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Stokes方程初边值问题的Phragmén-Lindelöf二择性原理*

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摘　要

本文对非定常的 Stokes 方程的初边值问题证明了 Phragmén-Lindelöf 二择性原理，即证明 Stokes 流函数的能量，随着与带状区域有限端距离的增加必定或者按指数率增长或者按指数率衰减。对能量衰减情况建立了 Stokes 流速度的最大模的点点估计。并提出求全能量上界的方法。

关键词　Stokes方程　初边值问题　Phragmén-Lindelöf二择性原理　能量耗散估计

一、引　言

在60～80年代，Saint-Venant 原理曾经是应用数学及力学的热门研究课题之一。大量研究结果的出现大大地拓展了这一领域(详见 C. O. Horgan 的专著[3,4]及其所列参考文献)。Saint-Venant 原理的共同特点是建立了能量随(距离柱体端面的)距离增加而呈指数率衰减的估计[1~5,9]。而且几乎所有文章都需要对所论的解加上在无穷远处衰减为零的先验假设。

进入90年代以来，Saint-Venant 原理的研究已逐步为 Phragmén-Lindelöf 二择性原理所取代。并且古典的 Phragmén-Lindelöf 二择性原理的概念和结果已经被大量的研究成果所拓广。显示了 Phragmén-Lindelöf 二择性原理在物理、力学等学科上的巨大应用前景。

本文对半无限带状区域上的非定常 Stokes 方程的初边值问题建立了 Phragmén-Lindelöf 二择性原理，使文[9]的结果得到较大的改进。

二、基本公式推导

假设R是平面上的半无限带状区域

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$$R=\{(x_1,x_2)\,|\,0\leqslant x_1<+\infty,\ 0<x_2<h\}\tag{2.1}$$

令R_z记R的子区域

$$R_z=\{(x_1,x_2)\,|\,0\leqslant z\leqslant x_1<+\infty,\ 0<x_2<h\}\tag{2.2}$$

显然$R_0\equiv R$。令L_z表示直线$x_1=z$与带状区域R相交的截线。

我们假设速度场$u_\alpha(x_1,x_2,t)(\alpha=1,2)$和压力$p(x_1,x_2,t)$是下面 Stokes 方程初边值问题的古典解。

$$\left.\begin{array}{l}\nu\Delta u_\alpha-p_{,\alpha}=u_{\alpha,t}\\ u_{\alpha,\alpha}=0\end{array}\right\}\qquad R\times[0,T]\text{内}\tag{2.3}$$

$$u_\alpha(x_1,0,t)=u_\alpha(x_1,h,t)=0\tag{2.4}$$

$$u_\alpha(0,x_2,t)=g_\alpha(x_2,t)\tag{2.5}$$

$$u_\alpha(x_1,x_2,0)=f_\alpha(x_1,x_2)\qquad(\alpha=1,2)\tag{2.6}$$

这里及下文，我们采用通常约定的求和符号，并使用逗号“，”表示求导运算．方程(2.3)中的ν是动力粘性系数．我们假设函数$g_\alpha(x_2,t)(\alpha=1,2)$满足适当的相容性条件，如$g_\alpha(0,t)=g_\alpha(h,t)=0$。

为了简便起见，本文只限于讨论零初值问题，即假设$f_\alpha(x_1,x_2)\equiv0$。事实上，本文结果对$f_\alpha\not\equiv0$时仍成立，只不过推导过程更为繁复而已。

为了在方程中消去压力项$p_{,\alpha}$，我们引入流函数$\varphi(x_1,x_2,t)$使得

$$u_1=\varphi_{,2},\quad u_2=-\varphi_{,1}$$

则问题(2.3)～(2.6)变为以下四阶初边值问题：

$$\nu\Delta^2\varphi=\frac{\partial(\Delta\varphi)}{\partial t},\qquad R\times[0,T]\text{内}\tag{2.7}$$

$$\left.\begin{array}{l}\varphi(x_1,0,t)=0,\ \varphi(x_1,h,t)=\int_0^h u_1dx_2\\ \varphi_{,2}(x_1,0,t)=0,\ \varphi_{,2}(x_1,h,t)=0\end{array}\right\}\tag{2.8}$$

$$\varphi_{,\alpha}(0,x_2,t)=\tilde{g}_\alpha(x_2,t)\qquad(\alpha=1,2)\tag{2.9}$$

$$\varphi_{,\alpha}(x_1,x_2,0)=0\qquad(\alpha=1,2)\tag{2.10}$$

这里$\tilde{g}_1=-g_2$，$\tilde{g}_2=g_1$。

应用散度定理及初边值条件(2.8)～(2.10)，可得

$$\begin{aligned}0&=\int_0^t\int_0^z\int_{L_\xi}\varphi(\nu\Delta^2\varphi-(\Delta\varphi)_{,\tau})dAd\tau\\&=\int_0^t\int_{L_z}(\nu\varphi\varphi_{,\alpha\alpha1}-\nu\varphi_{,\alpha}\varphi_{,\alpha1}-\varphi\varphi_{,1\tau})dx_2d\tau\\&\quad-\int_0^t\int_{L_0}(\nu\varphi\varphi_{,\alpha\alpha1}-\nu\varphi_{,\alpha}\varphi_{,\alpha1}-\varphi\varphi_{,1\tau})dx_2d\tau\\&\quad+\int_0^t\int_0^z\int_{L_\xi}\nu\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau+\frac{1}{2}\int_0^z\int_{L_\xi}\varphi_{,\alpha}\varphi_{,\alpha}dA\end{aligned}\tag{2.11}$$

由(2.11)，我们定义函数

$$\begin{aligned}\phi_1(z,t)&=-\int_0^t\int_{L_z}\left(\varphi_{,\alpha}\varphi_{,\alpha1}-\varphi\varphi_{,\alpha\alpha1}+\frac{1}{\nu}\varphi\varphi_{,1\tau}\right)dx_2d\tau\\&=-\int_0^t\int_0^z\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau-\frac{1}{2\nu}\int_0^z\int_{L_\xi}\varphi_{,\alpha}\varphi_{,\alpha}dA+\phi_1(0,t)\end{aligned}\tag{2.12}$$

如同(2.11)，我们可得到

$$0=\int_0^t\int_0^z\int_{L_\xi}\varphi_{,\tau}(\nu\Delta^2\varphi-\Delta\varphi_{,\tau})\,dA\,d\tau$$

$$=\int_0^t\int_{L_z}(\nu\varphi_{,\tau}\varphi_{,\alpha\alpha1}-\nu\varphi_{,\alpha\tau}\varphi_{,1\alpha}-\varphi_{,\tau}\varphi_{,1\tau})\,dx_2d\tau$$

$$-\int_0^t\int_{L_0}(\nu\varphi_{,\tau}\varphi_{,\alpha\alpha1}-\nu\varphi_{,\alpha\tau}\varphi_{,1\alpha}-\varphi_{,\tau}\varphi_{,1\tau})\,dx_2d\tau$$

$$+\int_0^t\int_0^z\int_{L_\xi}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}\,dA\,d\tau+\frac{1}{2}\int_0^z\int_{L_\xi}\nu\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}\,dA \tag{2.13}$$

由(2.13)定义函数

$$\phi_2(z,\ t)=-\int_0^t\int_{L_z}\left(\frac{1}{\nu^2}\varphi_{,\tau}\varphi_{,1\tau}+\frac{1}{\nu}\varphi_{,\alpha\tau}\varphi_{,1\alpha}-\frac{1}{\nu}\varphi_{,\tau}\varphi_{,\alpha\alpha1}\right)dx_2d\tau$$

$$=-\frac{1}{\nu^2}\int_0^t\int_0^z\int_{L_\xi}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}\,dA\,d\tau-\frac{1}{2\nu}\int_0^z\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}\,dA+\phi_2(0,t) \tag{2.14}$$

类似(2.12)，(2.14)，可定义函数

$$\phi_3(z,t)=-\int_0^t\int_{L_z}\left(\varphi_{,\alpha\alpha}\varphi_{,\alpha\alpha1}+\frac{1}{\nu}\varphi_{,12}\varphi_{,2\tau}\right)dx_2d\tau$$

$$=-\int_0^t\int_0^z\int_{L_\xi}\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}\,dA\,d\tau-\frac{1}{2\nu}\int_0^z\int_{L_\xi}\varphi_{,1\alpha}\varphi_{,1\alpha}\,dA+\phi_3(0,t) \tag{2.15}$$

现在，我们引入函数

$$\phi(z,t)=\phi_1(z,t)+\lambda\phi_2(z,t)+\gamma\phi_3(z,t) \tag{2.16}$$

这里λ，γ是待定正常数。

显然，由$\phi(z,t)$的定义，可得

$$\frac{\partial}{\partial z}\phi(z,t)\leqslant 0 \tag{2.17}$$

由定义(2.16)可知，对任意$z>z_0>0$，成立

$$\phi(z,t)-\phi(z_0,t)=-\int_0^t\int_{z_0}^z\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}\,dA\,d\tau-\frac{\lambda}{\nu^2}\int_0^t\int_{z_0}^z\int_{L_\xi}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}\,dA\,d\tau$$

$$-\gamma\int_0^t\int_{z_0}^z\int_{L_\xi}\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}\,dA\,d\tau-\frac{1}{2\nu}\int_{z_0}^z\int_{L_\xi}\varphi_{,\alpha}\varphi_{,\alpha}\,dA$$

$$-\frac{\lambda}{2\nu}\int_{z_0}^z\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}\,dA-\frac{\gamma}{2\nu}\int_{z_0}^z\int_{L_\xi}\varphi_{,1\alpha}\varphi_{,1\alpha}\,dA \tag{2.18}$$

由(2.18)得到，若当$z\to\infty$时，$\phi(z,t)\to 0$，那么

$$\phi(z,t)=\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}\,dA\,d\tau+\frac{\lambda}{\nu^2}\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}\,dA\,d\tau$$

$$+\gamma\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}\,dA\,d\tau+\frac{1}{2\nu}\int_z^\infty\int_{L_\xi}\varphi_{,\alpha}\varphi_{,\alpha}\,dA$$

$$+\frac{\lambda}{2\nu}\int_z^\infty\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}\,dA+\frac{\gamma}{2\nu}\int_z^\infty\int_{L_\xi}\varphi_{,1\alpha}\varphi_{1\alpha}\,dA \tag{2.19}$$

另一方面，由(2.18)，若$-\phi(z,t)$有正下界$\chi(z,t)$，那么

$$\int_0^t\int_0^z\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}\,dA\,d\tau+\frac{\lambda}{\nu^2}\int_0^t\int_0^z\int_{L_\xi}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}\,dA\,d\tau$$

$$+\gamma\int_0^t\int_0^z\int_{L_\xi}\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau+\frac{1}{2\nu}\int_0^z\int_{L_\xi}\varphi_{,\alpha}\varphi_{,\alpha}dA$$

$$+\frac{\lambda}{2\nu}\int_0^z\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dA+\frac{\gamma}{2\nu}\int_0^z\int_{L_\xi}\varphi_{,1\alpha}\varphi_{,1\alpha}dA\geqslant\chi(z,t)+\phi(0,t) \tag{2.20}$$

本文证明中将需要用到熟知的 Winginter 不等式[5]：假设 $\omega(x_2)\in C^1(0,h)$，且 $\omega(0)=\omega(h)=0$，那么

$$\int_0^h\omega^2dx_2\leqslant\frac{h^2}{\pi^2}\int_0^h\omega_{,2}^2dx_2 \tag{2.21}$$

由(2.21)，对问题(2.7)～(2.10)的解φ，显然有

$$\int_{L_z}\varphi_{,1}^2dx_2\leqslant\frac{h^2}{\pi^2}\int_{L_z}\varphi_{,12}^2dx_2 \tag{2.22}$$

$$\int_{L_z}\varphi_{,2}^2dx_2\leqslant\frac{h^2}{\pi^2}\int_{L_z}\varphi_{,22}^2dx_2 \tag{2.23}$$

另一方面，由于

$$\varphi(z,x_2,t)=\int_0^{x_2}\varphi_{,\eta}(z,\eta,t)d\eta$$

应用 Schwarz 不等式，易得

$$\int_{L_z}\varphi^2dx_2\leqslant h^2\int_{L_z}\varphi_{,2}^2dx_2 \tag{2.24}$$

三、Phragmén-Lindelöf二择结果

由$\phi(z,t)$的定义，可知

$$|\phi(z,t)|\leqslant\left|\int_0^t\int_{L_z}\left(\varphi_{,1}\varphi_{,1\alpha}+2\varphi_{,2}\varphi_{,21}-\varphi\varphi_{,111}+\frac{1}{\nu}\varphi\varphi_{,1\tau}\right)dx_2d\tau\right|$$

$$+\lambda\left|\int_0^t\int_{L_z}\left(\frac{1}{\nu^2}\varphi_{,\tau}\varphi_{,1\tau}+\frac{1}{\nu}\varphi_{,11}\varphi_{,1\tau}-\frac{1}{\nu}\varphi_{,\tau}\varphi_{,111}+\frac{2}{\nu}\varphi_{,21}\varphi_{,2\tau}\right)dx_2d\tau\right|$$

$$+\gamma\left|\int_0^t\int_{L_z}\left(\varphi_{,11}\varphi_{,111}+\varphi_{,22}\varphi_{,221}+\frac{1}{\nu}\varphi_{,12}\varphi_{,2\tau}\right)dx_2d\tau\right| \tag{3.1}$$

应用 Schwarz 不等式，不等式(2.22)～(2.24)及 Young 不等式，得到

$$\int_0^t\int_{L_z}\varphi_{,1}\varphi_{,11}dx_2d\tau\leqslant\frac{h}{\pi}\left(\frac{\varepsilon_1}{2}\int_0^t\int_{L_z}\varphi_{,11}^2dx_2d\tau+\frac{1}{2\varepsilon_1}\int_0^t\int_{L_z}\varphi_{,12}^2dx_2d\tau\right) \tag{3.2}$$

$$2\int_0^t\int_{L_z}\varphi_{,2}\varphi_{,21}dx_2d\tau\leqslant\frac{h}{\pi}\left(\varepsilon_2\int_0^t\int_{L_z}\varphi_{,12}^2dx_2d\tau+\frac{1}{\varepsilon_2}\int_0^t\int_{L_z}\varphi_{,22}^2dx_2d\tau\right) \tag{3.3}$$

$$\int_0^t\int_{L_z}\varphi\varphi_{,111}dx_2d\tau\leqslant\frac{h^2}{\pi\sqrt{\gamma}}\left(\frac{\varepsilon_3}{2}\gamma\int_0^t\int_{L_z}\varphi_{,111}^2dx_2d\tau+\frac{1}{2\varepsilon_3}\int_0^t\int_{L_z}\varphi_{,22}^2dx_2d\tau\right) \tag{3.4}$$

$$\frac{1}{\nu}\int_0^t\int_{L_z}\varphi\varphi_{,1\tau}dx_2d\tau\leqslant\frac{h^2}{\pi\sqrt{\lambda}}\left(\frac{\varepsilon_4}{2}\frac{\lambda}{\nu^2}\int_0^t\int_{L_z}\varphi_{,1\tau}^2dx_2d\tau+\frac{1}{2\varepsilon_4}\int_0^t\int_{L_z}\varphi_{,22}^2dx_2d\tau\right) \tag{3.5}$$

$$\frac{\lambda}{\nu^2}\int_0^t\int_{L_z}\varphi_{,\tau}\varphi_{,1\tau}dx_2d\tau\leqslant h\left(\frac{\varepsilon_5}{2}\frac{\lambda}{\nu^2}\int_0^t\int_{L_z}\varphi_{,1\tau}^2dx_2d\tau+\frac{1}{2\varepsilon_5}\frac{\lambda}{\nu^2}\int_0^t\int_{L_z}\varphi_{,2\tau}^2dx_2d\tau\right) \tag{3.6}$$

$$\frac{\lambda}{\nu}\int_0^t\int_{L_z}\varphi_{,11}\varphi_{,1\tau}dx_2d\tau\leqslant\frac{h}{\pi}\sqrt{\frac{\lambda}{\gamma}}\Big(\frac{\varepsilon_6}{2}\gamma\int_0^t\int_{L_z}\varphi^2_{,112}dx_2d\tau$$

$$+\frac{1}{2\varepsilon_6}\ \frac{\lambda}{\nu^2}\int_0^t\int_{L_z}\varphi^2_{,1\tau}dx_2d\tau\Big) \tag{3.7}$$

$$\frac{\lambda}{\nu}\int_0^t\int_{L_z}\varphi_{,\tau}\varphi_{,111}dx_2d\tau\leqslant h\sqrt{\frac{\lambda}{\gamma}}\Big(\frac{\varepsilon_7}{2}\ \frac{\lambda}{\nu^2}\int_0^t\int_{L_z}\varphi^2_{,2\tau}dx_2d\tau+\frac{1}{2\varepsilon_7}\gamma\int_0^t\int_{L_z}\varphi^2_{,211}dx_2d\tau\Big) \tag{3.8}$$

$$\frac{2\lambda}{\nu}\int_0^t\int_{L_z}\varphi_{,21}\varphi_{,2\tau}dx_2d\tau\leqslant\sqrt{\lambda}\Big(\varepsilon_8\int_0^t\int_{L_z}\varphi^2_{,12}dx_2d\tau+\frac{1}{\varepsilon_8}\ \frac{\lambda}{\nu^2}\int_0^t\int_{L_z}\varphi^2_{,2\tau}dx_2d\tau\Big) \tag{3.9}$$

$$\gamma\int_0^t\int_{L_z}\varphi_{,11}\varphi_{,111}dx_2d\tau\leqslant\sqrt{\gamma}\Big(\frac{\varepsilon_9}{2}\gamma\int_0^t\int_{L_z}\varphi^2_{,111}dx_2d\tau+\frac{1}{2\varepsilon_9}\int_0^t\int_{L_z}\varphi^2_{,11}dx_2d\tau\Big) \tag{3.10}$$

$$\gamma\int_0^t\int_{L_z}\varphi_{,22}\varphi_{,221}dx_2d\tau\leqslant\sqrt{\gamma}\Big(\frac{\varepsilon_{10}}{2}\gamma\int_0^t\int_{L_z}\varphi^2_{,221}dx_2d\tau+\frac{1}{2\varepsilon_{10}}\int_0^t\int_{L_z}\varphi^2_{,22}dx_2d\tau\Big) \tag{3.11}$$

$$\frac{\gamma}{\nu}\int_0^t\int_{L_z}\varphi_{,12}\varphi_{,2\tau}dx_2d\tau\leqslant\frac{\gamma}{\sqrt{\lambda}}\Big(\frac{\varepsilon_{11}}{2}\int_0^t\int_{L_z}\varphi^2_{,12}dx_2d\tau$$

$$+\frac{1}{2\varepsilon_{11}}\ \frac{\lambda}{\nu^2}\int_0^t\int_{L_z}\varphi^2_{,2\tau}dx_2d\tau\Big) \tag{3.12}$$

我们希望选取适当的λ，γ及$\varepsilon_i(i=1,2,\cdots,11)$使得增长或衰减率达到最大．经计算，选取$\lambda=\gamma=h^2/4$，以及

$$\varepsilon_1=\frac{3}{2},\quad \varepsilon_2=\frac{12}{7},\quad \varepsilon_3=1,\quad \varepsilon_4=1,\quad \varepsilon_5=\frac{4}{3},\quad \varepsilon_6=4$$

$$\varepsilon_7=\frac{3}{4},\quad \varepsilon_8=2,\quad \varepsilon_9=\frac{1}{2},\quad \varepsilon_{10}=4,\quad \varepsilon_{11}=1$$

结合(3.1)～(3.12)，得到

$$|\phi(z,t)|\leqslant-\frac{5h}{4}\ \frac{\partial}{\partial z}\phi(z,t),\qquad z\geqslant0,\ 0<t<T \tag{3.13}$$

(3.13)等价于

$$\frac{\partial}{\partial z}\Big[\exp\Big[\frac{4}{5h}z\Big]\phi(z,t)\Big]\leqslant0 \tag{3.14}$$

或

$$\frac{\partial}{\partial z}\Big[\exp\Big[-\frac{4}{5h}z\Big]\phi(z,t)\Big]\leqslant0 \tag{3.15}$$

将(3.14)在区间[0,z]上积分，得到

$$\phi(z,t)\leqslant\exp\Big[-\frac{4}{5h}z\Big]\phi(0,t) \tag{3.16}$$

可见，这时当$z\to\infty$时，$\phi(z,t)$按指数率衰减，由定义(2.19)，得到

$$\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau+\frac{h^2}{4}\int_0^t\int_z^\infty\int_{L_\xi}\Big(\frac{1}{\nu^2}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}+\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}\Big)dAd\tau$$

$$+\frac{1}{2\nu}\int_z^\infty\int_{L_\xi}\varphi_{,\alpha}\varphi_{,\alpha}dA+\frac{h^2}{8\nu}\int_z^\infty\int_{L_\xi}(\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}+\varphi_{,1\alpha}\varphi_{,1\alpha})dA$$

$$\leqslant \phi(0,t)\exp\left[-\frac{4}{5h}z\right] \tag{3.17}$$

应用 L'Hospital 法则，还可以进一步得到

$$\int_0^t\int_{L_z}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dx_2d\tau+\frac{1}{2\nu}\int_{L_z}\varphi_{,\alpha}\varphi_{,\alpha}dx_2\leqslant\frac{4}{5h}\phi(0,t)\exp\left[-\frac{4}{5h}z\right] \tag{3.18}$$

$$\int_0^t\int_{L_z}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}dx_2d\tau+\frac{\nu}{2}\int_{L_z}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dx_2\leqslant\frac{16\nu^2}{5h^3}\phi(0,t)\exp\left[-\frac{4}{5h}z\right] \tag{3.19}$$

$$\int_0^t\int_{L_z}\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dx_2d\tau+\frac{1}{2\nu}\int_{L_z}\varphi_{,1\alpha}\varphi_{,1\alpha}dx_2\leqslant\frac{16}{5h^3}\phi(0,t)\exp\left[-\frac{4}{5h}z\right] \tag{3.20}$$

另一方面，对任意$z>z_1>0$，将(3.15)式在区间$[z_1, z]$上积分，得到

$$-\phi(z,t)\geqslant-\phi(z_1,t)\exp\left[\frac{4}{5h}(z-z_1)\right] \tag{3.21}$$

由(2.21)式，可得

$$\lim_{z\to\infty}\left\{\exp\left[-\frac{4}{5h}z\right]\left[\int_0^t\int_0^z\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau+\frac{1}{2\nu}\int_0^z\int_{L_\xi}\varphi_{,\alpha}\varphi_{,\alpha}dA\right.\right.$$
$$+\frac{h^2}{4}\int_0^t\int_0^z\int_{L_\xi}\left(\frac{1}{\nu^2}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}+\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}\right)dAd\tau$$
$$\left.\left.+\frac{h^2}{8\nu}\int_0^z\int_{L_\xi}(\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}+\varphi_{,1\alpha}\varphi_{,1\alpha})dA\right]\right\}\geqslant C_1 \tag{3.22}$$

特别地，有

$$\lim_{z\to\infty}\left\{\exp\left[-\frac{4}{5h}z\right]\left[\int_0^t\int_{L_z}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dx_2d\tau+\frac{1}{2\nu}\int_{L_z}\varphi_{,\alpha}\varphi_{,\alpha}dx_2\right.\right.$$
$$+\frac{h^2}{4}\int_0^t\int_{L_z}\left(\frac{1}{\nu^2}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}+\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}\right)dx_2d\tau$$
$$\left.\left.+\frac{h^2}{8\nu}\int_{L_z}(\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}+\varphi_{,1\alpha}\varphi_{,1\alpha})dx_2\right]\right\}\frac{4}{5h}C_1 \tag{3.23}$$

在(3.22)、(3.23)中，C_1是一个依赖于z_1的正数

$$C_1=-\phi(z_1,t)\exp\left[-\frac{4}{5h}z_1\right]>0 \tag{3.24}$$

综上所述，我们得到 Stokes 方程的 Phragmén-Lindelöf 二择性原理：

定理1 假设φ是定义在半无限带状区域R上的 Stokes 方程初边值问题(2.7)～(2.10)的解，那么必定成立不等式或者(3.22)((3.23))或者(3.17)((3.18)～(3.20))。

四、Stokes流速度的点点衰减估计

作为定理1的应用，对于衰减情况，可以对 Stokes 流速度的最大模建立点点估计。

对固定的$z>0$，显然有

$$u_1^2(z,x_2,t)=\varphi_{,2}^2(z,x_2,t)=-2\int_{x_2}^h\varphi_{,2}\varphi_{,22}dx_2=2\int_0^{x_2}\varphi_{,2}\varphi_{,22}dx_2 \tag{4.1}$$

由(4.1) 应用 Schwarz 不等式及不等式(2.23)，得

$$u_1^2(z,x_2,t)\leqslant\int_{L_z}|\varphi_{,2}\varphi_{,22}|dx_2\leqslant\frac{h}{\pi}\int_{L_z}\varphi_{,22}^2dx_2 \tag{4.2}$$

类似地，有

$$u_2^2(z,x_2,t)\leqslant\frac{h}{\pi}\int_{L_z}\varphi_{,12}^2dx_2 \tag{4.3}$$

结合(4.2)、(4.3)及(3.19)，得 Stokes 流速度的最大模的点点估计

$$\max_{0\leqslant x_2\leqslant h}[u_1^2(z,x_2,t)+u_2^2(z,x_2,t)]\leqslant\frac{32\nu}{5h^2\pi}\phi(0,t)\exp\left[-\frac{4}{5h}z\right] \tag{4.4}$$

此外，还可以得到区域R内平行截线上的速度模的估计。事实上，由(3.18)直接得到

$$\int_{L_z}(u_1^2+u_2^2)dx_2=\int_{L_z}\varphi_{,\alpha}\varphi_{,\alpha}dx_2\leqslant\frac{8\nu}{5h}\phi(0,t)\exp\left[-\frac{4}{5h}z\right] \tag{4.5}$$

五、全 能 量 的 上 界

第三节和第四节的结果都涉及到全能量$\phi(0,t)$。因此，我们有必要求出$\phi(0,t)$的显式上界，证明$\phi(0,t)$的上界可以用边值条件$\tilde{g}_\alpha(\alpha=1,2)$来表达。

首先证明$\phi_3(0,t)$可以由$\phi_1(0,t)$及$\phi_2(0,t)$界定。

定理2 假设当$z\to\infty$时，$\phi(z,t)$衰减为零，那么对任意$z\geqslant0$，$z_0>0$，成立

$$z_0^2\phi_3(z+z_0,t)\leqslant2\phi_1(z,t)+\frac{4h^2}{\pi^2}\phi_2(z,t) \tag{5.1}$$

证明 为表达方便，我们采用如下符号

$$\tilde{\varphi}_3(z,t)=\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau+\frac{1}{2\nu}\int_z^\infty\int_{L_\xi}(\xi-z)^2\varphi_{,1\alpha}\varphi_{,1\alpha}dA \tag{5.2}$$

显然，有

$$\begin{aligned}0&=\nu\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2\varphi_{,11}\Delta^2\varphi dAd\tau-\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2\varphi_{,11}\Delta\varphi_{,\tau}dAd\tau\\&=\nu I_1+I_2\end{aligned} \tag{5.3}$$

重复应用分部积分，得到

$$I_1=-\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau+\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau$$

$$-2\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,12}^2dAd\tau \tag{5.4}$$

$$I_2=-\frac{1}{2}\int_z^\infty\int_{L_\xi}(\xi-z)^2\varphi_{,1\alpha}\varphi_{,1\alpha}dA+2\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)\varphi_{,12}\varphi_{,2\tau}dAd\tau \tag{5.5}$$

将(5.4)、(5.5)代入(5.3)，得到

$$\tilde{\varphi}_3(z,t)\leqslant\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau+\frac{2}{\nu}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)\varphi_{,12}\varphi_{,2\tau}dAd\tau \tag{5.6}$$

应用不等式(2.23)及 Schwarz 不等式，得

$$\frac{2}{\nu}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)\varphi_{,12}\varphi_{,2\tau}dAd\tau$$

$$\leqslant\varepsilon\frac{h^2}{\pi^2}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2\varphi_{,122}^2dAd\tau+\frac{1}{\varepsilon\nu^2}\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,2\tau}^2dAd\tau \tag{5.7}$$

取$\varepsilon=\pi^3/2h^2$，将(5.7)代入(5.6)得

$$\tilde{\phi}_3(z,t)\leqslant 2\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau+\frac{4h^2}{\pi^2\nu^2}\int_0^t\int_z^\infty\int_{L_\xi}\varphi_{,\alpha\tau}\varphi_{,\alpha\tau}dAd\tau$$

$$\leqslant 2\phi_1(z,t)+\frac{4h^2}{\pi^2}\phi_2(z,t) \tag{5.8}$$

注意到，对任意$z_0>0$，有

$$\tilde{\phi}_3(z,t)>\tilde{p}_3(z+z_0,t)>z_0^2\phi_3(z+z_0,t) \tag{5.9}$$

定理2得证。

定理2蕴涵$\phi_3(0,t)$可由$\phi_1(0,t)$及$\phi_2(0,t)$界定。

由$\phi_3(z,t)$的定义，知

$$\phi_3(0,t)=\int_0^t\int_R\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau+\frac{1}{2\nu}\int_R\varphi_{,1\alpha}\varphi_{,1\alpha}dA \tag{5.10}$$

由于

$$\frac{1}{2\nu}\int_R\varphi_{,1\alpha}\varphi_{,1\alpha}dA\leqslant\phi_2(0,t) \tag{5.11}$$

因此，为了求出$\phi_3(0,t)$的显式上界，关键在于求出(5.10)中右端第一个积分之上界.显然

$$\int_0^t\int_R\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau=\int_0^t\int_0^1\int_{L_\xi}\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau+\int_0^t\int_1^\infty\int_{L_\xi}\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau \tag{5.12}$$

在(5.1)中，取$z=0$，$z_1=1$，立得

$$\int_0^t\int_1^\infty\int_{L_\xi}\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau\leqslant 2\phi_1(0,t)+\frac{4h^2}{\pi^2}\phi_2(0,t) \tag{5.13}$$

由(5.8)式，可以得到

$$\int_0^t\int_0^\infty\int_{L_\xi}\xi^2\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau\leqslant 2\int_0^t\int_0^\infty\int_{L_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau+\frac{4h^2}{\pi^2}\phi_2(0,t) \tag{5.14}$$

在下文（见(5.26)），我们将看到，$\phi_2(0,t)$ 的上界事实上是一个常数，因此，(5.14) 等价于不等式

$$\int_0^t\int_0^\infty\int_{L_\xi}\xi^2\psi_{,\xi}\psi_{,\xi}dAd\tau\leqslant 2\int_0^t\int_0^\infty\int_{L_\xi}\psi^2dAd\tau+\frac{4h^2}{\pi^2}\phi_2(0,t) \tag{5.15}$$

这里$\psi(x_1,x_2,t)$是适当光滑函数。

对(5.12)右端第一个积分，作变量变换$\chi_1=1/\zeta$，并应用(5.15)(令$\varphi_{,\alpha\beta}=\psi$)可得

$$\int_0^t\int_0^1\int_{L_\xi}\varphi_{,1\alpha\beta}\varphi_{,1\alpha\beta}dAd\tau=\int_0^t\int_1^\infty\int_{L_\zeta}\zeta^2\varphi_{,\zeta\alpha\beta}\varphi_{,\zeta\alpha\beta}dAd\tau\leqslant\int_0^t\int_0^\infty\int_{L_\zeta}\zeta^2\varphi_{,\zeta\alpha\beta}\varphi_{,\zeta\alpha\beta}dAd\tau$$

$$\leqslant 2\int_0^t\int_0^\infty\int_{L_\zeta}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau+\frac{4h^2}{\pi^2}\phi_2(0,t)$$

$$\leqslant 2\phi_1(0,t)+\frac{4h^2}{\pi^2}\phi_2(0,t) \tag{5.16}$$

结合(5.10)、(5.11)、(5.13)及(5.16)，得

$$\phi_3(0,t)\leqslant 4\phi_1(0,t)+\left(\frac{8h^2}{\pi^2}+1\right)\phi_2(0,t) \tag{5.17}$$

由定义(2.16)及(5.17)知，要求$\phi(0,t)$的上界，只需求出$\phi_1(0,t)$，$\phi_2(0,t)$的显式上界.为此构造辅助函数

$$\omega(x_1,x_2,t)=\left\{\int_0^{x_2}\tilde{g}_2(\eta,t)d\eta+x_1\left[\tilde{g}_1(x_2,t)+\gamma\int_0^{x_2}\tilde{g}_2(\eta,t)d\eta\right]\right\}\exp[-\gamma x_1] \tag{5.18}$$

这里$\gamma>0$是待定常数.

容易验证ω满足与φ基本相同的初边条件:

$$\left.\begin{array}{l}\omega(x_1,0,t)=0,\ \omega(x_1,h,t)=(1+\gamma x_1)\exp[-\gamma x_1]\int_0^h\tilde{g}_2(\eta,t)d\eta \\ \omega_2(x_1,0,t)=\omega_{,2}(x_1,h,t)=0\end{array}\right\} \tag{5.19}$$

$$\omega_{,\alpha}(0,x_2,t)=\tilde{g}_\alpha(x_2,t)\qquad(\alpha=1,2) \tag{5.20}$$

$$\omega_{,\alpha}(x_1,x_2,0)=0\qquad(\alpha=1,2) \tag{5.21}$$

$$\omega,\omega_{,\alpha},\omega_{,\alpha\beta}\to 0\qquad(\text{对}x_2,t\text{一致})\text{当}x_1\to\infty \tag{5.22}$$

并且有

$$\omega(0,x_2,t)=\int_0^{x_2}\tilde{g}_2(\eta,t)\,d\eta=\varphi(0,x_2,t) \tag{5.23}$$

由定义(2.12)可知

$$\begin{aligned}\phi_1(0,t)&=-\int_0^t\int_{L_0}(\varphi_{,\alpha}\varphi_{,\alpha1}-\varphi\varphi_{,\alpha\alpha1}+\frac{1}{\nu}\varphi\varphi_{,1\tau})dx_2d\tau\\ &=\int_0^t\int_R(\omega_{,\alpha}\varphi_{,\alpha\beta}-\omega\varphi_{,\alpha\alpha\beta}+\frac{1}{\nu}\omega\varphi_{,\beta\tau})_{,\beta}dAd\tau\\ &=\int_0^t\int_R\omega_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau+\frac{1}{\nu}\int_R\omega_{,\beta}\varphi_{,\beta}dA-\frac{1}{\nu}\int_0^t\int_R\omega_{,\beta\tau}\varphi_{,\beta}dAd\tau\\ &\leqslant\frac{\varepsilon_1}{2}\int_0^t\int_R\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau+\frac{1}{2\varepsilon_1}\int_0^t\int_R\omega_{,\alpha\beta}\omega_{,\alpha\beta}dAd\tau+\frac{\varepsilon_2}{2\nu}\int_R\varphi_{,\beta}\varphi_{,\beta}dA\\ &\quad+\frac{1}{2\varepsilon_2\nu}\int_R\omega_{,\beta}\omega_{,\beta}dA+\frac{\varepsilon_3h^2}{2\pi^2}\int_0^t\int_R\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\tau\\ &\quad+\frac{1}{2\varepsilon_3\nu^2}\int_0^t\int_R\omega_{,\beta\tau}\omega_{,\beta\tau}dAd\tau\end{aligned} \tag{5.24}$$

在上式中取 $\varepsilon_1=1/2$, $\varepsilon_2=1/2$, $\varepsilon_3=\pi^2/2h^2$, 得到

$$\begin{aligned}\phi_1(0,t)&\leqslant 2\int_0^t\int_R\omega_{,\alpha\beta}\omega_{,\alpha\beta}dAd\tau+\frac{2}{\nu}\int_R\omega_{,\beta}\omega_{,\beta}dA\\ &\quad+\frac{2h^2}{\pi^2\nu^2}\int_0^t\int_R\omega_{,\beta\tau}\omega_{,\beta\tau}dAd\tau\end{aligned} \tag{2.25}$$

在(5.25)中, 选取适当的常数γ, 并积分, 即可得到以边界值$\tilde{g}_\alpha(\alpha=1,2)$表示的$\phi_1(0,t)$(在$R\times[0,t]$上)的最小上界.

类似地可得$\phi_2(0,t)$的显式上界

$$\begin{aligned}\phi_2(0,t)&\leqslant\frac{2}{\nu^2}\int_0^t\int_R\omega_{,\alpha\tau}\omega_{,\alpha\tau}dAd\tau+\frac{2}{\nu}\int_R\omega_{,\alpha\beta}\omega_{,\alpha\beta}dA\\ &\quad+\frac{2}{\nu}\int_0^t\int_{L_0}\omega_{,\alpha}\omega_{,\alpha\tau}dx_2d\tau+2\int_0^t\int_R\omega_{,\alpha\beta\beta}\omega_{,\alpha\beta\beta}dAd\tau\end{aligned} \tag{5.26}$$

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Phragmén-Lindelöf Alternative Results for the Initial Boundary Problem of Stokes Equation

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Abstract

In this paper we prove Phragmén-Lindelöf type alternative for the initial boundary problem of Stokes equation, i.e. we show that the energy expression for the solution of the initial boundary problem must either grow exponentially or decay exponentially with axial distance from the end of a semi-infinite strip. For the case of decay, we also establish the pointwise estimate for the maximum module of the Stokes flow and present a method for obtaining explicit bounds for the total energy.

Key words Stokes equation, initial boundary problem, Phragmén-Lindelöf alternative theorem, estimate of energy dissipation