

三角形REISSNER-MINDLIN板元*

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摘 要

本文提出构造无自锁现象的Reissner-Mindlin板元的一个一般性方法。此方法将剪切应变用它的适当的插值多项式代替, 当板厚趋于零时这对应于插值点的Kirchhoff条件, 因而单元无自锁现象。根据这种方法我们构造两个三角形元——一个3节点元和一个6节点元, 并给出数值结果。

关键词 Reissner-Mindlin板模型 自锁现象 三角形元

一、引 言

近年来很多努力用于发展可靠而有效的基于Reissner-Mindlin理论的板元, 困难是当 $t \rightarrow 0$ (t 是板厚) 时低阶 c^0 元出现自锁现象(locking)。基于台劳展式, 我们在[1]中给出自锁现象的一个数学分析: 剪切应变可分成两部分——当 $t \rightarrow 0$ 时一部分对应于Kirchhoff条件, 另一部分引起自锁。这样只要当 $t \rightarrow 0$ 时剪切应变只引导出Kirchhoff条件, 构造的单元就无自锁现象。按照此分析本文提出构造无自锁现象的R-M板元的一个简单方法, 方法是将剪切应变用它的适当的插值多项式代替, 当 $t \rightarrow 0$ 时它变成插值点的Kirchhoff条件。根据此方法我们构造两个三角形R-M板元。一个是三节点元, 挠度 w 和转角 $\beta = (\beta_x, \beta_y)^T$ 是分片线性函数, 剪切应变 $\beta - \nabla w$ 用它在单元中心点的零次插值多项式代替, 另一个是6节点元, w 和 β 是分片2次多项式, $\beta - \nabla w$ 用它在三边中点上的一次插值多项式代替, 并给出数值结果。

构造R-M板元的一类方法是投影方法^{[2][3]}, β 被映射到 ∇w 所在的空间, 这种方法的 w 的形函数空间通常与 β 的不同且投影算子复杂。

二、三角形R-M板元

我们考虑R-M模型的板弯曲问题: 求挠度 $w \in H_0^1(\Omega)$ 和转角 $\beta \in [H_0^1(\Omega)]^2$ 使下列板的能量泛函达极小。

$$J(\beta, w) = \frac{1}{2} \frac{E}{12(1-\nu^2)} \int_{\Omega} \left[\left(\frac{\partial \beta_x}{\partial x} \right)^2 + 2\nu \left(\frac{\partial \beta_x}{\partial x} \right) \left(\frac{\partial \beta_y}{\partial y} \right) + \left(\frac{\partial \beta_y}{\partial y} \right)^2 \right]$$

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$$\begin{aligned}
 & + \frac{(1-\nu)}{2} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \right)^2 \Big] dx dy + \frac{1}{2} \frac{Ek}{(1+\nu)t^2} \int_{\Omega} \left[\left(\beta_x - \frac{\partial w}{\partial x} \right)^2 \right. \\
 & \left. + \left(\beta_y - \frac{\partial w}{\partial y} \right)^2 \right] dx dy - \int_{\Omega} f w dx dy \quad (2.1)
 \end{aligned}$$

其中 E 是 Young 氏模量, ν 是 Poisson 比, t 是板厚, k 是剪力校正因子, Ω 是板占有的平面区域, f 是施加的荷载. (2.1) 的第一项是弯曲能量, 第二项是剪切能量.

设 Ω 是凸多边形, $\Omega = \cup_{K \in \Pi_h} K$ 是 Ω 的三角剖分, 满足正则性假定^[4]. 假定有限元空间是 $W_h \times B_h$, $W_h \subset H_0^1(\Omega)$, $B_h \subset [H_0^1(\Omega)]^2$, 则离散解 $(w_h, \beta_h) \in W_h \times B_h$ 是下列泛函的极小值.

$$\begin{aligned}
 J_h(\beta_h, W_h) = & \sum_{K \in \Pi_h} \left\{ \frac{1}{2} \frac{E}{12(1-\nu^2)} \int_K \left[\left(\frac{\partial \beta_x}{\partial x} \right)^2 + 2\nu \left(\frac{\partial \beta_x}{\partial x} \right) \left(\frac{\partial \beta_y}{\partial y} \right) \right. \right. \\
 & \left. \left. + \left(\frac{\partial \beta_y}{\partial y} \right)^2 + \frac{(1-\nu)}{2} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \right)^2 \right] dx dy + \frac{1}{2} \frac{Ek}{2(1+\nu)t^2} \int_K \left[\left(R \left(\beta_x - \frac{\partial w}{\partial x} \right) \right)^2 \right. \right. \\
 & \left. \left. + \left(R \left(\beta_y - \frac{\partial w}{\partial y} \right) \right)^2 \right] dx dy - \int_K f w dx dy \right\} \quad (2.2)
 \end{aligned}$$

其中 R 是插值算子满足

$$R\left(\frac{\partial w}{\partial x}\right) = \frac{\partial w}{\partial x}, \quad R\left(\frac{\partial w}{\partial y}\right) = \frac{\partial w}{\partial y}$$

实际应用中有不同的方法选择插值节点, 最好使它们具有某种对称性并尽可能选在单元边界上. 当 $t \rightarrow 0$ 时由 (2.2) 得

$$R\left(\beta_x - \frac{\partial w}{\partial x}\right) \Big|_K \rightarrow 0, \quad R\left(\beta_y - \frac{\partial w}{\partial y}\right) \Big|_K \rightarrow 0 \quad (\forall K \in \Pi_h) \quad (2.3)$$

这正是插值点上的 Kirchhoff 条件 $\beta = \nabla w$, 因而单元无自锁现象.

为按照上述方法构造三角形 R-M 板元, 我们采用等参方法^[5]. 设 K 是 x - y 平面上的一般性单元, 顶点是 $a_i(x_i, y_i)$ ($i=1, 2, 3$), 三边中点是

$$a_4\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right), \quad a_5\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right), \quad a_6\left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right)$$

\hat{K} 是参考元, 在 ξ - η 平面上, 顶点是 $\hat{a}_1(0, 1)$, $\hat{a}_2(0, 0)$, $\hat{a}_3(1, 0)$, 三边中点是 $\hat{a}_4(0, 1/2)$, $\hat{a}_5(1/2, 0)$, $\hat{a}_6(1/2, 1/2)$.

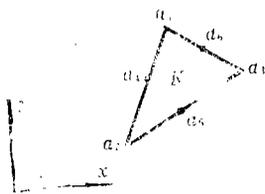


图 1

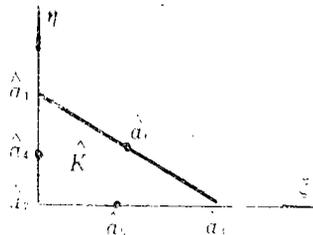


图 2

\hat{K} 的面积坐标是

$$\lambda_1 = \eta, \quad \lambda_2 = 1 - \xi - \eta, \quad \lambda_3 = \xi \quad (2.4)$$

从 K 到 \hat{K} 的变换是

$$x = \sum_{i=1}^3 x_i \lambda_i, \quad y = \sum_{i=1}^3 y_i \lambda_i$$

变换的Jacobi矩阵是

$$J = \begin{pmatrix} \partial x / \partial \xi & \partial x / \partial \eta \\ \partial y / \partial \xi & \partial y / \partial \eta \end{pmatrix} = \begin{pmatrix} x_{32} & x_{12} \\ y_{32} & y_{12} \end{pmatrix}$$

其中 $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$, 这样

$$|J| = \begin{vmatrix} x_{32} & x_{12} \\ y_{32} & y_{12} \end{vmatrix} = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad (2.5)$$

$$\begin{pmatrix} \partial \xi / \partial x & \partial \xi / \partial y \\ \partial \eta / \partial x & \partial \eta / \partial y \end{pmatrix} = J^{-1} = \frac{1}{|J|} \begin{pmatrix} y_{12} & x_{21} \\ y_{23} & x_{32} \end{pmatrix} \quad (2.6)$$

$$\left. \begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{y_{12}}{|J|} \frac{\partial}{\partial \xi} + \frac{y_{23}}{|J|} \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{x_{21}}{|J|} \frac{\partial}{\partial \xi} + \frac{x_{32}}{|J|} \frac{\partial}{\partial \eta} \end{aligned} \right\} \quad (2.7)$$

$P_m(\hat{K})$ 表示关于 ξ, η 的次数不超过 m 的多项式集合。

(1) SI₃元

这里SI是指剪切插值(Shear Interpolation)。

w 和 β 的形函数空间是 $P_1(\hat{K})$ 和 $[P_1(\hat{K})]^2$ 。

$$w = \sum_{i=1}^3 w_i \lambda_i, \quad \beta_x = \sum_{i=1}^3 \beta_{xi} \lambda_i, \quad \beta_y = \sum_{i=1}^3 \beta_{yi} \lambda_i \quad (2.8)$$

其中 $v_i = v(a_i)$ ($i=1, 2, 3$) 是 v 在一般性单元 K 的顶点上的函数值, $w_i, \beta_{xi}, \beta_{yi}$, ($i=1, 2, 3$) 是自由度. 在 Ω 的边界上, $w_i, \beta_{xi}, \beta_{yi}$ 取值为零. 显然由(2.8)确定的有限元空间 $W_h \times B_h$ 具有性质:

$$W_h \subset H_0^1(\Omega), \quad B_h \subset [H_0^1(\Omega)]^2$$

由(2.8), (2.7), (2.4)式我们可以得到:

$$\left. \begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{|J|} (y_{23} w_1 + y_{31} w_2 + y_{12} w_3) \\ \frac{\partial w}{\partial y} &= \frac{1}{|J|} (x_{32} w_1 + x_{13} w_2 + x_{21} w_3) \\ \frac{\partial \beta_x}{\partial x} &= \frac{1}{|J|} (y_{23} \beta_{x1} + y_{31} \beta_{x2} + y_{12} \beta_{x3}) \\ \frac{\partial \beta_x}{\partial y} &= \frac{1}{|J|} (x_{32} \beta_{x1} + x_{13} \beta_{x2} + x_{21} \beta_{x3}) \\ \frac{\partial \beta_y}{\partial x} &= \frac{1}{|J|} (y_{23} \beta_{y1} + y_{31} \beta_{y2} + y_{12} \beta_{y3}) \\ \frac{\partial \beta_y}{\partial y} &= \frac{1}{|J|} (x_{32} \beta_{y1} + x_{13} \beta_{y2} + x_{21} \beta_{y3}) \end{aligned} \right\} \quad (2.9)$$

$(\partial w / \partial x) \in P_0(\hat{K}), \partial w / \partial y \in P_0(\hat{K})$, 这样 $R: P_1(\hat{K}) \rightarrow P_0(\hat{K})$. 我们取 \hat{K} 的中心点—— $(1/3, 1/3)$ ——作为插值点。

$$\left. \begin{aligned} R\left(\frac{\partial w}{\partial x}\right) &= \frac{\partial w}{\partial x}, \quad R(\beta_x) = \beta_x \left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3} \sum_{i=1}^3 \beta_{x_i} \\ R\left(\frac{\partial w}{\partial y}\right) &= \frac{\partial w}{\partial y}, \quad R(\beta_y) = \beta_y \left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3} \sum_{i=1}^3 \beta_{y_i} \end{aligned} \right\} \quad (2.10)$$

由(2.2), (2.8)~(2.10)式利用通常的等参方法可以求得单元刚度矩阵.

(2) SI6元

w 和 β 的形函数空间是 $P_2(\hat{K})$ 和 $[P_2(\hat{K})]^2$.

$$w = \sum_{i=1}^6 w_i q_i(\xi, \eta), \quad \beta_x = \sum_{i=1}^6 \beta_{x_i} q_i(\xi, \eta), \quad \beta_y = \sum_{i=1}^6 \beta_{y_i} q_i(\xi, \eta) \quad (2.11)$$

其中 $q_1(\xi, \eta) = 2\lambda_1^2 - \lambda_1$, $q_2(\xi, \eta) = 2\lambda_2^2 - \lambda_2$, $q_3(\xi, \eta) = 2\lambda_3^2 - \lambda_3$, $q_4(\xi, \eta) = 4\lambda_1\lambda_2$, $q_5(\xi, \eta) = 4\lambda_2\lambda_3$, $q_6(\xi, \eta) = 4\lambda_3\lambda_1$, $v_i = v(a_i)$ ($i=1, \dots, 6$) 是 v 在一般性单元 k 的三个顶点和三边中点上的函数值. $w_i, \beta_{x_i}, \beta_{y_i}$ ($i=1, \dots, 6$) 是自由度.

在 Ω 的边界上, $w_i, \beta_{x_i}, \beta_{y_i}$ ($i=1, \dots, 6$) 取值为零. 假定 $W_h \times B_h$ 是由(2.11)生成的有限元空间, 则 $W_h \subset H_0^1(\Omega)$, $B_h \subset [H_0^1(\Omega)]^2$.

由(2.11)、(2.7)、(2.6)、(2.4)式我们有

$$\left. \begin{aligned} \frac{\partial q_1}{\partial \xi} &= 0, \quad \frac{\partial q_1}{\partial \eta} = 4\lambda_1 - 1, \quad \frac{\partial q_2}{\partial \xi} = \frac{\partial q_2}{\partial \eta} = -4\lambda_2 + 1, \quad \frac{\partial q_3}{\partial \xi} = 4\lambda_3 - 1 \\ \frac{\partial q_3}{\partial \eta} &= 0, \quad \frac{\partial q_4}{\partial \xi} = -4\lambda_1, \quad \frac{\partial q_4}{\partial \eta} = 4\lambda_2 - 4\lambda_1, \quad \frac{\partial q_5}{\partial \xi} = 4\lambda_2 - 4\lambda_3 \\ \frac{\partial q_5}{\partial \eta} &= -4\lambda_2, \quad \frac{\partial q_6}{\partial \xi} = 4\lambda_1, \quad \frac{\partial q_6}{\partial \eta} = 4\lambda_3 \end{aligned} \right\} \quad (2.12)$$

$$\left. \begin{aligned} \frac{\partial w}{\partial x} &= \sum_{i=1}^6 w_i \gamma_i(\xi, \eta), \quad \frac{\partial w}{\partial y} = \sum_{i=1}^6 w_i S_i(\xi, \eta) \\ \frac{\partial \beta_x}{\partial x} &= \sum_{i=1}^6 \beta_{x_i} \gamma_i(\xi, \eta), \quad \frac{\partial \beta_x}{\partial y} = \sum_{i=1}^6 \beta_{x_i} S_i(\xi, \eta) \\ \frac{\partial \beta_y}{\partial x} &= \sum_{i=1}^6 \beta_{y_i} \gamma_i(\xi, \eta), \quad \frac{\partial \beta_y}{\partial y} = \sum_{i=1}^6 \beta_{y_i} S_i(\xi, \eta) \end{aligned} \right\} \quad (2.13)$$

其中 $\gamma_i(\xi, \eta) = \frac{y_{12}}{|J|} \frac{\partial q_i}{\partial \xi} + \frac{y_{23}}{|J|} \frac{\partial q_i}{\partial \eta}$, $S_i(\xi, \eta) = \frac{x_{21}}{|J|} \frac{\partial q_i}{\partial \xi} + \frac{x_{32}}{|J|} \frac{\partial q_i}{\partial \eta}$ ($i=1, \dots, 6$).

$\frac{\partial w}{\partial x} \in P_1(\hat{K})$, $\frac{\partial w}{\partial y} \in P_1(\hat{K})$, 这样 $R: P_2(\hat{K}) \rightarrow P_1(\hat{K})$. 单元的三边中点取为插值点. 我们有

$$\left. \begin{aligned} R\left(\frac{\partial w}{\partial x}\right) &= \frac{\partial w}{\partial x} \\ R(\beta_x) &= \beta_{x4}(\lambda_1 + \lambda_2 - \lambda_3) + \beta_{x5}(-\lambda_1 + \lambda_2 + \lambda_3) + \beta_{x6}(\lambda_1 - \lambda_2 + \lambda_3) \\ R\left(\frac{\partial w}{\partial y}\right) &= \frac{\partial w}{\partial y} \\ R(\beta_y) &= \beta_{y4}(\lambda_1 + \lambda_2 - \lambda_3) + \beta_{y5}(-\lambda_1 + \lambda_2 + \lambda_3) + \beta_{y6}(\lambda_1 - \lambda_2 + \lambda_3) \end{aligned} \right\} \quad (2.14)$$

由(2,2), (2,11)~(2,14)用通常的方法可得到单元刚度矩阵。

三、数值结果

我们考虑一个简单的试验问题：均布荷载下的四周简支方板，Poisson比 $\nu=0.3$ ，板的边长取为1，剪力校正因子 $k=5/6$ ，由于对称性只有1/4板被离散，为检验自锁现象，我们分别取板厚 $t=10^{-2}, 10^{-3}, 10^{-4}$ ，对SI6元，正则剖分与非正则剖分均被采用。

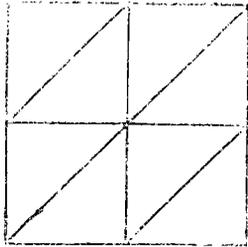


图3 $\frac{1}{4}$ 板正则剖分
(2×2)

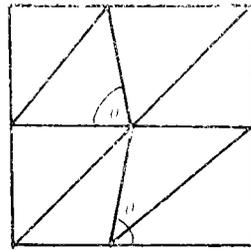


图4 $\frac{1}{4}$ 板非正则剖分
(2×2, $\text{tg}\theta=2.5$)

输出结果是板中心点的挠度 w 和弯矩 M_x, M_y 及板边中点的剪力 q_x, q_y ，它们的解析值可由[6]获得 ($w=0.443610 \cdot c, M_x=M_y=0.478860 \cdot c, q_x=q_y=0.337657 \cdot c$, c 是一个常数因子)。

相对误差列于表1中，由此表可看出SI3和SI6元均无自锁现象，均有很好的收敛性，对SI6元，正则剖分与非正则剖分下相对误差的差异不大，而非正则剖分的 w 的相对误差甚至比正则剖分的还要小。

表 1 SI3元的相对误差 (正则剖分)

	t	1/4板剖分		
		4×4	8×8	16×16
w	10^{-2}	0.2711	0.1414	0.07200
	10^{-3}	0.2706	0.1408	0.07143
	10^{-4}	0.2705	0.1408	0.07142
M	10^{-2}	0.08168	0.01900	0.004562
	10^{-3}	0.08172	0.01902	0.004544
	10^{-4}	0.08172	0.01902	0.004543
q	10^{-2}	0.2331	0.1350	0.07525
	10^{-3}	0.2332	0.1351	0.07542
	10^{-4}	0.2332	0.1351	0.07542

表 2

SI6元的相对误差 (正则剖分)

	t	1/4板剖分		
		2×2	4×4	8×8
w	10^{-2}	0.05975	0.01609	0.004539
	10^{-3}	0.05911	0.01547	0.003929
	10^{-4}	0.5910	0.01546	0.003923
M	10^{-2}	0.003487	0.001750	0.0005845
	10^{-3}	0.003719	0.001997	0.0008480
	10^{-1}	0.003722	0.002000	0.0008508
q	10^{-2}	0.2821	0.1300	0.05252
	10^{-3}	0.2816	0.1282	0.04611
	10^{-4}	0.2816	0.1282	0.04604

表 3

SI6元的相对误差 (非正则剖分)

	t	1/4板剖分		
		2×2	4×4	8×8
w	10^{-2}	0.05562	0.01443	0.004163
	10^{-3}	0.05495	0.01379	0.003533
	10^{-1}	0.05494	0.01378	0.003530
M	10^{-2}	0.01386	0.006718	0.001731
	10^{-3}	0.01416	0.007045	0.002069
	10^{-4}	0.01419	0.007050	0.002081
q	10^{-2}	0.3125	0.1366	0.05892
	10^{-3}	0.3120	0.1350	0.05363
	11^{-4}	0.3120	0.1350	0.05360

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Triangular Elements for Reissner-Mindlin Plate

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Abstract

A general method to construct locking free Reissner-Mindlin plate elements is presented. According to this method the shear strain is replaced by its proper interpolation polynomial, which corresponds to the Kirchhoff conditions at the interpolation points as the thickness of plate tends to zero, so the element is locking free. We construct two triangular elements by this method—a 3-node element and 6-node element. The numerical results are provided.

Key words Reissner-mindlin plate, locking, triangular element