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变厚度圆柱型正交各向异性圆形薄板的非线性非对称弯曲问题

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摘　要

本文首先导出变厚度圆柱型正交各向异性圆形薄板的非线性非对称弯曲的基本方程，利用"两变量法"，引进四个小参数，对厚度线性变化的圆柱型正交各向异性圆形薄板的非线性非对称弯曲问题进行研究，得到了挠度函数$W(r,\theta)$和应力函数$F(r,\theta)$对ε_1为N阶及对ε_2为M阶的一致有效渐近解。

关键词　变厚度正交各向异性圆板　非线性非对称弯曲　两变量法　一致有效渐近解

一、引　　言

圆形薄板的弯曲问题，在理论上和工程实践中都具有重要意义。但由于圆板的非线性非对称弯曲问题的基本微分方程是非线性的微分方程组，这就使求解析解成了一个难度非常大的问题。1961年 Fife 开始研究这一问题[1]。我国江福汝教授1982年研究了这方面的问题[2,3,4]，王新志、王林祥等人也进行了研究[5,6]，我们利用"两变量法"[7]和"混合摄动法"[18]也作了些工作[9,10]。以上的工作均为等厚度各向同性或正交各向异性圆板的弯曲问题。对于变厚度圆柱型正交各向异性圆形薄板的非线性非对称弯曲问题，国内外至今尚未见到有人作过研究。在本文中，我们仍然使用"两变量法"和"混合摄动法"，并引进四个小参数对厚度线性变化的圆柱型正交各向异性圆形薄板的非线性非对称弯曲问题进行研究，得到了挠度函数$W(r,\theta)$和应力函数$F(r,\theta)$对小参数ε_1为N阶及对小参数ε_2为M阶的一致有效渐近解。

二、变厚度正交各向异性圆形薄板的基本微分方程和边界条件

1．平衡方程

忽略体力，考虑了薄膜力N_r，N_θ和$N_{r\theta}$因转角而引起的z方向的分量后，沿z轴的力平衡条件为

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$$\frac{1}{r}\frac{\partial(Q_r\cdot r)}{\partial r}+\frac{1}{r}\frac{\partial Q_\theta}{\partial\theta}+q+N_r\frac{\partial^2W}{\partial r^2}+N_\theta\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}\right)$$
$$+2N_{r\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)=0 \tag{2.1}$$

切向轴力矩平衡条件为

$$\frac{\partial M_r}{\partial r}+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}+\frac{M_r-M_\theta}{r}-Q_r=0 \tag{2.2}$$

r 轴力矩平衡条件为

$$\frac{\partial M_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial M_\theta}{\partial\theta}+\frac{2M_{r\theta}}{r}-Q_\theta=0 \tag{2.3}$$

$q(r,\theta)$为非均布横向载荷．Q_r，Q_θ为横向剪力．M_r，M_θ为弯矩，$M_{r\theta}$为扭矩，它们分别为

$$M_r=-D_r\left[\frac{\partial^2W}{\partial r^2}+\mu_{\theta r}\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}\right)\right] \tag{2.4}$$

$$M_\theta=-D_\theta\left[\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}+\mu_{r\theta}\frac{\partial^2W}{\partial r^2}\right] \tag{2.5}$$

$$M_{r\theta}=-2D_k\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right) \tag{2.6}$$

其中

$$\left.\begin{aligned}&D_r=\frac{E_rh^3(r)}{12(1-\mu_{r\theta}\mu_{\theta r})},\ D_\theta=\frac{E_\theta h^3(r)}{12(1-\mu_{r\theta}\mu_{\theta r})},\ D_k=\frac{1}{12}Gh^3(r)\\&D_{r\theta}=D_r\mu_{\theta r}+2D_k=D_\theta\mu_{r\theta}+2D_k\end{aligned}\right\} \tag{2.7}$$

$h(r)$是圆板的厚度．E_r，E_θ分别为径向和环向的杨氏模量．$\mu_{r\theta}$，$\mu_{\theta r}$分别为径向和环向的泊松比．G为剪切模量．D_r，D_θ分别为径向和环向的抗弯刚度，$D_{r\theta}$为折合刚度，D_k是抗扭刚度．

将(2.4)～(2.6)代入(2.2)和(2.3)，而后和(2.7)一起再代入(2.1)得

$$D_r\frac{\partial^4W}{\partial r^4}+2D_{r\theta}\frac{1}{r^2}\frac{\partial^4W}{\partial r^2\partial\theta^2}+D_\theta\frac{1}{r^4}\frac{\partial^4W}{\partial\theta^4}+2D_r\frac{1}{r}\frac{\partial^3W}{\partial r^3}-2D_{r\theta}\frac{1}{r^3}\frac{\partial^3W}{\partial r\partial\theta^2}$$
$$-D_\theta\frac{1}{r^2}\frac{\partial^2W}{\partial r^2}+2(D_\theta+D_{r\theta})\frac{1}{r^4}\frac{\partial^2W}{\partial\theta^2}+D_\theta\frac{1}{r^3}\frac{\partial W}{\partial r}$$
$$+\frac{1}{r}\left[\frac{\partial^2W}{\partial r^2}+\mu_{\theta r}\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}\right)\right]\frac{d}{dr}\left(r\frac{dD_r}{dr}\right)$$
$$+\frac{dD_r}{dr}\frac{\partial}{\partial r}\left[\frac{\partial^2W}{\partial r^2}+\mu_{\theta r}\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}\right)\right]+\frac{dD_r}{dr}\left(\frac{\partial^3W}{\partial r^3}+\frac{1}{r}\frac{\partial^2W}{\partial r^2}\right)$$
$$+\frac{dD_{r\theta}}{dr}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial W}{\partial r}-\frac{W}{r}\right)-\frac{dD_\theta}{dr}\frac{1}{r^2}\left(\frac{\partial W}{\partial r}+\frac{1}{r}\frac{\partial^2W}{\partial\theta^2}\right)$$
$$+\frac{2}{r}\frac{dD_r}{dr}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)-\frac{\partial^2W}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F$$
$$-\frac{\partial^2F}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W+2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W}{\partial\theta}=q(r,\theta) \tag{2.8}$$

此处取[11] $N_r=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F$，$N_\theta=\frac{\partial^2F}{\partial r^2}$，$N_{r\theta}=-\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F}{\partial\theta}$．

2．协调方程

非对称大挠度问题的中面变形几何方程为

$$\left.\begin{aligned}&\varepsilon_r^0=\frac{\partial u_r}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2,\quad \varepsilon_\theta^0=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{1}{2r^2}\left(\frac{\partial W}{\partial \theta}\right)^2\\&\gamma_{r\theta}^0=\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial W}{\partial \theta}\frac{\partial W}{\partial r}\end{aligned}\right\}\tag{2.9}$$

其中 ε_r^0，ε_θ^0，$\gamma_{r\theta}^0$为中面的应变分量，u_r，u_θ分别为中面在径向和环向的位移分量。

薄膜力N_r，N_θ，$N_{r\theta}$与应变分量的关系为

$$\left.\begin{aligned}&N_r=\frac{E_r h(r)}{1-\mu_{r\theta}\mu_{\theta r}}\varepsilon_r^0+\frac{\mu_{r\theta}E_\theta h(r)}{1-\mu_{r\theta}\mu_{\theta r}}\varepsilon_\theta^0\\&N_\theta=\frac{\mu_{\theta r}E_r h(r)}{1-\mu_{r\theta}\mu_{\theta r}}\varepsilon_r^0+\frac{E_\theta h(r)}{1-\mu_{r\theta}\mu_{\theta r}}\varepsilon_\theta^0\\&N_{r\theta}=Gh(r)\gamma_{r\theta}^0\end{aligned}\right\}\tag{2.10}$$

由(2.9)式得协调方程

$$\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\varepsilon_r^0+\frac{\partial^2\varepsilon_\theta^0}{\partial r^2}-\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial \gamma_{r\theta}^0}{\partial \theta}-\frac{2}{r}\frac{\partial \varepsilon_r^0}{\partial r}+\frac{2}{r}\frac{\partial \varepsilon_\theta^0}{\partial r}-\frac{2}{r^2}\frac{\partial \gamma_{r\theta}^0}{\partial \theta}$$

$$=\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W}{\partial \theta}\right)^2-\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)W\tag{2.11}$$

将(2.10)代入上式，得到用挠度函数W和应力函数F表示的协调方程

$$\frac{1}{E_\theta}\frac{1}{h}\frac{\partial^4 F}{\partial r^4}+\frac{1}{E_r}\frac{1}{h}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)F$$

$$+\frac{1}{h}\left(\frac{1}{G}-\frac{2\mu_{\theta r}}{E_\theta}\right)\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F}{\partial \theta}+2\frac{1}{E_\theta h}\frac{1}{r}\frac{\partial^3 F}{\partial r^3}$$

$$-\frac{2}{E_r h}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)F+\frac{2}{h}\left(\frac{1}{G}-\frac{2\mu_{r\theta}}{E_r}\right)\frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F^2}{\partial \theta^2}$$

$$+\frac{1}{h^2}\frac{dh}{dr}\left[\frac{1+\mu_{r\theta}}{E_r}\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)F-\frac{2}{E_\theta}\left(\frac{1}{r}\frac{\partial^2 F}{\partial r^2}+\frac{\partial^3 F}{\partial r^3}\right)\right.$$

$$\left.-\frac{1}{G}\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2 F}{\partial \theta^2}\right]-\frac{2\mu_{r\theta}}{E_r}\frac{1}{h^3}\left(\frac{dh}{dr}\right)^2\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)F$$

$$+\frac{2}{E_\theta h^3}\left(\frac{dh}{dr}\right)^2\frac{\partial^2 F}{\partial r^2}-\frac{1}{h^2}\frac{d^2 h}{dr^2}\left[\frac{1}{E_\theta}\frac{\partial^2 F}{\partial r^2}-\frac{\mu_{r\theta}}{E_r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)F\right]$$

$$=\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W}{\partial \theta}\right)^2-\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)W\tag{2.12}$$

(2.8)和(2.12)构成了变厚度正交各向异性圆形薄板弯曲问题的基本方程。

板厚度的变化规律可以表达为

$$h(r)=h_0\left[1+b\frac{r}{C}+b_1\left(\frac{r}{C}\right)^2+b_2\left(\frac{r}{C}\right)^3+\cdots\right]$$

其中 h_0为板中心处的厚度，C为圆板的半径，且$|b+b_1+b_2+\cdots|<1$。为了简便，仅考虑厚度线性变化的圆板，即在级数展开中仅取两项

$$h(r)=h_0\left(1+b\frac{r}{C}\right)\qquad(|b|<1)\tag{2.13}$$

在本文中我们仅讨论周边可移夹支的情况，当然本文的方法可使用于任何支承情况。可移夹支的边界条件为[6]

$$W|_{r=C}=\frac{\partial W}{\partial r}\bigg|_{r=C}=N_r|_{r=C}=N_{r\theta}|_{r=C}=0 \qquad (2.14\text{a,b,c,d})$$

$$W|_{r=0},\ \frac{\partial W}{\partial r}\bigg|_{r=0},\ N_r|_{r=0},\ N_{r\theta}|_{r=0}\ \text{取有限值} \qquad (2.15\text{a,b,c,d})$$

引入无量纲变量[9]

$$\tilde{W}=\frac{W}{C},\ \tilde{r}=\frac{r}{C},\ \tilde{F}=\frac{F}{E_rC^2h_0},\ \tilde{q}=\frac{Cq}{E_rh_0}$$

将它们代入(2.8)，(2.12)式，并注意(2.13)，得（忽略符号“～”）

$$\begin{aligned}&(1+br)^3\Big[\varepsilon_1^2\frac{\partial^4W}{\partial r^4}+\varepsilon_2^2\frac{1}{r^2}\frac{\partial^4W}{\partial r^2\partial\theta^2}+\delta_2\varepsilon_1^2\frac{1}{r^4}\frac{\partial^4W}{\partial\theta^4}+2\varepsilon_1^2\frac{1}{r}\frac{\partial^3W}{\partial r^3}\\&-\varepsilon_2^2\frac{1}{r^3}\frac{\partial^3W}{\partial r\partial\theta^2}-\delta_2\varepsilon_1^2\frac{1}{r^2}\frac{\partial^2W}{\partial r^2}+(2\delta_2\varepsilon_1^2+\varepsilon_2^2)\frac{1}{r^4}\frac{\partial^2W}{\partial\theta^2}+\delta_2\varepsilon_1^2\frac{1}{r^3}\frac{\partial W}{\partial r}\Big]\\&+3b(1+br)^2\Big[\varepsilon_1^2\frac{1}{r}\frac{\partial^2W}{\partial r^2}+\mu_{\theta r}\varepsilon_1^2\frac{1}{r}\Big(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}\Big)\\&+\varepsilon_1^2\frac{\partial}{\partial r}\Big(\frac{\partial^2W}{\partial r^2}+\mu_{\theta r}\Big(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}\Big)\Big)+\frac{1}{2}\varepsilon_2^2\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big(\frac{\partial W}{\partial r}-\frac{W}{r}\Big)\\&-\delta_2\varepsilon_1^2\frac{1}{r}\Big(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}\Big)+\Big(\frac{1}{2}\varepsilon_2^2-\mu_{\theta r}\varepsilon_1^2\Big)\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W}{\partial\theta}\Big]\\&+6b^2(1+br)\varepsilon_1^2\Big[\frac{\partial^2W}{\partial r^2}+\mu_{\theta r}\Big(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}\Big)\Big]-\frac{\partial^2W}{\partial r^2}\Big(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2F}{\partial\theta^2}\Big)\\&-\frac{\partial^2F}{\partial r^2}\Big(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}\Big)+2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W}{\partial\theta}=q(r,\theta)\end{aligned} \qquad (2.16)$$

$$\begin{aligned}&(1+br)^2\Big[\frac{\partial^4F}{\partial r^4}+\delta_2\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)F+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F}{\partial\theta}\\&+2\frac{1}{r}\frac{\partial^3F}{\partial r^3}-2\delta_2\frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)F+2\delta_1\frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2F}{\partial\theta^2}\Big]\\&+b(1+br)\Big[(1+\mu_{r\theta})\delta_2\frac{1}{r}\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)F-2\Big(\frac{1}{r}\frac{\partial^2F}{\partial r^2}+\frac{\partial^3F}{\partial r^3}\Big)\\&-(\delta_1+2\delta_2\mu_{r\theta})\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2F}{\partial\theta^2}\Big]\\&+2b^2\Big[\frac{\partial^2F}{\partial r^2}-2\mu_{r\theta}\delta_2\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)F\Big]\\&=(1+br)^3\delta_2\Big[\Big(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W}{\partial\theta}\Big)^2-\frac{\partial^2W}{\partial r^2}\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)W\Big]\end{aligned} \qquad (2.17)$$

$$W|_{r=1}=\frac{\partial W}{\partial r}\bigg|_{r=1}=N_r|_{r=1}=N_{r\theta}|_{r=1}=0 \qquad (2.18\text{a,b,c,d})$$

$$W|_{r=0},\ \frac{\partial W}{\partial r}\bigg|_{r=0},\ N_r|_{r=0},\ N_{r\theta}|_{r=0},\ \text{取有限值} \qquad (2.19\text{a,b,c,d})$$

其中 $\varepsilon_1^2=\dfrac{h_0^2}{12(1-\mu_{r\theta}\mu_{\theta r})C^2}\ll 1,\ \varepsilon_2^2=2\left(\varepsilon_1^2\mu_{\theta r}+\dfrac{1}{6}\ \dfrac{Gh_0^2}{E_rC^2}\right)\ll 1$

$$\delta_1=E_\theta\left(\frac{1}{G}-2\frac{\mu_{\theta r}}{E_\theta}\right),\ \delta_2=\frac{E_\theta}{E_r}<1$$

三、微分算子展开

为了得到递推方程和递推边界条件，首先将微分算子展开．为此，我们在$r=1$的邻域引进如下变量

$$\xi=\frac{u(r,\theta)}{\varepsilon_1},\ \eta=r,\ \theta=\theta \tag{3.1}$$

$u(r,\theta)$是一个以后待定的函数，且$u(r,\theta)>0$，$u(1,\theta)=0$，$u(r,\theta)=u(r,\theta+2\pi)$．把$\xi,\eta,\theta$视为独立变量，将对$r$，$\theta$的偏导数换为对$\xi$，$\eta$，$\theta$的偏导数

$$\frac{\partial}{\partial r}=\frac{\partial}{\partial\xi}\ \frac{\partial\xi}{\partial r}+\frac{\partial}{\partial\eta}\ \frac{\partial\eta}{\partial r}$$

$$\frac{\partial}{\partial r^i}=\varepsilon_1^{-i}(A_{i,0}+\varepsilon_1A_{i,1}+\cdots+\varepsilon_1^iA_{i,i})\qquad(i=1,2,3,4) \tag{3.2}$$

其中：$A_{1,0}=u_{,r}\dfrac{\partial}{\partial\xi},\ A_{1,1}=\dfrac{\partial}{\partial\eta},\ A_{2,0}=u_{,r}^2\dfrac{\partial^2}{\partial\xi^2}$

$$A_{2,1}=2u_{,r}\frac{\partial^2}{\partial\xi\partial\eta}+u_{,rr}\frac{\partial}{\partial\xi},\ A_{2,2}=\frac{\partial^2}{\partial\eta^2}$$

$$A_{3,0}=u_{,r}^3\frac{\partial^3}{\partial\xi^3},\quad A_{3,1}=3u_{,r}^2\frac{\partial^3}{\partial\xi^2\partial\eta}+3u_{,r}u_{,rr}\frac{\partial^2}{\partial\xi^2}$$

$$A_{3,2}=3u_{,r}\frac{\partial^3}{\partial\xi\partial\eta^2}+3u_{,rr}\frac{\partial^2}{\partial\xi\partial\eta}+u_{,rrr}\frac{\partial}{\partial\xi},\ A_{3,3}=\frac{\partial^3}{\partial\eta^3}$$

$$A_{4,0}=u_{,r}^4\frac{\partial^4}{\partial\xi^4},\quad A_{4,1}=4u_{,r}^3\frac{\partial^4}{\partial\xi^3\partial\eta}+6u_{,r}^2u_{,rr}\frac{\partial^3}{\partial\xi^3}$$

$$A_{4,2}=6u_{,r}^2\frac{\partial^4}{\partial\xi^2\partial\eta^2}+12u_{,r}u_{,rr}\frac{\partial^3}{\partial\xi^2\partial\eta}+(3u_{,rr}^2+u_{,r}u_{,rrr})\frac{\partial^2}{\partial\xi^2}$$

$$A_{4,3}=4u_{,r}\frac{\partial^4}{\partial\xi\partial\eta^3}+6u_{,rr}\frac{\partial^3}{\partial\xi\partial\eta^2}+4u_{,rrr}\frac{\partial^2}{\partial\xi\partial\eta}+u_{,rrrr}\frac{\partial}{\partial\xi}$$

$$A_{4,4}=\frac{\partial^4}{\partial\eta^4},\ \frac{\partial}{\partial\theta}=\frac{\partial}{\partial\xi}\ \frac{\partial\xi}{\partial\theta}+\frac{\partial}{\partial\theta}\ \frac{\partial\theta}{\partial\theta}$$

$$\frac{\partial}{\partial\theta^i}=\varepsilon_1^{-i}(B_{i,0}+\varepsilon_1B_{i,1}+\cdots+\varepsilon_1^iB_{i,i})\qquad(i=1,2,3,4) \tag{3.3}$$

其中 各$B_{i,i}(i=1,\ 2,\ 3,\ 4)$与前面的各$A_{i,i}$相似，只需将各$A_{i,i}$中对η的偏导数和u对r的偏导数换为对θ的偏导数即可．

四、递推方程及递推边界条件

假设挠度函数$W(r,\theta)$和应力函数$F(r,\theta)$对ε_1为N阶和对ε_2为M阶的渐近展开式为[8]

$$W(r,\theta,\varepsilon_1,\varepsilon_2)=\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m W_{nm}(r,\theta)+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+2}\varepsilon_2^m v_{nm}(\xi,\eta,\theta) \tag{4.1}$$

$$F(r,\theta,\varepsilon_1,\varepsilon_2)=\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m F_{nm}(r,\theta)+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+4}\varepsilon_2^m \psi_{nm}(\xi,\eta,\theta) \tag{4.2}$$

v_{nm}，ψ_{nm}是边界层型函数．把微分算子(3.2)，(3.3)和挠度函数及应力函数的展开式(4.1)，(4.2)代入(2.16)～(2.19)式，然后比较$\varepsilon_1\varepsilon_2$的同次幂系数，得递推方程和递推边界条件

$$\frac{\partial^2W_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00}+\frac{\partial^2F_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}$$
$$-2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{00}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{00}}{\partial\theta}=-q(r,\theta) \tag{4.3}$$

$$(1+br)^2\left[\frac{\partial^4F_{00}}{\partial r^4}+\delta_2\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00}\right.$$
$$+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{00}}{\partial\theta}+\frac{2}{r}\frac{\partial^3F_{00}}{\partial r^3}-2\delta_2\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00}$$
$$\left.+2\delta_1\frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2F_{00}}{\partial\theta^2}\right]+b(1+br)\left[(1+\mu_{r\theta})\delta_2\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00}\right.$$
$$\left.-2\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}+\frac{\partial^3}{\partial r^3}\right)F_{00}-(\delta_1+2\delta_2\mu_{r\theta})\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2F_{00}}{\partial\theta^2}\right]+2b^2\left[\frac{\partial^2F_{00}}{\partial r^2}\right.$$
$$\left.-2\delta_2\mu_{r\theta}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00}\right]-\delta_2(1+br)^3\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{00}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{00}}{\partial\theta}\right.$$
$$\left.-\frac{\partial^2W_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\right]=0 \tag{4.4}$$

$$W_{00}|_{r=1}=\left.\frac{\partial W_{00}}{\partial r}\right|_{r=1}=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00}|_{r=1}=\left.\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{00}}{\partial\theta}\right|_{r=1}$$
$$=0 \tag{4.5a,b,c,d}$$

$W_{00}|_{r=0}$，$\left.\frac{\partial W_{00}}{\partial r}\right|_{r=0}$，$\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00}|_{r=0}$，$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}F_{00}|_{r=0}$，取有限值

(4.6a,b,c,d)

$$\frac{\partial^2W_{10}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00}+\frac{\partial^2W_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{10}$$
$$+\frac{\partial^2F_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{10}+\frac{\partial^2F_{10}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}$$
$$-2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{00}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{10}}{\partial\theta}-2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{10}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{00}}{\partial\theta}=0 \tag{4.7}$$

$$(1+br)^2\left[\frac{\partial^4F_{10}}{\partial r^4}+\delta_2\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{10}\right.$$
$$+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{10}}{\partial\theta}+\frac{1}{2r}\frac{\partial^3F_{10}}{\partial r^3}-2\delta_2\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{10}$$
$$\left.+2\delta_1\frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2F_{10}}{\partial\theta^2}\right]+b(1+br)\left[(1+\mu_{r\theta})\delta_2\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{10}\right.$$

$$-2\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}+\frac{\partial^3}{\partial r^3}\right)F_{10}-(\delta_1+2\delta_2\mu_{r\theta})\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2 F_{10}}{\partial\theta^2}\Bigg]$$

$$+2b^2\left[\frac{\partial^2 F_{10}}{\partial r^2}-2\mu_{r\theta}\delta_2\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{10}\right]$$

$$-\delta_2(1+br)^3\left[2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{10}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{00}}{\partial\theta}-\frac{\partial^2 W_{10}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\right.$$

$$\left.-\frac{\partial^2 W_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{10}\right]=0 \tag{4.8}$$

$$W_{10}|_{r=1}=\frac{\partial W_{10}}{\partial r}\bigg|_{r=1}+A_{1,0}\nu_{00}|_{\eta=1}=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{10}|_{r=1}$$

$$=\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{10}}{\partial\theta}\bigg|_{r=1}=0 \tag{4.9a,b,c,d}$$

$$\left.\begin{aligned}&W_{10}|_{r=0},\ \frac{\partial W_{10}}{\partial r}\bigg|_{r=0}+A_{1,0}\nu_{00}|_{\eta=0},\ \left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{10}|_{r=0},\\&\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{10}}{\partial\theta}\bigg|_{r=0}，\text{取有限值}\end{aligned}\right\} \tag{4.10a,b,c,d}$$

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$$(1+br)^3\left[\frac{\partial^4 W_{(n-2)m}}{\partial r^4}+\frac{1}{r^2}\frac{\partial^4 W_{n(m-2)}}{\partial r^2\partial\theta^2}+\delta_2\frac{1}{r^4}\frac{\partial^4 W_{(n-2)m}}{\partial\theta^4}+\frac{2}{r}\frac{\partial^3 W_{(n-2)m}}{\partial r^3}\right.$$

$$-\frac{1}{r^3}\frac{\partial^3 W_{n(m-2)}}{\partial r\partial\theta^2}-\delta_2\frac{1}{r^2}\frac{\partial^2 W_{(n-2)m}}{\partial r^2}+2\delta_2\frac{1}{r^4}\frac{\partial^2 W_{(n-2)m}}{\partial\theta^2}+\frac{1}{r^4}\frac{\partial^2 W_{n(m-2)}}{\partial\theta^2}$$

$$\left.+\delta_2\frac{1}{r^3}\frac{\partial W_{(n-2)m}}{\partial r}\right]+3b(1+br)^2\left[\frac{1}{r}\frac{\partial^2 W_{(n-2)m}}{\partial r^2}+\frac{\mu_{\theta r}}{r^2}\frac{\partial W_{(n-2)m}}{\partial r}\right.$$

$$+\frac{\mu_{\theta r}}{r^3}\frac{\partial^2 W_{(n-2)m}}{\partial\theta^2}-\frac{1}{2r^3}\frac{\partial^2 W_{n(m-2)}}{\partial\theta^2}-\delta_2\frac{1}{r^2}\frac{\partial W_{(n-2)m}}{\partial r}-\delta_2\frac{1}{r^3}\frac{\partial^2 W_{n(m-2)}}{\partial\theta^2}$$

$$\left.+\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{n(m-2)}}{\partial\theta}-\mu_{\theta r}\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{(n-2)m}}{\partial\theta}\right]$$

$$+6b^2(1+br)\left[\frac{\partial^2 W_{(n-2)m}}{\partial r^2}+\mu_{\theta r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{(n-2)m}\right]$$

$$-\left[\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial^2 W_{ij}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{(n-i)(m-j)}+\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial^2 F_{ij}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right.\right.$$

$$\left.\left.+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{(n-i)(m-j)}-2\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{ij}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}W_{(n-i)(m-j)}\right]=0 \tag{4.11}$$

$$(1+br)^2\left[\frac{\partial^4 F_{nm}}{\partial r^4}+\delta_2\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{nm}\right.$$

$$+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{nm}}{\partial\theta}+\frac{2}{r}\frac{\partial^3 F_{nm}}{\partial r^3}-2\delta_2\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{nm}$$

$$\left.+2\delta_1\frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2 F_{nm}}{\partial\theta^2}\right]+b(1+br)\left[(1+\mu_{r\theta})\delta_2\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{nm}\right.$$

$$-\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}+\frac{\partial^3}{\partial r^3}\right)F_{nm}-(\delta_1+2\delta_2\mu_{r\theta})\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2 F_{nm}}{\partial\theta^2}\Bigg]$$

$$+2b^2\left[\frac{\partial^2 F_{nm}}{\partial r^2}-2\delta_2\mu_{r\theta}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{nm}\right]$$

$$-\delta_2(1+br)^3\Bigg[\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{ij}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}W_{(n-i)(m-j)}$$

$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial^2 W_{ij}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{(n-i)(m-j)}\Bigg]=0 \tag{4.12}$$

$$W_{nm}|_{r=1}+\nu_{(n-2)m}|_{\eta=1}=0 \tag{4.13a}$$

$$\left.\frac{\partial W_{nm}}{\partial r}\right|_{r=1}+(A_{1,0}\nu_{(n-1)m}+A_{1,1}\nu_{(n-2)m})|_{\eta=1}=0 \tag{4.13b}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{nm}|_{r=1}+\left.\left(\frac{1}{\eta}\sum_{i=0}^{1}A_{1,i}\psi_{(n-2-i)m}+\frac{1}{\eta^2}\sum_{i=0}^{2}B_{2,i}\psi_{(n-2-i)m}\right)\right|_{\eta=1}=0 \tag{4.13c}$$

$$\left.\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{nm}}{\partial\theta}\right|_{r=1}+\left.\left(\sum_{i=0}^{1}\sum_{j=0}^{1}A_{1,i}\frac{1}{\eta}B_{1,j}\psi_{(n-2-i-j)m}\right)\right|_{\eta=1}=0 \tag{4.13d}$$

$$W_{nm}|_{r=0}+\nu_{(n-2)m}|_{\eta=0}\text{取有限值} \tag{4.14a}$$

$$\left.\frac{\partial W_{nm}}{\partial r}\right|_{r=0}+(A_{1,0}\nu_{(n-1)m}+A_{1,1}\nu_{(n-2)m})|_{\eta=0}\ \text{取有限值} \tag{4.14b}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{nm}|_{r=0}+\Bigg(\frac{1}{\eta}\sum_{i=0}^{1}A_{1,i}\psi_{(n-3-i)m}$$

$$\left.+\frac{1}{\eta^2}\sum_{i=0}^{2}B_{2,i}\psi_{(n-2-i)m}\Bigg)\right|_{\eta=0}\ \text{取有限值} \tag{4.14c}$$

$$\left.\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{nm}}{\partial\theta}\right)\right|_{r=0}+\left.\left(\sum_{i=0}^{1}\sum_{j=0}^{1}A_{1,i}\frac{1}{\eta}B_{1,j}\psi_{(n-2-i-j)m}\right)\right|_{\eta=0}\text{取有限值} \tag{4.14d}$$

边界层型函数ν_{nm}，ψ_{nm}的递推方程为

$$(1+b\eta)^3\left[A_{4,0}\nu_{00}+\delta_2\frac{1}{\eta^4}B_{4,0}\nu_{00}\right]-\Bigg[A_{2,2}F_{00}\frac{1}{\eta^2}B_{2,0}\nu_{00}$$

$$-2A_{1,1}\frac{1}{\eta}B_{1,1}F_{00}A_{1,0}\frac{1}{\eta}B_{1,0}\nu_{00}+A_{2,0}\nu_{00}\left(\frac{1}{\eta}A_{1,1}+\frac{1}{\eta^2}B_{2,2}\right)F_{00}\Bigg]=0 \tag{4.15}$$

$$(1+b\eta)^2\left[A_{4,0}\psi_{00}+\delta_2\frac{1}{\eta^2}B_{2,0}\frac{1}{\eta^2}B_{2,0}\psi_{00}+\delta_1 A_{1,0}\frac{1}{\eta}B_{1,0}A_{1,0}\frac{1}{\eta}B_{1,0}\psi_{00}\right]$$

$$-\delta_2(1+b\eta)^3\Bigg[2A_{1,1}\frac{1}{\eta}B_{1,1}W_{00}A_{1,0}\frac{1}{\eta}B_{1,0}\nu_{00}-A_{2,2}W_{00}\frac{1}{\eta^2}B_{2,0}\nu_{00}$$

$$-A_{2,0}\nu_{00}\left(\frac{1}{\eta}A_{1,1}+\frac{1}{\eta^2}B_{2,2}\right)W_{00}+A_{1,0}\frac{1}{\eta}B_{1,0}\nu_{00}A_{1,0}\frac{1}{\eta}B_{1,0}\nu_{00}$$

$$-A_{2,0}\nu_{00}\frac{1}{\eta^2}B_{2,0}\nu_{00}\Big]=0 \tag{4.16}$$

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$$(1+b\eta)^3\Big[\sum_{i=0}^{4}A_{4,i}\nu_{(n-i)m}+\frac{1}{\eta^2}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2,i}B_{2,j}\nu_{(n+2-i-j)(m-2)}$$

$$+\delta_2\frac{1}{\eta^4}\sum_{i=0}^{4}B_{4,i}\nu_{(n-i)m}+2\frac{1}{\eta}\sum_{i=0}^{3}A_{3,i}\nu_{(n-1-i)m}$$

$$-\frac{1}{\eta^3}\sum_{i=0}^{1}\sum_{j=0}^{2}A_{1,i}B_{2,j}\nu_{(n+1-i-j)(m-2)}-\delta_2\frac{1}{\eta^2}\sum_{i=0}^{2}A_{2,i}\nu_{(n-2-i)m}$$

$$+2\delta_2\frac{1}{\eta^4}\sum_{i=0}^{2}B_{2,i}\nu_{(n-2-i)m}+\frac{1}{\eta^4}\sum_{i=0}^{4}B_{2,i}\nu_{(n-i)(m-2)}$$

$$+\delta_2\frac{1}{\eta^3}\sum_{i=0}^{1}A_{1,i}\nu_{(n-3-i)m}\Big]+3b(1+b\eta)^3\Big[\frac{1}{\eta}\sum_{i=0}^{2}A_{2,i}\nu_{(n-2-i)m}$$

$$+\mu_{\theta r}\frac{1}{\eta^2}\sum_{i=0}^{1}A_{1,i}\nu_{(n-3-i)m}+\mu_{\theta r}\frac{1}{\eta^3}\sum_{i=0}^{2}B_{2,i}\nu_{(n-2-i)m}+\sum_{i=0}^{3}A_{3,i}\nu_{(n-1-i)m}$$

$$+\mu_{\theta r}\sum_{i=0}^{1}\sum_{j=0}^{1}A_{1,i}\frac{1}{\eta}A_{1,j}\nu_{(n-i-j)m}+\mu_{\theta r}\sum_{i=0}^{1}\sum_{j=0}^{2}A_{1,i}\frac{1}{\eta^2}B_{2,j}\nu_{(n-1-i-j)m}$$

$$+\frac{1}{2}\frac{1}{\eta^2}\sum_{i=0}^{2}\sum_{j=0}^{1}B_{2,i}A_{1,j}\nu_{(n+1-i-j)(m-2)}-\frac{1}{2}\frac{1}{\eta^3}\sum_{i=0}^{2}B_{2,i}\nu_{(n-i)(m-2)}$$

$$-\delta_2\frac{1}{\eta^2}\sum_{i=0}^{1}A_{1,i}\nu_{(n-2-i)m}-\delta_2\frac{1}{\eta^3}\sum_{i=0}^{2}B_{2,i}\nu_{(n-2-i)m}$$

$$+\frac{1}{2}\frac{1}{\eta}\sum_{i=0}^{1}\sum_{j=0}^{1}A_{1,i}\frac{1}{\eta}B_{1,j}\nu_{(n-i-j)(m-2)}-\mu_{\theta r}\frac{1}{\eta}\sum_{i=0}^{1}\sum_{j=0}^{1}A_{1,i}\frac{1}{\eta}B_{1,j}\nu_{(n-2-i-j)m}\Big]$$

$$+6b^2(1+b\eta)\Big[\sum_{i=0}^{2}A_{2,i}\nu_{(n-2-i)m}+\mu_{\theta r}\frac{1}{\eta}\sum_{i=0}^{1}A_{1,i}\nu_{(n-3-i)m}+\mu_{\theta r}\frac{1}{\eta^2}\sum_{i=0}^{2}B_{2,i}\nu_{(n-2-i)m}\Big]$$

$$-\Big[\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{1}A_{2,2}W_{lr}\frac{1}{\eta}A_{1,i}\psi_{(n-3-i-l)(m-r)}+\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}A_{2,2}W_{lr}\frac{1}{\eta^2}B_{2,i}\psi_{(n-2-i-l)(m-r)}$$

$$+\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{1}A_{2,2}F_{lr}\frac{1}{\eta}A_{1,i}\nu_{(n-1-l-i)(m-r)}+\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}A_{2,2}F_{lr}\frac{1}{\eta^2}B_{2,i}\nu_{(n-l-i)(m-r)}$$

$$-2\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}A_{1,1}\frac{1}{\eta}B_{1,1}F_{lr}A_{1,i}\frac{1}{\eta}B_{1,j}\nu_{(n-l-i-j)(m-r)}$$

$$+\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2'}\left(\frac{1}{\eta}A_{1,1}+\frac{1}{\eta^2}B_{2,2}\right)F_{lr}A_{2,i}\nu_{(n-i-l)(m-r)}$$

$$+\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{1}A_{2,i}\nu_{lr}\frac{1}{\eta}A_{1,j}\psi_{(n-3-i-j-l)(m-r)}$$

$$+\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2,i}\nu_{lr}\frac{1}{\eta^2}B_{2,j}\psi_{(n-2-i-j-l)(m-r)}$$

$$+\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{1}A_{2,i}\psi_{lr}\frac{1}{\eta}A_{1,j}\nu_{(n-3-i-j-l)(m-r)}$$

$$+\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2,i}\psi_{lr}\frac{1}{\eta^2}B_{2,j}\nu_{(n-2-i-j-l)(m-r)}$$

$$-2\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{s=0}^{1}A_{1,i}\frac{1}{\eta}B_{1,j}'\psi_{lr}A_{1,k}\frac{1}{\eta}B_{1,s}\nu_{(n-2-l-i-j-k-s)(m-r)}\Bigg]=0 \qquad (4.17)$$

$$(1+b\eta)^2\Bigg[\sum_{i=0}^{4}A_{4,i}\psi_{(n-i)m}+\delta_2\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{1}{\eta}A_{1,i}\frac{1}{\eta}A_{1,j}\psi_{(n-2-i-j)m}$$

$$+\delta_2\sum_{i=0}^{1}\sum_{j=0}^{2}\frac{1}{\eta}A_{1,i}\frac{1}{\eta^2}B_{2,j}\psi_{(n-1-i-j)m}+\delta_2\sum_{i=0}^{2}\sum_{j=0}^{1}\frac{1}{\eta^2}B_{2,i}\frac{1}{\eta}A_{1,j}\psi_{(n-1-i-j)m}$$

$$+\delta_2\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{1}{\eta^2}B_{2,i}\frac{1}{\eta^2}B_{2,j}\psi_{(n-i-j)m}+\delta_1\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{s=0}^{1}A_{1,i}\frac{1}{\eta}B_{1,j}A_{1,k}$$

$$\cdot\frac{1}{\eta}B_{1,s}\psi_{(n-i-j-k-s)m}-2\delta_2\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{1}{\eta}A_{1,i}\frac{1}{\eta}A_{1,j}\psi_{(n-2-i-j)m}-2\delta_2\sum_{i=0}^{1}\sum_{j=0}^{2}\frac{1}{\eta}A_{1,i}$$

$$\cdot\frac{1}{\eta^2}B_{2,j}\psi_{(n-1-i-j)m}+2\delta_1\sum_{i=0}^{1}\sum_{j=0}^{2}\frac{1}{\eta^2}A_{1,i}\frac{1}{\eta}B_{2,j}\psi_{(n-1-i-j)m}\Bigg]$$

$$+b(1+b\eta)\Bigg[(1+\mu_{r\theta})\delta_2\left(\sum_{i=0}^{1}\frac{1}{\eta^2}A_{1,i}\psi_{(n-3-i)m}+\sum_{i=0}^{2}\frac{1}{\eta^3}B_{2,i}\psi_{(n-2-i)m}\right)$$

$$-2\left(\sum_{i=0}^{2}\frac{1}{\eta}A_{2,i}\psi_{(n-2-i)m}+\sum_{i=0}^{3}A_{3,i}\psi_{(n-1-i)m}\right)$$

$$-(\delta_1+2\delta_2\mu_{r\theta})\sum_{i=0}^{1}\sum_{j=0}^{2}\frac{1}{\eta}A_{1,i}\frac{1}{\eta}B_{2,j}\psi_{(n-1-i-j)m}\Bigg]$$

$$+2b^2\Bigg[\sum_{i=0}^{2}A_{2,i}\psi_{(n-2-i)m}-2\mu_{r0}\delta_2\left(\sum_{i=0}^{1}\frac{1}{\eta}A_{1,i}\psi_{(n-3-i)m}+\sum_{i=0}^{2}\frac{1}{\eta^2}B_{2,i}\psi_{(n-2-i)m}\right)\Bigg]$$

$$-\delta_2(1+b\eta)^3\Big[2\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}A_{1,1}\frac{1}{\eta}B_{1,1}W_{lr}A_{1,i}\frac{1}{\eta}B_{1,j}\nu_{(n-i-j-l)(m-r)}$$

$$-\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{1}A_{2,2}W_{lr}\frac{1}{\eta}A_{1,i}\nu_{(n-1-i-l)(m-r)}-\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}A_{2,2}W_{lr}\frac{1}{\eta^2}B_{2,i}\nu_{(n-i-l)(m-r)}$$

$$-\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}\frac{1}{\eta}A_{1,1}W_{lr}A_{2,i}\nu_{(n-i-l)(m-r)}-\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}\frac{1}{\eta^2}B_{2,2}W_{lr}A_{2,i}\nu_{(n-i-l)(m-r)}$$

$$+\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{s=0}^{1}A_{1,i}\frac{1}{\eta}B_{1,j}\nu_{lr}A_{1,k}\frac{1}{\eta}B_{1,s}\nu_{(n-i-j-k-s-l)(m-r)}$$

$$-\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{1}A_{2,i}\nu_{lr}\frac{1}{\eta}A_{1,j}\nu_{(n-1-i-j-l)(m-r)}$$

$$-\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2,i}\nu_{lr}\frac{1}{\eta^2}B_{2,j}\nu_{(n-i-j-l)(m-r)}\Big]=0 \qquad (4.18)$$

五、挠度函数和应力函数的渐近解

在方程(4.4)中，$\delta_2<1$，将其视为小参数．利用正则摄动法求解W_{00}和F_{00}．为此我们令

$$W_{00}=\sum_{i=0}^{p}\delta_2^iW_{00i},\quad F_{00}=\sum_{i=0}^{p}\delta_2^iF_{00i} \qquad (5.1\sim5.2)$$

将(5.1)和(5.2)代入(4.3)～(4.6)式，然后比较δ_2同次幂系数，得W_{00}和F_{00}的递推方程和递推边界条件

$$\frac{\partial^2W_{000}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{000}+\frac{\partial^2F_{000}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{000}$$

$$-2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{000}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{000}}{\partial\theta}=-q(r,\theta) \qquad (5.3)$$

$$(1+br)^2\left[\frac{\partial^4F_{000}}{\partial r^4}+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{000}}{\partial\theta}+\frac{2}{r}\frac{\partial^3F_{000}}{\partial r^3}+2\delta_1\frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2F_{000}}{\partial\theta^2}\right]$$

$$-b(1+br)\left[2\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}+\frac{\partial^3}{\partial r^3}\right)F_{000}+\delta_1\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2F_{000}}{\partial\theta^2}\right]+2b^2\frac{\partial^2F_{000}}{\partial r^2}=0$$

$$(5.4)$$

$$W_{000}|_{r=1}=\frac{\partial W_{000}}{\partial r}\Big|_{r=1}=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{000}|_{r=1}=\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{000}}{\partial\theta}\Big|_{r=1}=0$$

(5.5a,b,c,d)

$W_{000}|_{r=0}$，$\dfrac{\partial W_{000}}{\partial r}\Big|_{r=0}$，$\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right)F_{000}|_{r=0}$，$\dfrac{\partial}{\partial r}\dfrac{1}{r}\dfrac{\partial F_{000}}{\partial\theta}\Big|_{r=0}$ 取有限值

(5.6a,b,c,d)

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$$\sum_{i=0}^{n}\frac{\partial^2 W_{00i}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00(n-i)}+\sum_{i=0}^{n}\frac{\partial^2 F_{00i}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00(n-i)}$$

$$-2\sum_{i=0}^{n}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{00i}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{00(n-i)}}{\partial\theta}=0 \tag{5.7}$$

$$(1+br)^2\left[\frac{\partial^4 F_{00n}}{\partial r^4}+\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00(n-1)}\right.$$

$$+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial F_{00n}}{\partial\theta}+\frac{2}{r}\frac{\partial^3 F_{00n}}{\partial r^3}-\frac{2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00(n-1)}$$

$$\left.+2\delta_1\frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2 F_{00n}}{\partial\theta^2}\right]+b(1+br)\left[(1+\mu_{r\theta})\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00(n-1)}\right.$$

$$\left.-2\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}+\frac{\partial^3}{\partial r^3}\right)F_{00n}-\delta_1\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2 F_{00n}}{\partial\theta^2}-2\mu_{r\theta}\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial^2 F_{00(n-1)}}{\partial\theta^2}\right]$$

$$+2b^2\left[\frac{\partial^2 F_{00n}}{\partial r^2}-2\mu_{r\theta}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00(n-1)}\right]$$

$$+(1+br)^3\left[\sum_{i=0}^{n}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W_{00i}}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}W_{00(n-1-i)}\right.$$

$$\left.-\sum_{i=0}^{n}\frac{\partial^2 W_{00i}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00(n-1-i)}\right]=0 \tag{5.8}$$

$$W_{00n}|_{r=1}=\frac{\partial W_{00n}}{\partial r}\bigg|_{r=1}=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00n}|_{r=1}=\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}F_{00n}|_{r=1}=0$$

(5.9a,b,c,d)

$W_{00n}|_{r=0}$, $\frac{\partial W_{00n}}{\partial r}\Big|_{r=0}$, $\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00n}|_{r=0}$, $\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}F_{00n}|_{r=0}$取有限值

(5.10a,b,c,d)

利用分离变量法求解(5.4)式，为此令

$$F_{000}=R(r)\Phi(\theta) \tag{5.11}$$

将(5.11)代入(5.4)式，得

$$(1+br)^2\left(\frac{d^4R}{dr^4}+\frac{2}{r}\frac{d^3R}{dr^3}-\frac{\lambda\delta_1}{r^2}\frac{d^2R}{dr^2}+\frac{\lambda\delta_1}{r^3}\frac{dR}{dr}-\frac{\lambda\delta_1}{r^4}R\right)$$

$$-b(1+br)\left(2\frac{d^3R}{dr^3}+\frac{2}{r}\frac{d^2R}{dr^2}-\frac{\lambda\delta_1}{r^2}\frac{dR}{dr}+\frac{\lambda\delta_1}{r^3}R\right)+2b^2\frac{d^2R}{dr^2}=0 \tag{5.12}$$

$$\frac{d^2\Phi}{d\theta^2}+\lambda\Phi(\theta)=0 \tag{5.13}$$

方程(5.13)满足自然周期性边界条件的解为

$$\Phi(\theta)=A\cos k\theta+B\sin k\theta\qquad(k=1,\ 2,\ 3,\ \cdots) \tag{5.14}$$

其中 $k^2=\lambda$. 在(5.12)中，$|b|<1$，可视为小参数. 利用正则摄动法可以求得$R(r)$.为此令

$$R(r)=\sum_{i=0}^{\infty}b^iR_i(r) \tag{5.14}$$

将(5.14)代入(5.12)，然后比较b的同次幂系数得

$$\frac{d^4R_0}{dr^4}+\frac{2}{r}\frac{d^3R_0}{dr^3}-\frac{A\delta_1}{r^2}\frac{d^2R_0}{dr^2}+\frac{\lambda\delta_1}{r^3}\frac{dR_0}{dr}-\frac{\lambda\delta_1}{r^4}R_0=0 \tag{5.15}$$

$$\frac{d^4R_1}{dr^4}+\frac{2}{r}\frac{d^3R_1}{dr^3}-\frac{\lambda\delta_1}{r^2}\frac{d^2R_1}{dr^2}+\frac{\lambda\delta_1}{r^3}\frac{dR_1}{dr}-\frac{\lambda\delta_1}{r^4}R_1$$
$$-\left(2\frac{d^3R_0}{dr^3}+\frac{2}{r}\frac{d^2R^0}{dr^2}-\frac{\lambda\delta_1}{r^2}\frac{dR_0}{dr}+\frac{\lambda\delta_1}{r^3}R_0\right)=0 \tag{5.16}$$

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截至b的一级近似并满足边界条件(5.5c,d)和(5.6c,d)的F_{000}为

$$F_{000}=[(c_{01}+bc_{11})r-(c_{03}+bc_{13})(r^{\sqrt{1+\delta_1}+1}-\sqrt{1+\delta_1}\,r)$$
$$+bQc_{03}(r^{\sqrt{1+\delta_1}+2}-(1+\sqrt{1+\delta_1})r)](A\cos\theta+B\sin\theta) \tag{5.17}$$

其中
$$Q=\frac{\sqrt{1+\delta_1}(2+\delta_1+2\sqrt{1+\delta_1})}{(1+\sqrt{1+\delta_1})^2(1+2\sqrt{1+\delta_1})}.$$

将(5.17)式代入(5.3)式，便可得到W_{000}的偏微分方程．如果$q(r,\theta)$给定，可以用付里叶级数法求出W_{000}；再利用(5.7)～(5.10)可以求得所有W_{00n}和F_{00n}，即求得了W_{00}和F_{00}．将W_{00}和F_{00}代入(4.11)和(4.12)(取$n=1$，$m=0$，且带负下标的量均取0)，可以获得关于W_{10}和F_{10}的微分方程．解微分方程，并利用边界条件可求得W_{10}和F_{10}.这样可逐次求得W_{nm}和F_{nm}($n=0, 1, 2, \cdots, N$; $m=0, 1, 2, \cdots, M$).

将F_{00}代入方程(4.15)得

$$(1+b\eta)^3\left[u_{,r}^4\frac{\partial^4\nu_{00}}{\partial\xi^4}+\delta_2\frac{1}{\eta^4}u_{,\theta}^4\frac{\partial^4\nu_{00}}{\partial\xi^4}\right]-\left[\frac{1}{\eta^2}\frac{\partial^2F_{00}}{\partial\eta^2}u_{,\theta}^2\frac{\partial^2\nu_{00}}{\partial\xi^2}\right.$$
$$\left.-2\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial F_{00}}{\partial\theta}\frac{1}{\eta}u_{,r}u_{,\theta}\frac{\partial^2\nu_{00}}{\partial\xi^2}+u_{,r}^2\frac{\partial^2\nu_{00}}{\partial\xi^2}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00}\right]=0 \tag{5.18}$$

今取
$$(1+b\eta)^3\left(\left(\frac{\partial u}{\partial r}\right)^4+\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^4\right)=\frac{1}{\eta^2}\left(\frac{\partial u}{\partial\theta}\right)^2\frac{\partial^2F_{00}}{\partial\eta^2}-2\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\frac{1}{\eta}\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial F_{00}}{\partial\theta}$$
$$+\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)F_{00} \tag{5.19}$$

方程(5.19)是关于$u(r,\theta)$的一阶非线性偏微分方程．于是(5.18)式变为

$$\frac{\partial^4\nu_{00}}{\partial\xi^4}-\frac{\partial^2\nu_{00}}{\partial\xi^2}=0 \tag{5.20}$$

由上式及边界条件(4.9b)，可得到边界层型函数ν_{00}

$$\nu_{00}=C_0(\eta,\theta)\exp(-\xi) \tag{5.21}$$

其中　$C_0(\eta,\theta)$是待定函数．将ν_{00}，W_{00}代入方程(4.16)，得到关于ψ_{00}的微分方程

$$\left[\left(\frac{\partial u}{\partial r}\right)^4+\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial\theta}\right)^4+\delta_1\frac{1}{\eta^2}\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial u}{\partial\theta}\right)^2\right]\frac{\partial^4\psi_{00}}{\partial\xi^4}$$
$$=\delta_2(1+b\eta)\left[2\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)\frac{1}{\eta}\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial W_{00}}{\partial\theta}-\left(\frac{\partial u}{\partial\theta}\right)^2\frac{1}{\eta^2}\frac{\partial^2W_{00}}{\partial\eta^2}\right.$$
$$\left.-\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\right]\frac{\partial^2\nu_{00}}{\partial\xi^2} \tag{5.22}$$

解方程得

$$\psi_{00}=\frac{1}{H}\delta_2(1+b\eta)\left[2\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial \theta}\right)\frac{1}{\eta}\frac{\partial}{\partial \eta}\frac{1}{\eta}\frac{\partial W_{00}}{\partial \theta}-\left(\frac{\partial u}{\partial \theta}\right)^2\frac{1}{\eta^2}\frac{\partial^2 W_{00}}{\partial \eta^2}\right.$$
$$\left.-\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial \eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2}\right)W_{00}\right]C_0(\eta,\theta)\exp(-\xi) \tag{5.23}$$

其中 $$H=\left(\frac{\partial u}{\partial r}\right)^4+\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial \theta}\right)^4+\delta_1\frac{1}{\eta^2}\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial u}{\partial \theta}\right)^2.$$

将ν_{00}，F_{00}和F_{10}代入方程(4.17)（取$n=1$，$m=0$，且带负下标的项取零），可以得到ν_{10}的微分方程

$$(1+b\eta)^3\left(\left(\frac{\partial u}{\partial r}\right)^4+\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial \theta}\right)^4\right)\frac{\partial^4\nu_{10}}{\partial \xi^4}+\left[3b(1+b\eta)^2\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)^2\right.$$
$$\left.-\frac{1}{\eta^2}\left(\frac{\partial u}{\partial \theta}\right)^2\frac{\partial^2 F_{00}}{\partial \eta^2}-\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial \eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2}\right)F_{00}+\frac{2}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial}{\partial \eta}\frac{1}{\eta}\frac{\partial F_{00}}{\partial \theta}\right]\frac{\partial^2\nu_{10}}{\partial \xi^2}$$
$$=\left\{\left[(1+b\eta)^3\left(4\left(\frac{\partial u}{\partial r}\right)^3\frac{\partial C_0(\eta,\theta)}{\partial \eta}+4\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial \theta}\right)^3\frac{\partial C_0(\eta,\theta)}{\partial \theta}\right.\right.\right.$$
$$\left.\left.+6\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial^2 u}{\partial r^2}\right)C_0(\eta,\theta)+6\left(\frac{\partial u}{\partial \theta}\right)^2\left(\frac{\partial^2 u}{\partial \theta^2}\right)\frac{\partial C_0(\eta,\theta)}{\partial \eta}\right)\right]$$
$$+3b(1+b\eta)^2\left[\mu_{\theta r}\left(\frac{\partial u}{\partial r}\right)\frac{\partial}{\partial \eta}\left(\frac{1}{\eta}C_0(\eta,\theta)\right)+\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial C_0(\eta,\theta)}{\partial \eta}\right.$$
$$\left.+\left(\left(\frac{\partial u}{\partial r}\right)^3+\mu_{\theta r}\frac{1}{\eta^2}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial \theta}\right)^2\right)C_0(\eta,\theta)\right]+\left[2\left(\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial}{\partial \eta}\frac{1}{\eta}\frac{\partial F_{00}}{\partial \theta}\right.\right.$$
$$\left.-\left(\frac{\partial u}{\partial \theta}\right)\frac{1}{\eta^2}\frac{\partial^2 F_{00}}{\partial \eta^2}\right)\frac{\partial C_0(\eta,\theta)}{\partial \theta}+2\left(\frac{\partial u}{\partial \theta}\right)\left(\frac{\partial}{\partial \eta}\frac{1}{\eta}\frac{\partial F_{00}}{\partial \theta}\right)\frac{\partial}{\partial \eta}\left(\frac{1}{\eta}C_0(\eta,\theta)\right)$$
$$-2\left(\frac{\partial u}{\partial r}\right)\left(\left(\frac{1}{\eta}\frac{\partial}{\partial \eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2}\right)F_{00}\right)\frac{\partial C_0(\eta,\theta)}{\partial \eta}$$
$$+\left(\left(\frac{\partial^2 u}{\partial r^2}\right)\frac{1}{\eta^2}\frac{\partial^2 F_{00}}{\partial \eta^2}+\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial \eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2}\right)F_{00}-\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial^2 F_{00}}{\partial \eta^2}\right.$$
$$+\left(\frac{\partial^2 u}{\partial r^2}\right)\left(\frac{1}{\eta}\frac{\partial}{\partial \eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2}\right)F_{00}-2\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial}{\partial \eta}\frac{1}{\eta}\frac{\partial F_{00}}{\partial \theta}$$
$$\left.\left.\left.-\frac{1}{\eta^2}\left(\frac{\partial u}{\partial \theta}\right)^2\frac{\partial^2 F_{00}}{\partial \eta^2}\right)C_0(\eta,\theta)\right]\right\}\exp(-\xi) \tag{5.24}$$

若令(5.24)式右端关于$\exp[-\xi]$的系数等于零，则可得到$C_0(\eta,\theta)$的一阶线性偏微分方程．$C_0(\eta,\theta)$的边界条件可由条件(4.9b)获得

$$C_0(\eta,\theta)|_{\eta=1}=\frac{1}{\left(\frac{\partial u}{\partial r}\right)}\frac{\partial W_{10}}{\partial r}\bigg|_{r=1} \tag{5.25}$$

并且$C_0(\eta,\theta)$满足自然周期性条件$C_0(\eta,\theta)=C_0(\eta,\theta+2\pi)$．同时方程(5.24)化为

$$(1+b\eta)^3\left(\left(\frac{\partial u}{\partial r}\right)^4+\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial \theta}\right)^4\right)\frac{\partial^4\nu_{10}}{\partial \xi^4}$$
$$-\left[\frac{1}{\eta^2}\left(\frac{\partial u}{\partial \theta}\right)^2\frac{\partial^2 F_{00}}{\partial \eta^2}+\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{1}{\eta}\frac{\partial}{\partial \eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2}\right)F_{00}\right.$$

$$-2\frac{1}{\eta}\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial}{\partial \eta}\ \frac{1}{\eta}\ \frac{\partial F_{00}}{\partial \theta}-3b(1+b\eta)^2\left(\frac{\partial u}{\partial r}\right)^2\Bigg]\frac{\partial^2 \nu_{10}}{\partial \xi^2}=0 \tag{5.26}$$

利用(5.19)式可得

$$\frac{\partial^4 \nu_{10}}{\partial \xi^4}-\left(1-\frac{3b}{1+b\eta}\ \frac{1}{\eta}\ \frac{\left(\frac{\partial u}{\partial r}\right)^2}{\left(\frac{\partial u}{\partial r}\right)^4+\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial \theta}\right)^4}\right)\frac{\partial^2 \nu_{10}}{\partial \xi^2}=0 \tag{5.27}$$

解此方程得到边界层型函数ν_{10}

$$\nu_{10}=\frac{1}{I}C_1(\eta,\theta)\exp(-\sqrt{I}\,\xi) \tag{5.28}$$

其中 $C_1(\eta,\theta)$ 是待定函数，$I=1-\frac{3b}{1+b\eta}\ \frac{1}{\eta}\ \frac{\left(\frac{\partial u}{\partial r}\right)^2}{\left(\frac{\partial u}{\partial r}\right)^4+\delta_2\frac{1}{\eta^4}\left(\frac{\partial u}{\partial \theta}\right)^4}$.

将ψ_{00}，ν_{00}，ν_{10}，W_{00}，W_{10}代入方程（4.25）（取$n=1$，$m=0$，且带负下标的项取零），可以得到关于ψ_{10}的微分方程，然后积分可得到边界层型函数ψ_{10}。且和求$C_0(\eta,\theta)$的方法类似地可求出$C_1(\eta,\theta)$。这样可逐次求得ν_{nm}，$\psi_{nm}(n=0,\ 1,\ 2,\ \cdots,\ N;\ m=0,\ 1,\ 2,\ \cdots,\ M)$。

把上述所得的W_{nm}，F_{nm}，ν_{nm}，$\psi_{nm}(n=0,\ 1,\ 2,\ \cdots,\ N;\ m=0,\ 1,\ 2,\ \cdots,\ M)$分别代入(2.16)和(2.17)式，便可得到关于边值问题(2.18)和(2.19)的关于ε_1为N阶和关于ε_2为M阶的一致有效渐近解。

六、讨　　论

变厚度圆形板在机械设计中有着非常重要的应用。因此给出它的弯曲问题的一个渐近解有着重要的工程实践意义。

以上我们获得了变厚度圆柱型正交各向异性圆形薄板的非线性非对称大变形弯曲问题的对于ε_1为N阶和对ε_2为M阶的一致有效渐近解。将文献[9]和[10]的研究范围又向外扩展了一步。当然，当$b=0$时，变厚度圆形薄板退化为等厚度圆形薄板的情形，令(5.17)、(5.21)和(5.28)式中的$b=0$，所得结果与文献[10]中的基本一致。显然，本文的方法可以很容易推广到厚度非线性变化的情况。

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The Problem of the Non-Linear Unsymmetrical Bending for Cylindrically Orthotropic Circular Thin Plate with Variable Thickness

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Abstract

To begin with, in this paper, the governing equations of the problem of the non-linear unsymmetrical bending for cylindrically orthotropic circular thin plate with variable thickness are derived. By using "the method of two-variable" and introducing four small parameters, the problem of the non-linear unsymmetrical bending for cylindrically orthotropic circular thin plate with linear variable thickness is studied, and the uniformly valid asymptotic solution of Nth-order for ε_1 and Mth-order for ε_2 is obtained.

Key words orthotropic circular plate with variable thickness, non-linear unsymmetrical bending, method of two-variable, the uniformly valid asymptotic solution.