应用数学和力学，第18卷 第11期（1997年11月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

非线性非完整约束空间准坐标表示的系统的基本动力学方程

邱 荣[1]

（汪家䜣推荐，1996年1月22日收到，1997年6月5日收到修改稿）

摘 要

用与准坐标表示的一阶非线性非完整约束超曲面的基矢量共线的量和米歇尔斯基方程点乘作为一阶非线性非完整约束变质量系统的基本动力学方程．由此可导出用准坐标表示的各种形式的运动微分方程．和约登(Jourdain)原理相容．举了例子．

关键词 非线性非完整约束 准坐标 基本动力学方程

N个质点组成的力学系统，第i个质点的矢径$\boldsymbol{r}_i$可表示为广义坐标q_s及时间t的函数．力学系统的位形由广义坐标确定，受到g个一阶非线性非完整约束，即

$$f_\beta(q_s,\ \dot{q}_s;\ t)=0 \qquad (s=1,\ 2,\ \cdots,\ n) \tag{1}$$

取准速度$\dot{\pi}$，使得

$$\left.\begin{aligned}\dot{\pi}_{\varepsilon+\beta}&=f_\beta(q_s,\ \dot{q}_s;\ t)=0\\ \dot{\pi}_\sigma&=\dot{\pi}_\sigma(q_s,\ \dot{q}_s;\ t)\end{aligned}\right\} \tag{2}$$

$$(s=1,\ 2,\ \cdots,\ n;\ \beta=1,\ 2,\ \cdots,\ g;\ \sigma=1,\ 2,\ \cdots,\ \varepsilon;\ \varepsilon=n-g)$$

设由方程(2)可反解出$\dot{q}_s$

$$\dot{q}_s=\dot{q}_s(q_s,\ \dot{\pi}_\sigma;\ t) \tag{3}$$

各$\dot{\pi}_\sigma(\sigma=1,\ 2,\ \cdots,\ \varepsilon)$彼此独立．并令$\tilde{\dot{\boldsymbol{r}}}_i$表示$\dot{\boldsymbol{r}}_i$中的$\dot{q}_s$已用$\dot{\pi}_\sigma$表示出来，即

$$\tilde{\dot{\boldsymbol{r}}}_i(q_s,\ \dot{\pi}_\sigma;\ t)=\dot{\boldsymbol{r}}_i(q_s,\ \dot{q}_s(q_s,\ \dot{\pi}_\sigma;\ t);\ t) \qquad (i=1,\ 2,\ \cdots,\ N) \tag{4}$$

方程(4)是一阶非线性非完整约束用准坐标表示的超曲面的参数方程．因此我们可引入用准坐标表示的一阶非线性非完整约束超曲面上的与其基矢量共线的量$(\partial\tilde{\dot{\boldsymbol{r}}}_i/\partial\dot{\pi}_\sigma)$．该矢量与米歇尔斯基方程点乘，即

$$\sum_{i=1}^{N}\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i}{\partial\dot{\pi}_\sigma}\cdot(\boldsymbol{F}_i+\boldsymbol{R}_i-m_i\ddot{\boldsymbol{r}}_i)=0 \tag{5}$$

作为用准坐标表示的一阶非线性非完整约束的变质量系统的基本动力学方程．或写为

$$\widetilde{(Q_F)}_\sigma+\widetilde{(Q_R)}_\sigma+\widetilde{(Q_{m\ddot{r}})}_\sigma=0 \tag{5'}$$

式中

1 福州大学西禅南村3—203，福州 350002．

$$\widetilde{(Q_F)}_\sigma=\sum_{i=1}^{N}F_i\cdot\frac{\partial\tilde{\dot{r}}_i}{\partial\dot{\pi}_\sigma},\quad \widetilde{(Q_R)}_\sigma=\sum_{i=1}^{N}R_i\cdot\frac{\partial\tilde{\dot{r}}_i}{\partial\dot{\pi}_\sigma},\quad \widetilde{(Q_{m\ddot{r}})}_\sigma=-\sum_{i=1}^{N}m_i\ddot{r}_i\cdot\frac{\partial\tilde{\dot{r}}_i}{\partial\dot{\pi}_\sigma}$$

$\widetilde{(Q_F)}$为准坐标表示下的广义 的广义力，$\widetilde{(Q_R)}$为准坐标表示的广义的广义反推力，$\widetilde{(Q_{m\ddot{r}})}$称它为准坐标表示的广义惯性力．$\boldsymbol{R}_i$是反推力，$\dot{\pi}_\sigma$是准速度．

将(4)式对$\dot{\pi}_\sigma$求偏微商，得

$$\left(\frac{\partial\tilde{\dot{r}}_i}{\partial\dot{\pi}_\sigma}\right)=\sum_{s=1}^{n}\frac{\partial\dot{r}_i}{\partial\dot{q}_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}\tag{6}$$

又由于
$$\dot{r}_i=\frac{dr_i}{dt}=\sum_{s=1}^{n}\frac{\partial r_i}{\partial q_s}\dot{q}_s+\frac{\partial r_i}{\partial t}\tag{7}$$

将(7)式对$\dot{\pi}_\sigma$求偏微商，并利用(3)式，得：

$$\frac{\partial\tilde{\dot{r}}_i}{\partial\dot{\pi}_\sigma}=\sum_{s=1}^{n}\frac{\partial r_i}{\partial q_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}\tag{8}$$

将(6)或(8)式代入(5)式，得：

$$\sum_{i=1}^{N}\sum_{s=1}^{n}\frac{\partial\dot{r}_i}{\partial\dot{q}_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}\cdot(F_i+R_i-m_i\ddot{r}_i)=0\tag{9}$$

或
$$\sum_{i=1}^{N}\sum_{s=1}^{n}\frac{\partial r_i}{\partial q_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}\cdot(F_i+R_i-m_i\ddot{r}_i)=0$$

如$\boldsymbol{R}_i=0$，由此式得到凯恩(Kane)方程[1]．

现由(5)式导出用准坐标表示的阿沛尔型方程．将$\tilde{\dot{r}}_i(q_s,\ \dot{\pi}_\sigma;\ t)$对时间$t$求全微商，得：

$$\tilde{\ddot{r}}_i=\frac{d}{dt}\tilde{\dot{r}}_i=\sum_{s=1}^{n}\frac{\partial\tilde{\dot{r}}_i}{\partial q_s}\dot{q}_s+\sum_{\sigma=1}^{\varepsilon}\frac{\partial\tilde{\dot{r}}_i}{\partial\dot{\pi}_\sigma}\ddot{\pi}_\sigma+\frac{\partial\tilde{\dot{r}}_i}{\partial t}\tag{10}$$

因而

$$\frac{\partial\tilde{\ddot{r}}_i}{\partial\ddot{\pi}_\sigma}=\frac{\partial\tilde{\dot{r}}_i}{\partial\dot{\pi}_\sigma}\tag{11}$$

$\tilde{\ddot{r}}_i$表示$\ddot{r}_i$中的$\dot{q}_s$和$\ddot{q}_s$已各用$\dot{\pi}_\sigma$和$\ddot{\pi}_\sigma$消去．将(11)式代入(5)式，得

$$\sum_{i=1}^{N}\frac{\partial\tilde{\ddot{r}}_i}{\partial\ddot{\pi}_\sigma}\cdot(F_i+R_i-m_i\ddot{r}_i)=0\tag{12}$$

引入系统的加速度能量函数

$$S^*=\frac{1}{2}\sum_{i=1}^{N}m_i\tilde{\ddot{r}}_i^2\tag{13}$$

S^*为S中的$\dot{q}_s$和$\ddot{q}_s$已各用$\dot{\pi}_\sigma$和$\ddot{\pi}_\sigma$消去的在准坐标表示下的表达式，于是(12)式可写成

$$\frac{\partial S^*}{\partial\ddot{\pi}_\sigma}=P^*_\sigma\tag{14}$$

式中

$$P_{\sigma}^{*}=\sum_{i=1}^{N}(\boldsymbol{F}_i+\boldsymbol{R}_i)\cdot\frac{\partial\widetilde{\dot{\boldsymbol{r}}}_i}{\partial\dot{\pi}_{\sigma}}=\sum_{i=1}^{N}(\boldsymbol{F}_i+\boldsymbol{R}_i)\cdot\sum_{s=1}^{n}\frac{\partial\boldsymbol{r}_i}{\partial q_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_{\sigma}}=\sum_{s=1}^{n}(Q_F+Q_R)\frac{\partial\dot{q}_s}{\partial\dot{\pi}_{\sigma}} \tag{15}$$

P_{σ}^{*}用准坐标表示，而Q_F为广义力，Q_R为广义反推力．(14)式即准坐标表示的阿沛尔型方程．

下面由(5)式导出准坐标表示的尼尔松型方程．系统的动能为

$$T=\frac{1}{2}\sum_{i=1}^{N}m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i \tag{16}$$

其时间微商为

$$\dot{T}=\sum_{i=1}^{N}m_i\dot{\boldsymbol{r}}_i\cdot\ddot{\boldsymbol{r}}_i \tag{17}$$

令T^*和$\dot{T}^*$为T和$\dot{T}$中各消去$\dot{q}_s$和$\ddot{q}_s$所得的表示式，即：

$$T^*(q_s,\ \dot{\pi}_{\sigma};\ t)=T[q_s,\ \dot{q}_s(q_s,\ \dot{\pi}_{\sigma};\ t);\ t] \tag{18}$$

$$\dot{T}^*(q_s,\ \dot{\pi}_{\sigma},\ \ddot{\pi}_{\sigma};\ t)=\dot{T}[q_s,\ \dot{q}_s(q_s,\ \dot{\pi}_{\sigma};\ t),\ \ddot{q}_s(q_s,\ \dot{\pi}_{\sigma},\ \ddot{\pi}_{\sigma};\ t);\ t] \tag{19}$$

由(18)式有 $\dfrac{\partial T^*}{\partial q_s}=\dfrac{\partial T}{\partial q_s}+\sum\limits_{k=1}^{n}\dfrac{\partial T}{\partial\dot{q}_k}\dfrac{\partial\dot{q}_k}{\partial q_s}$

两边乘以$\partial\dot{q}_s/\partial\dot{\pi}_{\sigma}$，并对$s$求和，得

$$\sum_{s=1}^{n}\frac{\partial T^*}{\partial q_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}}=\sum_{s=1}^{n}\frac{\partial T}{\partial q_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}}+\sum_{s=1}^{n}\sum_{k=1}^{n}\frac{\partial T}{\partial\dot{q}_k}\frac{\partial\dot{q}_k}{\partial q_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}} \tag{20}$$

利用运算

$$\frac{\partial}{\partial\pi_s}=\sum_{k=1}^{n}\frac{\partial\dot{q}_k}{\partial\dot{\pi}_s}\frac{\partial}{\partial q_k}$$

(20)式变为

$$\frac{\partial T^*}{\partial\pi_{\sigma}}=\sum_{s=1}^{n}\frac{\partial T}{\partial q_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_{\sigma}}+\sum_{k=1}^{n}\frac{\partial T}{\partial\dot{q}_k}\frac{\partial\dot{q}_k}{\partial\pi}=\sum_{s=1}^{n}\sum_{i=1}^{N}m_i\dot{\boldsymbol{r}}_i\cdot\frac{\partial\dot{\boldsymbol{r}}_i}{\partial q_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_{\sigma}}+\sum_{k=1}^{n}\frac{\partial T}{\partial\dot{q}_k}\frac{\partial\dot{q}_k}{\partial\pi_{\sigma}} \tag{21}$$

由(17)和(19)式，得

$$\frac{\partial\dot{T}^*}{\partial\dot{\pi}_{\sigma}}=\sum_{s=1}^{n}\frac{\partial\dot{T}}{\partial\dot{q}_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_{\sigma}}+\sum_{s=1}^{n}\frac{\partial\dot{T}}{\partial\ddot{q}_s}\frac{\partial\ddot{q}_s}{\partial\dot{\pi}}=\sum_{s=1}^{n}\frac{\partial\dot{T}}{\partial\dot{q}_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_{\sigma}}+\sum_{s=1}^{n}\frac{\partial T}{\partial\dot{q}_s}\frac{\partial\ddot{q}_s}{\partial\dot{\pi}_{\sigma}}$$

$$\simeq\sum_{s=1}^{n}\Big[\sum_{i=1}^{N}\Big(m_i\ddot{\boldsymbol{r}}_i\cdot\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_s}+m_i\dot{\boldsymbol{r}}_i\cdot\frac{\partial\ddot{\boldsymbol{r}}_i}{\partial\dot{q}_s}\Big)\frac{\partial\dot{q}_s}{\partial\dot{\pi}_{\sigma}}+\sum_{s=1}^{n}\frac{\partial T}{\partial\dot{q}_s}\frac{\partial\ddot{q}_s}{\partial\dot{\pi}_{\sigma}}$$

$$=\sum_{s=1}^{n}\sum_{i=1}^{N}m_i\Big(\ddot{\boldsymbol{r}}_i\cdot\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_s}+2\dot{\boldsymbol{r}}_i\cdot\frac{\partial\dot{\boldsymbol{r}}_i}{\partial q_s}\Big)\frac{\partial\dot{q}_s}{\partial\dot{\pi}_{\sigma}}+\sum_{s=1}^{n}\frac{\partial T}{\partial\dot{q}_s}\frac{\partial\ddot{q}_s}{\partial\dot{\pi}_{\sigma}} \tag{22}$$

将(21)和(22)式代入(5)式，并利用(6)式，得：

$$\frac{\partial\dot{T}^*}{\partial\dot{\pi}_{\sigma}}-2\frac{\partial T^*}{\partial\pi_{\sigma}}-\sum_{s=1}^{n}\frac{\partial T}{\partial\dot{q}_k}\Big(\frac{\partial\ddot{q}_s}{\partial\dot{\pi}_{\sigma}}-2\frac{\partial\dot{q}_s}{\partial\pi_{\sigma}}\Big)=P_{\sigma}^{*} \tag{23}$$

方程(23)即一阶非线性非完整约束变质量系统在准坐标下的广义尼尔松型方程.

将T^*对$\dot{\pi}_\sigma$求偏微商后再将$(\partial T^*/\partial\dot{\pi}_\sigma)$对时间微商，得

$$\frac{d}{dt}\frac{\partial T^*}{\partial\dot{\pi}_\sigma}=\frac{d}{dt}\left(\sum_{s=1}^{n}\frac{\partial T}{\partial\dot{q}_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}\right)=\sum_{s=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_s}\right)\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}+\sum_{s=1}^{n}\frac{\partial T}{\partial\dot{q}_s}\frac{d}{dt}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}$$

$$=\sum_{s=1}^{n}\sum_{i=1}^{N}\left(m_i\ddot{\boldsymbol{r}}_i\cdot\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_s}+m_i\dot{\boldsymbol{r}}_i\cdot\frac{d}{dt}\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_s}\right)\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}+\sum_{s=1}^{n}\frac{\partial T}{\partial\dot{q}_s}\frac{d}{dt}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma} \tag{24}$$

(24)式减去(21)式后，代入(5)式，并利用(6)式，得

$$\frac{d}{dt}\frac{\partial T^*}{\partial\dot{\pi}_\sigma}-\frac{\partial T^*}{\partial\pi_\sigma}=P_\sigma^*+\sum_{s=1}^{n}\frac{\partial T}{\partial\dot{q}_s}\left(\frac{d}{dt}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}-\frac{\partial\dot{q}_s}{\partial\pi_\sigma}\right) \tag{25}$$

已利用$\frac{d}{dt}\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_s}=\frac{d}{dt}\frac{\partial\boldsymbol{r}_i}{\partial q_s}=\frac{\partial\dot{\boldsymbol{r}}_i}{\partial q_s}$.(25)式即用准坐标表示的一阶非线性非完整约束变质量系的查浦雷金型方程.

由(18)式知

$$\frac{\partial T}{\partial\dot{q}_s}=\sum_{k=1}^{n}\frac{\partial T^*}{\partial\dot{\pi}_k}\frac{\partial\dot{\pi}_k}{\partial\dot{q}_s}\qquad\text{（未考虑非完整约束）}$$

故 $$\sum_{s=1}^{n}\frac{\partial T}{\partial\dot{q}_s}\left(\frac{d}{dt}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}-\frac{\partial\dot{q}_s}{\partial\pi_\sigma}\right)=\sum_{s=1}^{n}\sum_{k=1}^{n}\frac{\partial T^*}{\partial\dot{\pi}_k}\frac{\partial\dot{\pi}_k}{\partial\dot{q}_s}\left(\frac{d}{dt}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}-\frac{\partial\dot{q}_s}{\partial\pi_\sigma}\right) \tag{26}$$

因 $$\frac{\partial\dot{\pi}_k}{\partial\dot{q}_s}\frac{d}{dt}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}=\frac{d}{dt}\left(\frac{\partial\dot{\pi}_k}{\partial\dot{q}_s}\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}\right)-\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}\frac{d}{dt}\left(\frac{\partial\dot{\pi}_k}{\partial\dot{q}_s}\right)=-\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}\frac{d}{dt}\left(\frac{\partial\dot{\pi}_k}{\partial\dot{q}_s}\right) \tag{27}$$

又 $$\frac{\partial\dot{q}_s}{\partial\pi_\sigma}=\frac{d}{dt}\frac{\partial q_s}{\partial\pi_\sigma},\quad\frac{\partial\dot{\pi}_k}{\partial\dot{q}_s}=\frac{\partial\pi_k}{\partial q_s}$$

故 $$\frac{\partial\dot{\pi}_k}{\partial\dot{q}_s}\frac{\partial\dot{q}_s}{\partial\pi_\sigma}=\frac{\partial\pi_k}{\partial q_s}\frac{d}{dt}\frac{\partial q_s}{\partial\pi_\sigma}=\frac{d}{dt}\left(\frac{\partial\pi_k}{\partial q_s}\frac{\partial q_s}{\partial\pi_\sigma}\right)-\frac{\partial q_s}{\partial\pi_\sigma}\frac{d}{dt}\left(\frac{\partial\pi_k}{\partial q_s}\right)$$

$$=-\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}\frac{\partial\dot{\pi}_k}{\partial q_s} \tag{27'}$$

把(27)和(27)′式代入(26)式，然后代入(25)式，得

$$\frac{d}{dt}\frac{\partial T^*}{\partial\dot{\pi}_\sigma}-\frac{\partial T^*}{\partial\pi_\sigma}+\sum_{s=1}^{n}\sum_{k=1}^{n}\frac{\partial T^*}{\partial\dot{\pi}_k}\left(\frac{d}{dt}\frac{\partial\dot{\pi}_k}{\partial\dot{q}_s}-\frac{\partial\dot{\pi}_k}{\partial q_s}\right)\frac{\partial\dot{q}_s}{\partial\dot{\pi}_\sigma}=P_\sigma^* \tag{28}$$

(28)式即用准坐标表示的一阶非线性完整变质量系统的波尔兹曼-海默尔型方程.

由 $$\frac{\partial T^*}{\partial\dot{\pi}_\sigma}=\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i\cdot\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i}{\partial\dot{\pi}_\sigma} \tag{29}$$

$$\frac{\partial T^*}{\partial\pi_\sigma}=\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i\cdot\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i}{\partial\pi_\sigma}$$

得

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_\sigma}-\frac{\partial T^*}{\partial \pi_\sigma}=\sum_{i=1}^{N} m_i\ddot{\boldsymbol{r}}_i\cdot\frac{\partial \tilde{\dot{\boldsymbol{r}}}_i}{\partial \dot{\pi}_\sigma}+\sum_{i=1}^{N} m_i\tilde{\dot{\boldsymbol{r}}}_i\cdot\left(\frac{d}{dt}\frac{\partial \tilde{\dot{\boldsymbol{r}}}_i}{\partial \dot{\pi}_\sigma}-\frac{\partial \tilde{\dot{\boldsymbol{r}}}_i}{\partial \pi_\sigma}\right)$$

将上式代入(5)式，得

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_\sigma}-\frac{\partial T^*}{\partial \pi_\sigma}=P_\sigma^*+\sum_{i=1}^{N} m_i\tilde{\dot{\boldsymbol{r}}}_i\cdot\left(\frac{d}{dt}\frac{\partial \tilde{\dot{\boldsymbol{r}}}_i}{\partial \dot{\pi}_\sigma}-\frac{\partial \tilde{\dot{\boldsymbol{r}}}_i}{\partial \pi_\sigma}\right) \tag{30}$$

这即用准坐标表示的广义的麦克-米朗型方程.

将(29)式的第一式对时间求微商后代入(5)式得

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_\sigma}=P_\sigma^*+\sum_{i=1}^{N} m_i\tilde{\dot{\boldsymbol{r}}}_i\cdot\frac{d}{dt}\frac{\partial \tilde{\dot{\boldsymbol{r}}}_i}{\partial \dot{\pi}_\sigma}\qquad (\sigma=1,\ 2,\ \cdots,\ \varepsilon) \tag{31}$$

将(22)式代入(9)式，得

$$\frac{\partial \dot{T}^*}{\partial \dot{\pi}_\sigma}=P_\sigma^*+2\sum_{s=1}^{n}\left(\frac{\partial T}{\partial q_s}\right)\frac{\partial \dot{q}_s}{\partial \dot{\pi}_\sigma}+\sum_{s=1}^{n}\frac{\partial T}{\partial \dot{q}_s}\frac{\partial \ddot{q}_s}{\partial \dot{\pi}_\sigma}\qquad (\sigma=1,\ 2,\ \cdots,\ \varepsilon) \tag{32}$$

(31)式和(32)式是两组新方程式组.

现在，我们来证明基本动力学方程式(5)与约登原理是一致的.

因(4)式为约束超曲面的参数方程，t，q_s为参数，$\dot{\pi}_\sigma$为自变量，所以有变分$\delta t=\delta q_s=0$，而$\delta\dot{\pi}_\sigma\neq 0$，又因

$$\left.\begin{aligned}&\delta \boldsymbol{r}_i=\sum_{s=1}^{n}\frac{\partial \boldsymbol{r}_i}{\partial q_s}\delta q_s=0\\ \text{和}\quad&\sum_{\sigma=1}^{\varepsilon}\frac{\partial \tilde{\dot{\boldsymbol{r}}}_i}{\partial \dot{\pi}_\sigma}\delta\dot{\pi}_\sigma=\sum_{\sigma=1}^{\varepsilon}\frac{\partial \tilde{\dot{\boldsymbol{r}}}_i}{\partial \dot{\pi}_\sigma}\delta\dot{\pi}_\sigma+\sum_{s=1}^{n}\frac{\partial \tilde{\dot{\boldsymbol{r}}}_i}{\partial q_s}\delta q_s=\delta\tilde{\dot{\boldsymbol{r}}}_i=\delta\dot{\boldsymbol{r}}_i\end{aligned}\right\} \tag{33a}$$

将(5)式乘以$\delta\dot{\pi}_\sigma$后对σ求和及应用(33a)便得约登方程:

$$\left.\begin{aligned}&\sum_{i=1}^{N}\delta\dot{\boldsymbol{r}}_i\cdot(\boldsymbol{F}_i+\boldsymbol{R}_i-m_i\ddot{\boldsymbol{r}}_i)=0\\ &\delta t=\delta\boldsymbol{r}_i=0,\quad \delta\dot{\boldsymbol{r}}_i\neq 0\end{aligned}\right\} \tag{33}$$

反之，可由(33)式得(5)式.

例题 设一质量为m的质点在三维空间中运动，所受约束力为

$$\dot{q}_3^2=\dot{q}_1^2+\dot{q}_2^2 \tag{34}$$

其中 $q_1=x$，$q_2=y$，$q_3=z$，所受的主动力为Q_1，Q_2，Q_3，试由(5)、(31)和(32)式分别建立质点的运动微分方程. $\dot{m}=0$.

解 (Ⅰ) 利用(5)式建立质点的运动微分方程

质点的矢径为$\boldsymbol{r}=q_1\boldsymbol{i}+q_2\boldsymbol{j}+q_3\boldsymbol{k}$，$\dot{\boldsymbol{r}}=\dot{q}_1\boldsymbol{i}+\dot{q}_2\boldsymbol{j}+\dot{q}_3\boldsymbol{k}$，取准速度

$$\dot{\pi}_3=\frac{1}{2}(\dot{q}_3^2-\dot{q}_1^2-\dot{q}_2^2)=0,\ \dot{\pi}_2=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2),\ \dot{\pi}_1=\mathrm{tg}^{-1}\frac{\dot{q}_2}{\dot{q}_1} \tag{35}$$

由此反解，得

$$\dot{q}_1=\sqrt{\dot{\pi}_2-\dot{\pi}_3}\cos\dot{\pi}_1=\sqrt{\dot{\pi}_2}\cos\dot{\pi}_1,\ \dot{q}_2=\sqrt{\dot{\pi}_2}\sin\dot{\pi}_1,\ \dot{q}_3=\sqrt{\dot{\pi}_2+\dot{\pi}_3}=\sqrt{\dot{\pi}_2} \tag{36}$$

将$\dot{\boldsymbol{r}}_i$中的$\dot{q}_s$用$\dot{\pi}_\sigma$表示，得

$$\tilde{\boldsymbol{r}}=\sqrt{\dot{\pi}_2}(\cos\dot{\pi}_1\boldsymbol{i}+\sin\dot{\pi}_1\boldsymbol{j}+\boldsymbol{k}) \tag{37}$$

又

$$\ddot{\boldsymbol{r}}=\frac{d\tilde{\boldsymbol{r}}}{dt}=\left(\frac{\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\cos\dot{\pi}_1-\ddot{\pi}_1\sqrt{\dot{\pi}_2}\sin\dot{\pi}_1\right)\boldsymbol{i}+\left(\frac{\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\sin\dot{\pi}_1+\ddot{\pi}_1\sqrt{\dot{\pi}_2}\cos\dot{\pi}_1\right)\boldsymbol{j}+\frac{\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\boldsymbol{k} \tag{38}$$

而

$$\frac{\partial\tilde{\boldsymbol{r}}}{\partial\dot{\pi}_1}=-\sqrt{\dot{\pi}_2}\sin\dot{\pi}_1\boldsymbol{i}+\sqrt{\dot{\pi}_2}\cos\dot{\pi}_1\boldsymbol{j} \tag{39}$$

$$\frac{\partial\tilde{\boldsymbol{r}}}{\partial\dot{\pi}_2}=\frac{1}{2\sqrt{\dot{\pi}_2}}(\cos\dot{\pi}_1\boldsymbol{i}+\sin\dot{\pi}_1\boldsymbol{j}+\boldsymbol{k}) \tag{40}$$

将(38)、(39)和(40)式代入(5)，并用矩阵表示，得

$$\begin{pmatrix}-\sqrt{\dot{\pi}_2}\sin_1\dot{\pi}, & \sqrt{\dot{\pi}_2}\cos\dot{\pi}_1, & 0\\ \frac{1}{2\sqrt{\dot{\pi}_2}}\cos\dot{\pi}_1, & \frac{1}{2\sqrt{\dot{\pi}_2}}\sin\dot{\pi}_1, & \frac{1}{2\sqrt{\dot{\pi}_2}}\end{pmatrix}\begin{pmatrix}Q_1-\frac{m\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\cos\dot{\pi}_1+m\ddot{\pi}_1\sqrt{\dot{\pi}_2}\sin\dot{\pi}_1\\ Q_2-\frac{m\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\sin\dot{\pi}_1-m\ddot{\pi}_1\sqrt{\dot{\pi}_2}\cos\dot{\pi}_1\\ Q_3-\frac{m\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\end{pmatrix}=0$$

由此得

$$-\sqrt{\dot{\pi}_2}\sin\dot{\pi}_1\left(Q_1-\frac{m\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\cos\dot{\pi}_1+m\ddot{\pi}_1\sqrt{\dot{\pi}_2}\sin\dot{\pi}_1\right)+\sqrt{\dot{\pi}_2}\cos\dot{\pi}_1\left(Q_2-\frac{m\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\sin\dot{\pi}_1-m\ddot{\pi}_1\sqrt{\dot{\pi}_2}\cos\dot{\pi}_1\right)=0$$

和

$$\frac{1}{2\sqrt{\dot{\pi}_2}}\left[\cos\dot{\pi}_1\left(Q_1-\frac{m\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\cos\dot{\pi}_1+m\ddot{\pi}\sqrt{\dot{\pi}_2}\sin\dot{\pi}_1\right)+\sin\dot{\pi}_1\left(Q_1-\frac{m\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\sin\dot{\pi}_1-m\ddot{\pi}_1\sqrt{\dot{\pi}_2}\cos\dot{\pi}_1\right)+Q-\frac{m\ddot{\pi}_2}{2\sqrt{\dot{\pi}_2}}\right]=0$$

化简后，即得质点的运动微分方程组为:

$$\left.\begin{aligned}m\dot{\pi}_2\ddot{\pi}_1&=\sqrt{\dot{\pi}_2}(-Q_1\sin\dot{\pi}_1+Q_2\cos\dot{\pi}_1)\\ m\frac{\ddot{\pi}_2}{\sqrt{\dot{\pi}_2}}&=Q_1\cos\dot{\pi}_1+Q_2\sin\dot{\pi}_1+Q_3\end{aligned}\right\} \tag{41}$$

（Ⅱ） 利用(31)式建立质点的运动微分方程

$$T=\frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2),\quad T^*=m\dot{\pi}_2 \tag{42}$$

$$\frac{\partial T^*}{\partial\dot{\pi}_1}=0,\quad \frac{\partial T^*}{\partial\dot{\pi}_2}=m;\quad \frac{d}{dt}\frac{\partial T^*}{\partial\dot{\pi}_1}=\frac{d}{dt}\frac{\partial T^*}{\partial\dot{\pi}_2}=0 \tag{43}$$

由(39)和(40)式，得

$$P_1^*=\sum_{i=1}^{N}\boldsymbol{F}_i\cdot\frac{\partial\tilde{\dot{\boldsymbol{r}}}}{\partial\dot{\boldsymbol{\pi}}_1}=(Q_1\boldsymbol{i}+Q_2\boldsymbol{j}+Q_3\boldsymbol{k})\cdot\sqrt{\dot{\boldsymbol{\pi}}_2}(-\sin\dot{\boldsymbol{\pi}}_1\boldsymbol{i}+\cos\dot{\boldsymbol{\pi}}_1\boldsymbol{j})$$

$$=\sqrt{\dot{\boldsymbol{\pi}}_2}(-Q_1\sin\dot{\boldsymbol{\pi}}_1+Q_2\cos\dot{\boldsymbol{\pi}}_1) \tag{44}$$

$$P_2^*=\frac{1}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}(Q_1\cos\dot{\boldsymbol{\pi}}_1+Q_2\sin\dot{\boldsymbol{\pi}}_1+Q_3) \tag{45}$$

由(37)和(39)、(40)式，得

$$m\tilde{\dot{\boldsymbol{r}}}\cdot\frac{d}{dt}\frac{\partial\tilde{\dot{\boldsymbol{r}}}}{\partial\dot{\boldsymbol{\pi}}_1}=-m\dot{\boldsymbol{\pi}}_2\ddot{\pi}_1;\ m\tilde{\dot{\boldsymbol{r}}}\cdot\frac{d}{dt}\frac{\partial\tilde{\dot{\boldsymbol{r}}}}{\partial\dot{\boldsymbol{\pi}}_2}=-\frac{m\ddot{\pi}_2}{2\dot{\boldsymbol{\pi}}_2} \tag{46}$$

将(43)、(44)和(46)式的第一式及(43)、(45)及(46)式第二式分别代入(31)式，得

$$\left.\begin{aligned}0&\simeq\sqrt{\dot{\boldsymbol{\pi}}_2}(-Q_1\sin\dot{\boldsymbol{\pi}}_1+Q_2\cos\dot{\boldsymbol{\pi}}_1)-m\dot{\boldsymbol{\pi}}_2\ddot{\pi}_1\\0&=\frac{1}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}(Q_1\cos\dot{\boldsymbol{\pi}}_1+Q_2\sin\dot{\boldsymbol{\pi}}_1+Q_3)-\frac{m\ddot{\pi}_2}{2\dot{\boldsymbol{\pi}}_2}\end{aligned}\right\}$$

所以也得(41)式.

（Ⅲ） 利用(32)式求质点的运动微分方程

$$\dot{T}^*=m\ddot{\pi}_2 \tag{47}$$

$$\frac{\partial\dot{T}^*}{\partial\dot{\boldsymbol{\pi}}_1}=\frac{\partial\dot{T}^*}{\partial\dot{\boldsymbol{\pi}}_2}=0;\quad\frac{\partial T}{\partial q_1}=\frac{\partial T}{\partial q_2}=\frac{\partial T}{\partial q_3}=0 \tag{48}$$

$$\frac{\partial T}{\partial\dot{q}_1}=m\dot{q}_1=m\sqrt{\dot{\boldsymbol{\pi}}_2}\cos\dot{\boldsymbol{\pi}}_1,\ \frac{\partial T}{\partial\dot{q}_2}=m\dot{q}_2=m\sqrt{\dot{\boldsymbol{\pi}}_2}\sin\dot{\boldsymbol{\pi}}_1,\ \frac{\partial T}{\partial\dot{q}_3}=m\dot{q}_3=m\sqrt{\dot{\boldsymbol{\pi}}_2} \tag{49}$$

对(36)式求时间微商，得

$$\left.\begin{aligned}\ddot{q}_1&=\frac{d}{dt}(\sqrt{\dot{\boldsymbol{\pi}}_2}\cos\dot{\boldsymbol{\pi}}_1)=\frac{\ddot{\pi}_2}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}\cos\dot{\boldsymbol{\pi}}_1-\sqrt{\dot{\boldsymbol{\pi}}_2}\ddot{\pi}_1\sin\dot{\boldsymbol{\pi}}_1\\\ddot{q}_2&=\frac{d}{dt}(\sqrt{\dot{\boldsymbol{\pi}}_2}\sin{}_1\dot{\boldsymbol{\pi}})=\frac{\ddot{\pi}_2}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}\sin\dot{\boldsymbol{\pi}}_1+\sqrt{\dot{\boldsymbol{\pi}}_2}\ddot{\pi}_1\cos\dot{\boldsymbol{\pi}}_1\\\ddot{q}_3&=\frac{d}{dt}\sqrt{\dot{\boldsymbol{\pi}}_2}=\frac{\ddot{\pi}_2}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}\end{aligned}\right\} \tag{50}$$

于是

$$\left.\begin{aligned}&\frac{\partial\ddot{q}_1}{\partial\dot{\boldsymbol{\pi}}_1}=-\frac{\ddot{\pi}_2}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}\sin\dot{\boldsymbol{\pi}}_1-\sqrt{\dot{\boldsymbol{\pi}}_2}\ddot{\pi}_1\cos\dot{\boldsymbol{\pi}},\qquad\frac{\partial\ddot{q}_1}{\partial\dot{\boldsymbol{\pi}}_2}=-\frac{\ddot{\pi}_2\cos\dot{\boldsymbol{\pi}}_1}{4\dot{\boldsymbol{\pi}}_2^{3/2}}-\frac{\ddot{\pi}_1\sin\dot{\boldsymbol{\pi}}_1}{2\sqrt{\dot{\boldsymbol{\pi}}_1}}\\&\frac{\partial\ddot{q}_2}{\partial\dot{\boldsymbol{\pi}}_1}=\frac{\ddot{\pi}_2\cos\dot{\boldsymbol{\pi}}_1}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}\cos\dot{\boldsymbol{\pi}}_1-\ddot{\pi}_1\sqrt{\dot{\boldsymbol{\pi}}_2}\sin\dot{\boldsymbol{\pi}}_1,\qquad\frac{\partial\ddot{q}_2}{\partial\dot{\boldsymbol{\pi}}_2}=-\frac{\ddot{\pi}_2\sin\dot{\boldsymbol{\pi}}_1}{4\dot{\boldsymbol{\pi}}_2^{3/2}}+\frac{\ddot{\pi}_1\cos\dot{\boldsymbol{\pi}}_1}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}\\&\frac{\partial\ddot{q}_3}{\partial\dot{\boldsymbol{\pi}}_1}=0,\qquad\frac{\partial\ddot{q}_3}{\partial\dot{\boldsymbol{\pi}}_2}=-\frac{\ddot{\pi}_2}{4\dot{\boldsymbol{\pi}}_2^{3/2}}\end{aligned}\right\} \tag{51}$$

所以由(49)与(51)式，得

$$\sum_{s=1}^{3}\frac{\partial T}{\partial\dot{q}_s}\frac{\partial\ddot{q}_s}{\partial\dot{\boldsymbol{\pi}}_1}=-m\sqrt{\dot{\boldsymbol{\pi}}_2}\cos\dot{\boldsymbol{\pi}}_1\left(-\frac{\ddot{\pi}_2}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}\sin\dot{\boldsymbol{\pi}}_1+\sqrt{\dot{\boldsymbol{\pi}}_2}\ddot{\pi}_1\cos\dot{\boldsymbol{\pi}}_1\right)$$

$$-m\sqrt{\dot{\boldsymbol{\pi}}_2}\sin\dot{\boldsymbol{\pi}}_1\left(\frac{\ddot{\pi}_2}{2\sqrt{\dot{\boldsymbol{\pi}}_2}}\cos\dot{\boldsymbol{\pi}}_1+\ddot{\pi}_1\sqrt{\dot{\boldsymbol{\pi}}_2}\sin\dot{\boldsymbol{\pi}}_1\right)+0=-m\dot{\boldsymbol{\pi}}_2\ddot{\pi}_1 \tag{52}$$

$$\sum_{s=1}^{3}\frac{\partial T}{\partial \dot{q}_s}\frac{\partial \ddot{q}_s}{\partial \dot{\pi}_2}=-\frac{m\ddot{\pi}_2}{2\dot{\pi}_1} \tag{53}$$

将(44)、(48)和(52)式及(45)、(48)和(53)式分别代入(32)式也得(41)式.

由此可知，利用三种新方程组计算所得结果相同，用其他方程计算也一样. 但用基本动力学方程(5)计算最为简便.

致谢 感谢汪家诛教授的帮助.

参 考 文 献

[1] 刘书振、陈书勤、罗绍凯，《分析力学》，河南大学版出社，开封 (1992)，289.

The Fundamental Equations of Dynamics Using Representation of Quasi-Coordinates in the Space of Non-Linear Non-Holonomic Constraints

Qiu Rong

(*Fuzhou University, Fuzhou* 350002, *P. R. China*)

Abstract

The dot product of the bases vectors on the super-surface of the non-linear nonholonomic constraints with one order, expressed by quasi-coodinates, and Mishirskii equations are regarded as the fundamental equations of dynamics with non-linear and nonholonomic constraints in one order for the system of the variable mass. From these the variant differential-equations of dynamics expressed by quasi-coordinates are derived. The fundamental equations of dynamics are compatible with the principle of Jourdain. A case is cited.

Key words non-linear non-holonomic constraints, quasi-coordinates, fundamental equations of dynamics