bar z^k=h_{j\bar k}a_{l'}^j\bar a_{s'}^kdz'^l\otimes d\bar z'^s=h'_{l\bar s}dz'^l\otimes d\bar z'^s=\varphi_*h=(\varphi^{-1})^*h=h'$$

$$\Rightarrow\quad h'_{l\bar s}=a_{l'}^jh_{j\bar k}\bar a_{s'}^k,\quad H'=A^{-1}H(\bar A^{-1})^T\tag{4.4}$$

但
$$\begin{aligned}dH'&=dA^{-1}\cdot H\cdot(\bar A^{-1})^T+A^{-1}\cdot dH\cdot(\bar A^{-1})^T+A^{-1}\cdot H\cdot d(\bar A^{-1})^T\\&=dA^{-1}\cdot H\cdot(\bar A^{-1})^T+A^{-1}(\omega\cdot H+H\cdot\bar\omega^T)\cdot(\bar A^{-1})^T+A^{-1}\cdot H\cdot d(\bar A^{-1})^T\\&=dA^{-1}\cdot H\cdot(\bar A^{-1})^T+A^{-1}\cdot\omega\cdot H\cdot(\bar A^{-1})^T+A^{-1}\cdot H\cdot\bar\omega^T\cdot(\bar A^{-1})^T\\&\quad+A^{-1}\cdot H\cdot d(\bar A^{-1})^T\\&=(dA^{-1}+A^{-1}\cdot\omega)\cdot H\cdot(\bar A^{-1})^T+A^{-1}\cdot H\cdot\overline{((A^{-1}\cdot\omega)^T+dA^{-1T})}\\&=\omega'\cdot A^{-1}\cdot H(\bar A^{-1})^T+A^{-1}\cdot H\cdot\overline{\omega'\cdot A^{-1T}}=\omega'\cdot H'+H'\cdot\bar\omega'^T\end{aligned}$$

即变换后，在新坐标系(U',z'^j)下度量h'与D相容。

定理1 在全纯坐标变换φ下，$\varphi:z^j\to z'^k$，连络D与度量$h'=\varphi_*h=(\varphi^{-1})^*h=h\circ\varphi^{-1}$相容。度量矩阵的变换为(4.4)，$D$在$h'$下的连络矩阵$\omega'$与其在$h$下的连络矩阵$\omega$的关系为(4.2)与(4.3)。

因A全纯，$dA^{-1}=\partial A^{-1}$，$\omega=\partial H\cdot H^{-1}$，

故
$$\omega'=(\partial A^{-1}+A^{-1}\cdot\partial H\cdot H^{-1})A,\quad \omega_{j'}^{l'}=(\partial a_{j'}^s+a_{j'}^l\omega_l^s)a_s^{l'}$$

故
$$\Gamma_{j'r'}^{l'}dz'^r=\left(\frac{\partial a_{j'}^s}{\partial z'^r}dz'^r+a_{j'}^l\Gamma_{lt}^sa_{r'}^tdz'^r\right)a_s^{l'}$$

$$\Gamma_{j'r'}^{l'}=\left(\frac{\partial a_{j'}^s}{\partial z'^r}+a_{j'}^l\Gamma_{lt}^sa_{r'}^t\right)a_s^{l'},\quad \Gamma_{lt}^s=\frac{\partial h_{l\bar k}}{\partial z^t}h^{\bar ks}\tag{4.5}$$

而$\Gamma^{i'}_{\bar{j}'\bar{r}'}=\overline{\Gamma}^{l'}_{j'r'}$。限于篇幅略述全纯变换$\varphi: z\to z'$下速度、加速度、力场、动能之变化。

速度 $V=v^j\partial/\partial z^j+v^{\bar{j}}\partial/\partial\bar{z}^j\in\mathscr{X}(M)$, $U\subset M^n$上 (4.6)

$$\varphi_*V=v^j\circ\varphi^{-1}(z')a^{k'}_j\partial/\partial z'^k+v^{\bar{j}}\circ\varphi^{-1}(z')\bar{a}^{k'}_j\partial/\partial\bar{z}'^k=v'^k\partial/\partial z'^k+v'^{\bar{k}}\partial/\partial\bar{z}'^k=V' \tag{4.7}$$

$$\begin{aligned}D\varphi_*V&=D(v^ja^{k'}_j)\partial/\partial z'^k+D(v^{\bar{j}}\bar{a}^{k'}_j)\partial/\partial\bar{z}'^k\\&=(d(v^ja^{k'}_j)+v^ja^{l'}_j\omega^{\ k}_l)\partial/\partial z'^k+(d(v^{\bar{j}}\bar{a}^{k'}_j)+v^{\bar{j}}\bar{a}^{l'}_j\bar{\omega}^{\ k}_l)\partial/\partial\bar{z}'^k\end{aligned} \tag{4.8}$$

$$\begin{aligned}\varphi_*DV&=(Dv^j)a^{k'}_j\partial/\partial z'^k+(Dv^{\bar{j}})\bar{a}^{k'}_j\partial/\partial\bar{z}'^k\\&=(dv^j+v^l\omega^j_l)a^{k'}_j\partial/\partial'^k+(dv^{\bar{j}}+v^{\bar{l}}\bar{\omega}^j_l)\bar{a}^{k'}_j\partial/\partial\bar{z}'^k\end{aligned} \tag{4.9}$$

而
$$\begin{aligned}d(v^ja^{k'}_j)+v^ja^{l'}_j\omega^{\ k}_l&=dv^ja^{k'}_j+v^jda^{k'}_j+v^ja^{l'}_j(da^s_l+a^{k}_{l'}\omega^s_k)a^{k'}_s\\&=(dv^j+v^l\omega^j_l)a^{k'}_j\end{aligned} \tag{4.10}$$

同理
$$d(v^{\bar{j}}\bar{a}^{k'}_j)+v^{\bar{j}}\bar{a}^{l'}_j\bar{\omega}^{\ k}_l=(dv^{\bar{j}}+v^{\bar{l}}\bar{\omega}^j_l)\bar{a}^{k'}_j \tag{4.11}$$

故
$$\varphi_*DV=D\varphi_*V,\quad \varphi_*\frac{DV}{dt}=\frac{D\varphi_*V}{dt} \tag{4.12}$$

设力场 $F=b_jdz^j+b_{\bar{j}}d\bar{z}^j\in\mathscr{F}'(M)$ $U\subset M^n$上

则
$$\varphi_*F=b_j\circ\varphi^{-1}(z')a^j_{k'}dz'^k+b_{\bar{j}}\circ\varphi^{-1}(z')\bar{a}^j_{k'}d\bar{z}'^k=b'_kdz'^k+b'_{\bar{k}}d\bar{z}'^k=F'$$

$$DF=Db_jdz^j+Db_{\bar{j}}d\bar{z}^j=(db_j-b_l\omega^l_j)dz^j+(db_{\bar{j}}-b_{\bar{l}}\bar{\omega}^l_j)d\bar{z}^j \tag{4.13}$$

$$\varphi_*DF=(db_j-b_l\omega^l_j)a^j_{k'}dz'^k+(db_{\bar{j}}-b_{\bar{l}}\bar{\omega}^l_j)\bar{a}^j_{k'}d\bar{z}'^k \tag{4.14}$$

$$\begin{aligned}D\varphi_*F&=D(b_ja^j_{k'})dz'^k+D(b_{\bar{j}}\bar{a}^j_{k'})d\bar{z}'^k\\&=(d(b_ja^j_{k'})-b_ja^j_{l'}\omega^{\ l}_k)dz'^k+(d(b_{\bar{j}}\bar{a}^j_{k'})-b_{\bar{j}}\bar{a}^j_{l'}\bar{\omega}^{\ l}_k)d\bar{z}'^k\end{aligned} \tag{4.15}$$

而
$$\begin{aligned}d(b_ja^j_{k'})-b_ja^j_{l'}\omega^{\ l}_k&=db_ja^j_{k'}+da^j_{k'}b_j-b_ja^j_{l'}(da^s_{k'}+a^l_{k'}\omega^{\ s}_l)a^{l'}_s\\&=a^j_{k'}(db_j-b_s\omega^s_j)\end{aligned}$$

同理 $d(b_{\bar{j}}-\bar{a}^j_{k'})-b_{\bar{j}}\bar{a}^j_{l'}\bar{\omega}^{\ l}_k=\bar{a}^j_{k'}(db_{\bar{j}}-b_{\bar{s}}\bar{\omega}^s_j)$

故
$$\varphi^*DF=D\varphi_*F \tag{4.16}$$

定理2 在全纯变换$\varphi: z\to z'$下，诱导映射，$\varphi_*=(\varphi^{-1})^*$，$\varphi^*=\varphi_*^{-1}$与绝对微分算子$D$作用顺序可交换：

$$\varphi_*DV=D\varphi_*V,\quad \varphi_*DF=D\varphi_*F,\quad \forall V\in\mathscr{X}(M),\ F\in\mathscr{F}'(M)$$

且
$$\left.\begin{aligned}&\varphi_*\frac{DV}{dt}=\frac{D\varphi_*V}{dt},\quad v'^k=v^ja^{k'}_j,\quad b'_k=a^j_{k'}b_j\\&\qquad v'^{\bar{k}}=v_{\bar{j}}\bar{a}^{k'}_j,\quad b'_{\bar{k}}=\bar{a}^j_{k'}b_{\bar{j}}\end{aligned}\right\} \tag{4.17}$$

推论 $\varphi_*D(V^b)=D\varphi_*(V^b)$, $\varphi_*D(F^\#)=D\varphi_*(F^\#)$ (4.18)

详细推算可知

$$D\left(\frac{\partial}{\partial z^j}\right)^b=\frac{1}{2}(dh_{j\bar{k}}-h_{j\bar{l}}\bar{\omega}^l_k)d\bar{z}^k$$

$$\varphi_*\left(\frac{\partial}{\partial z^j}\right)^b=\frac{1}{2}h_{j\bar{k}}\bar{a}^k_{l'}d\bar{z}'^l$$

$$\varphi_*D\left(\frac{\partial}{\partial z^j}\right)^b=\frac{1}{2}(dh_{j\bar{k}}-h_{j\bar{l}}\bar{\omega}^l_k)\bar{a}^k_{s'}d\bar{z}'^s \tag{4.19}$$

$$D\varphi_*\left(\frac{\partial}{\partial z^j}\right)^b=\frac{1}{2}(d(h_{j\bar{r}}\bar{a}^k_{l'})-h_{j\bar{k}}\bar{a}^k_{l'}\bar{\omega}^{\ l}_s)d\bar{z}'^s \tag{4.20}$$

且
$$d(h_{j\bar{k}}\bar{a}^k_{s'})-h_{j\bar{k}}\bar{a}^k_{l'}\bar{\omega}^{\ l}_s=(dh_{j\bar{k}}-h_{j\bar{l}}\bar{\omega}^l_k)\bar{a}^h_{s'}$$

即 $\varphi_* D(\partial/\partial z^j)^b = D\varphi_*(\partial/\partial z^j)^b$. 类似，$\varphi_* D(\partial/\partial \bar z^j)^b = D\varphi_*(\partial/\partial \bar z^j)^b$.

$\varphi_* D(dz^{j*}) = D\varphi_*(dz^{j*})$，$\varphi_* D(d\bar z^{j*}) = D\varphi_*(d\bar z^{j*})$

但 $(D\partial/\partial z^j)^b \neq D(\partial/\partial z^j)^b$

又
$$\varphi_*\left(D\frac{\partial}{\partial z^j}\right)^b = \frac{1}{2}\omega_j^k h_{k\bar l}\bar a_{\bar s}^{\bar l}{}' d\bar z'^s \tag{4.21}$$

$$\left(D\varphi_*\frac{\partial}{\partial z^j}\right)^b = (da_j^k{}' + a_j^l{}'\omega_l^{\,\prime k})\frac{1}{2}h'_{k\bar s}d\bar z'^s \tag{4.22}$$

$$\begin{aligned}(da_j^k{}' + a_j^l{}'\omega_l'^k)h'_{k\bar s} &= [da_j^k{}' + a_j^l{}'(da_l^r{}' + a_l^k{}'\omega_i^r)a_r^k{}']h'_{k\bar s}\\ &= \omega_j^r h_{r\bar l}\bar a_{\bar s}^{\bar l}{}'\end{aligned}$$

故 $\varphi_*(D\partial/\partial z^j)^b = (D\varphi_*\partial/\partial z^j)^b$，$\varphi_*(D\partial/\partial \bar z^j)^b = (D\varphi_*\partial/\partial \bar z^j)^b$

一般地，

$$\varphi_*(DV)^b = \frac{1}{2}Dv^j h_{j\bar k}\bar a_{\bar l}^{\bar k}{}' d\bar z'^l + \frac{1}{2}Dv^{\bar j}h_{k\bar j}a_l^k{}' dz'^l \tag{4.23}$$

$$(D_*DV)^b = \frac{1}{2}Dv^j a_j^r{}' h'_{r\bar l}d\bar z'^l + \frac{1}{2}Dv^{\bar j}\bar a_{\bar j}^{\bar r}{}' h'_{l\bar r}dz'^l \tag{4.24}$$

而 $a_j^r{}' h'_{r\bar l} = a_j^r{}' h_{j\bar k}a_r^j{}'\bar a_{\bar l}^{\bar k}{}' = h_{j\bar k}\bar a_{\bar l}^{\bar k}{}'$，$\bar a_{\bar j}^{\bar r}{}' h'_{l\bar r} = \bar a_{\bar j}^{\bar r}{}' h_{k\bar j}a_l^k{}'\bar a_{\bar r}^{\bar j}{}' = h_{k\bar j}a_l^k{}'$

综上述及定理2

$$\varphi_*(DV)^b = (\varphi_* DV)^b = (D\varphi_* V)^b$$

定理3 全纯变换$\varphi: z \mapsto z'F$，φ_*与b可交换：

$$\varphi_*(DV)^b = (\varphi_* DV)^b = (D\varphi_* V)^b \tag{4.25}$$

且
$$\varphi_*\left(\frac{DV}{dt}\right)^b = \left(\varphi_*\frac{DV}{dt}\right)^b = \left(\frac{D\varphi_* V}{dt}\right)^b \tag{4.26}$$

又 $\varphi: z \mapsto z'$，$h' = \varphi_* h = (\varphi^{-1})^* h = h\circ\varphi^{-1}$

动能
$$T' = \frac{1}{2}h'(V', V') = \frac{1}{2}(\varphi^{-1})^* h(V', V') = \frac{1}{2}h(\varphi_*^{-1}V', \varphi_*^{-1}V') \tag{4.27}$$

若$V' = \varphi_* V$，则$V = \varphi_*^{-1}V'$，于是

$$T' = \frac{1}{2}h'(V', V') = \frac{1}{2}h(\varphi_*^{-1}V', \varphi_*^{-1}V') = \frac{1}{2}h(V, V) = T$$

定理4 质点在力场F作用下，于 Kähler 流形M^n上运动产生的动能，不随局部的变更而变化。计算公式如(4.27)。

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Newtonian Mechanics on Kähler Manifold

Zhang Rongye

(*Institute of Mathematics, Academia Sinica, Beijing* 100080, *P. R. China*)

Abstract

In this paper we discuss Newtonian Mechanics on Kähler Manifold, and also give the complex mathematical aspects of Newton's law, the law of kinetic energy, the law of kinetic quantity, the equation of motion and the "general equation of dynamics", and so on.

Key words Kähler Manifold, a connection, an absolute differential, the duality pairing