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高阶剪切变形板理论Kármán型方程及在热后屈曲分析中的应用

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摘　要

本文基于Reddy高阶剪切变形板理论导出Kármán型非线性大挠度方程并用于层合板热后屈曲分析．分析中计及板初始几何缺陷和热效应．给出了四边简支，对称正交铺设层合板在均匀或非均匀抛物型热分布作用下的后屈曲分析．采用摄动-Galerkin混合法确定板的热屈曲载荷与热后屈曲平衡路径．同时讨论了横向剪切变形，板长宽比，铺层数以及初始几何缺陷等各种参数变化的影响．

关键词　层合板　高阶理论　热后屈曲　摄动-Galerkin混合法

一、引　言

复合材料层合板广泛用于核能、石化和航空工业中．此种板状构件不可避免地存在不同程度的初始几何缺陷．由于边界约束，温度变化将会在板内产生应力并导致屈曲．因此，对于非完善层合板热屈曲和热后屈曲性态必须有充分的认识．

追随von Kármán的作法，Stavsky(1963)[1]导出了以横向挠度$\overline{W}$和应力函数$\overline{F}$表示的四阶偏微分耦合方程组，用以表征复合材料层合薄板的非线性大挠度．这一工作基于经典层合板理论，并包括热效应．沈惠申和林忠钦(1995)[2]将这一工作推广用于完善和非完善复合材料层合薄板，在均匀或非均匀热分布作用下的后屈曲分析．

近代，复合材料层合板研究结果表明：（1）与各向同性板相比，复合材料层合板其板厚有明显的影响；（2）由于横向剪切弹性模量比面内弹性模量低得多，因此，在复合材料层合板中横向剪切变形的影响更显著．鉴于以上两点，在层合板的分析中必须采用剪切变形板理论．

本文基于Reddy高阶剪切变形板理论导出Kármán型非线性大挠度方程，其中包括热效应．作为应用给出了四边简支对称正交铺设层合板，在均匀或非均匀抛物型热分布作用下的后屈曲分析．采用摄动-Galerkin混合法确定板的热屈曲载荷和热后屈曲平衡路径．板的材料性能常数假定与温度变化无关．分析中计及板的初始几何缺陷，其形式取作和板小挠度

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屈曲模态一致。

二、Kármán型大挠度方程

考虑长为a，宽为b，厚度为t 的任意铺设层合板，其铺层数为 N，承受机械荷载或热荷载作用。$\bar{U}$，$\bar{V}$，$\bar{W}$分别为对应右手坐标系(X, Y, Z)的板的位移分量。$\bar{\Psi}_x$和 $\bar{\Psi}_y$ 分别为板中面法线相对于Y和X轴的转角。以$\bar{W}^*(X, Y)$和$\bar{W}(X, Y)$ 分别表示板的初始的和附加的挠度，以$\bar{F}(X, Y)$表示应力函数，即有

$$\bar{N}_1=\bar{F}_{,yy},\quad \bar{N}_2=\bar{F}_{,xx},\quad \bar{N}_6=-\bar{F}_{,xy} \tag{2.1}$$

根据Reddy高阶剪切变形板理论[3]，我们有位移场

$$\left.\begin{aligned} &u_1=\bar{U}+Z[\bar{\Psi}_x-4(Z/t)^2(\bar{\Psi}_x+\partial\bar{W}/\partial X)/3]\\ &u_2=\bar{V}+Z[\bar{\Psi}_y-4(Z/t)^2(\bar{\Psi}_y+\partial\bar{W}/\partial Y)/3],\quad u_3=\bar{W}\end{aligned}\right\} \tag{2.2}$$

对应于(2.2)式位移场的Kárman应变关系为

$$\left.\begin{aligned} &\epsilon_1=\epsilon_1^0+Z(\kappa_1^0+Z^2\kappa_1^2),\quad \epsilon_2=\epsilon_2^0+Z(\kappa_2^0+Z^2\kappa_2^2),\quad \epsilon_3=0\\ &\epsilon_4=\epsilon_4^0+Z^2\kappa_4^2,\quad \epsilon_5=\epsilon_5^0+Z^2\kappa_5^2,\quad \epsilon_6=\epsilon_6^0+Z(\kappa_6^0+Z^2\kappa_6^2)\end{aligned}\right\} \tag{2.3}$$

其中

$$\left.\begin{aligned} &\epsilon_1^0=\partial\bar{U}/\partial X+(\partial\bar{W}/\partial X)^2/2+(\partial\bar{W}/\partial X)(\partial\bar{W}^*/\partial X)\\ &\kappa_1^0=\partial\bar{\Psi}_x/\partial X,\quad \kappa_1^2=-(4/3t^2)(\partial\bar{\Psi}_x/\partial X+\partial^2\bar{W}/\partial X^2)\\ &\epsilon_2^0=\partial\bar{V}/\partial Y+(\partial\bar{W}/\partial Y)^2/2+(\partial\bar{W}/\partial Y)(\partial\bar{W}^*/\partial Y)\\ &\kappa_2^0=\partial\bar{\Psi}_y/\partial Y,\quad \kappa_2^2=-(4/3t^2)(\partial\bar{\Psi}_y/\partial Y+\partial^2\bar{W}/\partial Y^2)\\ &\epsilon_4^0=\bar{\Psi}_y+\partial\bar{W}/\partial Y,\quad \kappa_4^2=-(4/t^2)(\bar{\Psi}_y+\partial\bar{W}/\partial Y)\\ &\epsilon_5^0=\bar{\Psi}_x+\partial\bar{W}/\partial X,\quad \kappa_5^2=-(4/t^2)(\bar{\Psi}_x+\partial\bar{W}/\partial X)\\ &\epsilon_6^0=\partial\bar{U}/\partial Y+\partial\bar{V}/\partial X+(\partial\bar{W}/\partial X)(\partial\bar{W}/\partial Y)\\ &\qquad+(\partial\bar{W}/\partial X)(\partial\bar{W}^*/\partial Y)+(\partial\bar{W}/\partial Y)(\partial\bar{W}^*/\partial X)\\ &\kappa_6^0=\partial\bar{\Psi}_x/\partial Y+\partial\bar{\Psi}_y/\partial X\\ &\kappa_6^2=-(4/3t^2)(\partial\bar{\Psi}_x/\partial Y+\partial\bar{\Psi}_y/\partial X+2\partial^2\bar{W}/\partial X\partial Y)\end{aligned}\right\} \tag{2.4}$$

应力合力$\bar{N}_i$，$\bar{M}_i$，$\bar{P}_i$，Q_i和R_i与中面应变及曲率的关系为

$$\begin{Bmatrix}\bar{\mathbf{N}}\\ \bar{\mathbf{M}}\\ \bar{\mathbf{P}}\end{Bmatrix}=\begin{bmatrix}\mathbf{A}&\mathbf{B}&\mathbf{E}\\ \mathbf{B}&\mathbf{D}&\mathbf{F}\\ \mathbf{E}&\mathbf{F}&\mathbf{H}\end{bmatrix}\begin{Bmatrix}\underset{\sim}{\epsilon}^0\\ \boldsymbol{\kappa}^0\\ \boldsymbol{\kappa}^2\end{Bmatrix}+\begin{Bmatrix}\bar{\mathbf{N}}^T\\ \bar{\mathbf{M}}^T\\ \bar{\mathbf{P}}^T\end{Bmatrix},\quad \begin{Bmatrix}\mathbf{Q}\\ \mathbf{R}\end{Bmatrix}=\begin{bmatrix}\mathbf{A}&\mathbf{D}\\ \mathbf{D}&\mathbf{F}\end{bmatrix}\begin{Bmatrix}\underset{\sim}{\epsilon}^0\\ \boldsymbol{\kappa}^2\end{Bmatrix} \tag{2.5a,b}$$

其中应力合力$\bar{N}_i$，$\bar{M}_i$，$\bar{P}_i$，Q_i和R_i定义为

$$\left.\begin{aligned} &(\bar{N}_i,\ \bar{M}_i,\ \bar{P}_i)=\int_{-t/2}^{t/2}\sigma_i(1,\ Z,\ Z^3)dZ\qquad (i=1,\ 2,\ 6)\\ &(Q_2,\ R_2)=\int_{-t/2}^{t/2}\sigma_4(1,Z^2)dZ,\quad (Q_1,R_1)=\int_{-t/2}^{t/2}\sigma_5(1,Z^2)dZ\end{aligned}\right\} \tag{2.6a~c}$$

A_{ij}，B_{ij}等为板的刚度系数，定义为

$$\left.\begin{aligned}&(A_{ij},\ B_{ij},\ D_{ij},\ E_{ij},\ F_{ij},\ H_{ij})\\&\quad=\sum_{k=1}\int_{t_{k-1}}^{t_k}(\bar{Q}_{ij})_k(1,\ Z,\ Z^2,\ Z^3,\ Z^4,\ Z^6)dZ\qquad(i,\ j=1,\ 2,\ 6)\\&(A_{ij},\ D_{ij},\ F_{ij})=\sum_{k=1}\int_{t_{k-1}}^{t_k}(\bar{Q}_{ij})_k(1,\ Z^2,\ Z^4)dZ\qquad(i,j=4,5)\end{aligned}\right\}\tag{2.7a,b}$$

其中$\bar{Q}_{ij}$为转换弹性常数，定义为

$$\begin{Bmatrix}\bar{Q}_{11}\\\bar{Q}_{12}\\\bar{Q}_{22}\\\bar{Q}_{16}\\\bar{Q}_{26}\\\bar{Q}_{66}\end{Bmatrix}=\begin{bmatrix}c^4&2c^2s^2&s^4&4c^2s^2\\c^2s^2&c^4+s^4&c^2s^2&-4c^2s^2\\s^4&2c^2s^2&c^4&4c^2s^2\\c^3s&cs^3-c^3s&-cs^3&-2cs(c^2-s^2)\\cs^3&c^3s-cs^3&-c^3s&2cs(c^2-s^2)\\c^2s^2&-2c^2s^2&c^2s^2&(c^2-s^2)^2\end{bmatrix}\begin{Bmatrix}Q_{11}\\Q_{12}\\Q_{22}\\Q_{66}\end{Bmatrix}\tag{2.8a}$$

和

$$\begin{Bmatrix}\bar{Q}_{44}\\\bar{Q}_{45}\\\bar{Q}_{55}\end{Bmatrix}=\begin{bmatrix}c^2&s^2\\-cs&cs\\s^2&c^2\end{bmatrix}\begin{Bmatrix}Q_{44}\\Q_{55}\end{Bmatrix}\tag{2.8b}$$

其中

$$\left.\begin{aligned}&Q_{11}=\frac{E_{11}}{(1-\nu_{12}\nu_{21})},\quad Q_{22}=\frac{E_{22}}{(1-\nu_{12}\nu_{21})},\quad Q_{12}=\frac{\nu_{21}E_{11}}{(1-\nu_{12}\nu_{21})}\\&Q_{44}=G_{23},\quad Q_{55}=G_{13},\quad Q_{66}=G_{12}\end{aligned}\right\}\tag{2.8c}$$

且

$$c=\cos\theta,\quad s=\sin\theta\tag{2.8d}$$

其中θ为铺层角.

在(2.5a)式中热力、热弯矩和高阶热弯矩定义为

$$\begin{bmatrix}N_x^T, & M_x^T, & P_x^T\\N_y^T, & M_y^T, & P_y^T\\N_{xy}^T, & M_{xy}^T, & P_{xy}^T\end{bmatrix}=\sum_{k=1}\int_{t_{k-1}}^{t_k}(1,\ Z,\ Z^3)\begin{bmatrix}A_x\\A_y\\A_{xy}\end{bmatrix}_k T(X,Y,Z)dZ\tag{2.9a}$$

及

$$\begin{Bmatrix}S_x^T\\S_y^T\\S_{xy}^T\end{Bmatrix}=\begin{Bmatrix}M_x^T\\M_y^T\\M_{xy}^T\end{Bmatrix}-\frac{4}{3t^2}\begin{Bmatrix}P_x^T\\P_y^T\\P_{xy}^T\end{Bmatrix}\tag{2.9b}$$

在(2.9a)式中

$$\begin{Bmatrix}A_x\\A_y\\A_{xy}\end{Bmatrix}=-\begin{bmatrix}\bar{Q}_{11}&\bar{Q}_{12}&\bar{Q}_{16}\\\bar{Q}_{12}&\bar{Q}_{22}&\bar{Q}_{26}\\\bar{Q}_{16}&\bar{Q}_{26}&\bar{Q}_{66}\end{bmatrix}\begin{bmatrix}c^2&s^2\\s^2&c^2\\2cs&-2cs\end{bmatrix}\begin{Bmatrix}\alpha_{11}\\\alpha_{22}\end{Bmatrix}\tag{2.10}$$

其中α_{11}和α_{22}为第k层板的热膨胀系数.

方程(2.5a)的部分求逆得出

$$\begin{Bmatrix}\boldsymbol{\varepsilon}^0\\\mathbf{M}^*\\\mathbf{P}^*\end{Bmatrix}=\begin{bmatrix}\mathbf{A}^*&\mathbf{B}^*&\mathbf{E}^*\\-(\mathbf{B}^*)^T&\mathbf{D}^*&(\mathbf{F}^*)^T\\-(\mathbf{E}^*)^T&\mathbf{F}^*&\mathbf{H}^*\end{bmatrix}\begin{Bmatrix}\mathbf{N}^*\\\boldsymbol{\kappa}^0\\\boldsymbol{\kappa}^2\end{Bmatrix}\tag{2.11}$$

其中

$$\mathbf{N}^*=\bar{\mathbf{N}}-\bar{\mathbf{N}}^T,\ \mathbf{M}^*=\bar{\mathbf{M}}-\bar{\mathbf{M}}^T,\ \mathbf{P}^*=\bar{\mathbf{P}}-\bar{\mathbf{P}}^T \tag{2.12a}$$

刚度矩阵$\mathbf{A}^*$，$\mathbf{B}^*$等定义为

$$\left.\begin{array}{l}\mathbf{A}^*=\mathbf{A}^{-1},\ \mathbf{B}^*=-\mathbf{A}^{-1}\mathbf{B},\ \mathbf{D}^*=\mathbf{D}-\mathbf{B}\mathbf{A}^{-1}\mathbf{B},\ \mathbf{E}^*=-\mathbf{A}^{-1}\mathbf{E}\\ \mathbf{F}^*=\mathbf{F}-\mathbf{E}\mathbf{A}^{-1}\mathbf{B},\ \mathbf{H}^*=\mathbf{H}-\mathbf{E}\mathbf{A}^{-1}\mathbf{E}\end{array}\right\} \tag{2.12b}$$

通常$\mathbf{A}^*$，$\mathbf{D}^*$和$\mathbf{H}^*$是对称矩阵，而$\mathbf{B}^*$，$\mathbf{E}^*$和$\mathbf{F}^*$不一定是对称矩阵.

将(2.1)和(2.11)式代入平衡方程

$$\left.\begin{array}{l}\dfrac{\partial N_1}{\partial X}+\dfrac{\partial N_6}{\partial Y}=0,\ \dfrac{\partial N_6}{\partial X}+\dfrac{\partial N_2}{\partial Y}=0\\ \dfrac{\partial Q_1}{\partial X}+\dfrac{\partial Q_2}{\partial Y}-\dfrac{4}{t^2}\left[\dfrac{\partial R_1}{\partial X}+\dfrac{\partial R_2}{\partial Y}\right]+\dfrac{4}{3t^2}\left[\dfrac{\partial^2 P_1}{\partial X^2}+2\dfrac{\partial^2 P_6}{\partial X\partial Y}+\dfrac{\partial^2 P_2}{\partial Y^2}\right]\\ \quad+\dfrac{\partial}{\partial X}\left[N_1\dfrac{\partial}{\partial X}(\bar{W}+\bar{W}^*)+N_6\dfrac{\partial}{\partial Y}(\bar{W}+\bar{W}^*)\right]\\ \quad+\dfrac{\partial}{\partial Y}\left[N_6\dfrac{\partial}{\partial X}(\bar{W}+\bar{W}^*)+N_2\dfrac{\partial}{\partial Y}(\bar{W}+\bar{W}^*)\right]+q=0\\ \dfrac{\partial M_1}{\partial X}+\dfrac{\partial M_6}{\partial Y}-Q_1+\dfrac{4}{t^2}R_1-\dfrac{4}{3t^2}\left[\dfrac{\partial P_1}{\partial X}+\dfrac{\partial P_6}{\partial Y}\right]=0\\ \dfrac{\partial M_6}{\partial X}+\dfrac{\partial M_2}{\partial Y}-Q_2+\dfrac{4}{t^2}R_2-\dfrac{4}{3t^2}\left[\dfrac{\partial P_6}{\partial X}+\dfrac{\partial P_2}{\partial Y}\right]=0\end{array}\right\} \tag{2.13a～e}$$

另外，考虑协调方程

$$\frac{\partial^2\epsilon_1^0}{\partial Y^2}+\frac{\partial^2\epsilon_2^0}{\partial X^2}-\frac{\partial^2\epsilon_6^0}{\partial X\partial Y}=\left[\frac{\partial^2\bar{W}}{\partial X\partial Y}\right]^2-\frac{\partial^2\bar{W}}{\partial X^2}\frac{\partial^2\bar{W}}{\partial Y^2}+2\left[\frac{\partial^2\bar{W}^*}{\partial X\partial Y}\right]^2-\frac{\partial^2\bar{W}}{\partial X^2}\frac{\partial^2\bar{W}^*}{\partial Y^2}$$
$$-\frac{\partial^2\bar{W}^*}{\partial X^2}\frac{\partial^2\bar{W}}{\partial Y^2} \tag{2.14}$$

那么可以导得Kárman型大挠度方程

$$L_{11}(\bar{W})-L_{12}(\bar{\Psi}_x)-L_{13}(\bar{\Psi}_y)+L_{14}(\bar{F})-L_{15}(N^T)-L_{16}(M^T)$$
$$=L(\bar{W}+\bar{W}^*,\ \bar{F})+q \tag{2.15}$$

$$L_{21}(\bar{F})+L_{22}(\bar{\Psi}_x)+L_{23}(\bar{\Psi}_y)-L_{24}(\bar{W})-L_{25}(N^T)=-L(\bar{W}+2\bar{W}^*,\ \bar{W})/2 \tag{2.16}$$

$$L_{31}(\bar{W})+L_{32}(\bar{\Psi}_x)+L_{33}(\bar{\Psi}_y)+L_{34}(\bar{F})-L_{35}(N^T)-L_{36}(S^T)=0 \tag{2.17}$$

$$L_{41}(\bar{W})+L_{42}(\bar{\Psi}_x)+L_{43}(\bar{\Psi}_y)+L_{44}(\bar{F})-L_{45}(N^T)-L_{46}(S^T)=0 \tag{2.18}$$

其中各算子定义为

$$\begin{array}{l}L_{11}(\quad)=\dfrac{4}{3t^2}\left[F_{11}^*\dfrac{\partial^4}{\partial X^4}+2(F_{16}^*+F_{61}^*)\dfrac{\partial^4}{\partial X^3\partial Y}+(F_{12}^*+F_{21}^*+4F_{66}^*)\dfrac{\partial^4}{\partial X^2\partial Y^2}\right.\\ \qquad\left.+2(F_{26}^*+F_{62}^*)\dfrac{\partial^4}{\partial X\partial Y^3}+F_{22}^*\dfrac{\partial^4}{\partial Y^4}\right]\\ L_{12}(\quad)=\left[D_{11}^*-\dfrac{4}{3t^2}F_{11}^*\right]\dfrac{\partial^3}{\partial X^3}+\left[3D_{16}^*-\dfrac{4}{3t^2}(F_{61}^*+2F_{16}^*)\right]\dfrac{\partial^3}{\partial X^2\partial Y}\\ \qquad+\left[(D_{12}^*+2D_{66}^*)-\dfrac{4}{3t^2}(F_{12}^*+2F_{66}^*)\right]\dfrac{\partial^3}{\partial X\partial Y^2}+\left[D_{26}^*-\dfrac{4}{3t^2}F_{26}^*\right]\dfrac{\partial^3}{\partial Y^3}\end{array}$$

$$L_{13}(\quad)=\left[D^*_{16}-\frac{4}{3t^2}F^*_{16}\right]\frac{\partial^3}{\partial X^3}+\left[(D^*_{12}+2D^*_{66})-\frac{4}{3t^2}(F^*_{21}+2F^*_{66})\right]\frac{\partial^3}{\partial X^2\partial Y}$$

$$+\left[3D^*_{26}-\frac{4}{3t^2}(F^*_{62}+2F^*_{26})\right]\frac{\partial^3}{\partial X^2\partial Y}+\left[D^*_{22}-\frac{4}{3t^2}F^*_{22}\right]\frac{\partial^3}{\partial Y^3}$$

$$L_{14}(\quad)=B^*_{21}\frac{\partial^4}{\partial X^4}+(2B^*_{26}-B^*_{61})\frac{\partial^4}{\partial X^3\partial Y}+(B^*_{11}+B^*_{22}-2B^*_{66})\frac{\partial^4}{\partial X^2\partial Y^2}$$

$$+(2B^*_{16}-B^*_{62})\frac{\partial^4}{\partial X\partial Y^3}+B^*_{12}\frac{\partial^4}{\partial Y^4}$$

$$L_{15}(N^T)=\frac{\partial^2}{\partial X^2}(B^*_{11}N^T_x+B^*_{21}N^T_y+B^*_{61}N^T_{xy})+\frac{\partial^2}{\partial X\partial Y}(B^*_{16}N^T_x+B^*_{26}N^T_y+B^*_{66}N^T_{xy})$$

$$+\frac{\partial^2}{\partial Y^2}(B^*_{12}N^T_x+B^*_{22}N^T_y+B^*_{62}N^T_{xy})$$

$$L_{16}(M^T)=\frac{\partial^2}{\partial X^2}(M^T_x)+2\frac{\partial^2}{\partial X\partial Y}(M^T_{xy})+\frac{\partial^2}{\partial Y^2}(M^T_y)$$

$$L_{21}(\quad)=A^*_{22}\frac{\partial^4}{\partial X^4}-2A^*_{26}\frac{\partial^4}{\partial X^3\partial Y}+(2A^*_{12}+A^*_{66})\frac{\partial^4}{\partial X^2\partial Y^2}-2A^*_{16}\frac{\partial^4}{\partial X\partial Y^3}$$

$$+A^*_{11}\frac{\partial^4}{\partial Y^4}$$

$$L_{22}(\quad)=\left[B^*_{21}-\frac{4}{3t^2}E^*_{21}\right]\frac{\partial^3}{\partial X^3}+\left[(B^*_{26}-B^*_{61})-\frac{4}{3t^2}(E^*_{26}-E^*_{61})\right]\frac{\partial^3}{\partial X^2\partial Y}$$

$$+\left[(B^*_{11}-B^*_{66})-\frac{4}{3t^2}(E^*_{11}-E^*_{66})\right]\frac{\partial^3}{\partial X\partial Y^2}+\left[B^*_{16}-\frac{4}{3t^2}E^*_{16}\right]\frac{\partial^3}{\partial Y^3}$$

$$L_{23}(\quad)=\left[B^*_{26}-\frac{4}{3t^2}E^*_{26}\right]\frac{\partial^3}{\partial X^3}+\left[(B^*_{22}-B^*_{66})-\frac{4}{3t^2}(E^*_{22}-E^*_{66})\right]\frac{\partial^3}{\partial X^2\partial Y}$$

$$+\left[(B^*_{16}-B^*_{62})-\frac{4}{3t^2}(E^*_{16}-E^*_{62})\right]\frac{\partial^3}{\partial X\partial Y^2}+\left[B^*_{12}-\frac{4}{3t^2}E^*_{12}\right]\frac{\partial^3}{\partial Y^3}$$

$$L_{24}(\quad)=\frac{4}{3t^2}\left[E^*_{21}\frac{\partial^4}{\partial X^4}+(2E^*_{26}-E^*_{61})\frac{\partial^4}{\partial X^3\partial Y}+(E^*_{11}+E^*_{22}-2E^*_{66})\frac{\partial^4}{\partial X^2\partial Y^2}\right.$$

$$\left.+(2E^*_{16}-E^*_{62})\frac{\partial^4}{\partial X\partial Y^3}+E^*_{12}\frac{\partial^4}{\partial Y^4}\right]$$

$$L_{25}(N^T)=\frac{\partial^2}{\partial X^2}(A^*_{12}N^T_x+A^*_{22}N^T_y+A^*_{26}N^T_{xy})-\frac{\partial^2}{\partial X\partial Y}(A^*_{16}N^T_x+A^*_{26}N^T_y$$

$$+A^*_{66}N^T_{xy})+\frac{\partial^2}{\partial Y^2}(A^*_{11}N^T_x+A^*_{12}N^T_y+A^*_{16}N^T_{xy})$$

$$L_{31}(\quad)=\left[A_{55}-\frac{8}{t^2}D_{55}+\frac{16}{t^4}F_{55}\right]\frac{\partial}{\partial X}+\left[A_{45}-\frac{8}{t^2}D_{45}+\frac{16}{t^4}F_{45}\right]\frac{\partial}{\partial Y}$$

$$+\frac{4}{3t^2}\left[(F^*_{11}-\frac{4}{3t^2}H^*_{11})\frac{\partial^3}{\partial X^3}+\left[(F^*_{16}+2F^*_{61})-\frac{4}{t^2}H^*_{16}\right]\frac{\partial^3}{\partial X^2\partial Y}\right.$$

$$+\left[(F^*_{21}+2F^*_{66})-\frac{4}{3t^2}(H^*_{12}+2H^*_{66})\right]\frac{\partial^3}{\partial X\partial Y^2}$$

$$+\left(F^{*}_{26}-\frac{4}{3t^{2}}H^{*}_{26}\right)\frac{\partial^{3}}{\partial Y^{3}}\Big]$$

$$L_{32}(\quad)=\Big[A_{55}-\frac{8}{t^{2}}D_{55}+\frac{16}{t^{4}}F_{55}\Big]-\Big[D^{*}_{11}-\frac{8}{3t^{2}}F^{*}_{11}+\frac{16}{9t^{4}}H^{*}_{11}\Big]\frac{\partial^{2}}{\partial X^{2}}$$

$$-2\Big[D^{*}_{16}-\frac{4}{3t^{2}}(F^{*}_{16}+F^{*}_{61})+\frac{16}{9t^{4}}H^{*}_{16}\Big]\frac{\partial^{2}}{\partial X\partial Y}$$

$$-\Big[D^{*}_{66}-\frac{8}{3t^{2}}F^{*}_{66}+\frac{16}{9t^{4}}H^{*}_{66}\Big]\frac{\partial^{2}}{\partial Y^{2}}$$

$$L_{33}(\quad)=\Big[A_{45}-\frac{8}{t^{2}}D_{45}+\frac{16}{t^{4}}F_{45}\Big]-\Big[D^{*}_{16}-\frac{4}{3t^{2}}(F^{*}_{16}+F^{*}_{61})+\frac{16}{9t^{4}}H^{*}_{16}\Big]\frac{\partial^{2}}{\partial X^{2}}$$

$$-\Big[(D^{*}_{12}+D^{*}_{66})-\frac{4}{3t^{2}}(F^{*}_{12}+F^{*}_{21}+2F^{*}_{66})+\frac{16}{9t^{4}}(H^{*}_{12}+H^{*}_{66})\Big]\frac{\partial^{2}}{\partial X\partial Y}$$

$$-\Big[D^{*}_{26}-\frac{4}{3t^{2}}(F^{*}_{26}+F^{*}_{62})+\frac{16}{9t^{4}}H^{*}_{26}\Big]\frac{\partial^{2}}{\partial Y^{2}}$$

$$L_{34}(\quad)=L_{22}(\quad)$$

$$L_{35}(N^{T})=\frac{\partial}{\partial X}\Big[\left(B^{*}_{11}-\frac{4}{3t^{2}}E^{*}_{11}\right)N^{T}_{x}+\left(B^{*}_{21}-\frac{4}{3t^{2}}E^{*}_{21}\right)N^{T}_{y}+\Big(B^{*}_{61}$$

$$-\frac{4}{3t^{2}}E^{*}_{61}\Big)N^{T}_{xy}\Big]+\frac{\partial}{\partial Y}\Big[\left(B^{*}_{16}-\frac{4}{3t^{2}}E^{*}_{16}\right)N^{T}_{x}+\left(B^{*}_{26}-\frac{4}{3t^{2}}E^{*}_{26}\right)N^{T}_{y}$$

$$+\left(B^{*}_{66}-\frac{4}{3t^{2}}E^{*}_{66}\right)N^{T}_{xy}\Big]$$

$$L_{36}(S^{T})=\frac{\partial}{\partial X}(S^{T}_{x})+\frac{\partial}{\partial Y}(S^{T}_{xy})$$

$$L_{41}(\quad)=\Big[A_{45}-\frac{8}{t^{2}}D_{45}+\frac{16}{t^{4}}F_{45}\Big]\frac{\partial}{\partial X}+\Big[A_{44}-\frac{8}{t^{2}}D_{44}+\frac{16}{t^{4}}F_{44}\Big]\frac{\partial}{\partial Y}$$

$$+\frac{4}{3t^{2}}\Big[\left(F^{*}_{16}-\frac{4}{3t^{2}}H^{*}_{16}\right)\frac{\partial^{3}}{\partial X^{3}}+\Big[(F^{*}_{12}+2F^{*}_{66})$$

$$-\frac{4}{3t^{2}}(H^{*}_{12}+2H^{*}_{66})\Big]\frac{\partial^{3}}{\partial X^{2}\partial Y}+\Big[(F^{*}_{26}+2F^{*}_{62})-\frac{4}{t^{2}}H^{*}_{26}\Big]\frac{\partial^{3}}{\partial X\partial Y^{2}}$$

$$+\left(F^{*}_{22}-\frac{4}{3t^{2}}H^{*}_{22}\right)\frac{\partial^{3}}{\partial Y^{3}}\Big]$$

$$L_{42}(\quad)=L_{33}(\quad)$$

$$L_{43}(\quad)=\Big[A_{44}-\frac{8}{t^{2}}D_{44}+\frac{16}{t^{4}}F_{44}\Big]-\Big[D^{*}_{66}-\frac{8}{3t^{2}}F^{*}_{66}+\frac{16}{9t^{4}}H^{*}_{66}\Big]\frac{\partial^{2}}{\partial X^{2}}$$

$$-2\Big[D^{*}_{26}-\frac{4}{3t^{2}}(F^{*}_{26}+F^{*}_{62})+\frac{16}{9t^{4}}H^{*}_{26}\Big]\frac{\partial^{2}}{\partial X\partial Y}$$

$$-\Big[D^{*}_{22}-\frac{8}{3t^{2}}F^{*}_{22}+\frac{16}{9t^{4}}H^{*}_{22}\Big]\frac{\partial^{2}}{\partial Y^{2}}$$

$$L_{44}(\quad)=L_{2}(\quad)$$

$$L_{45}(N^{T})=\frac{\partial}{\partial X}\Big[\left(B^{*}_{16}-\frac{4}{3t^{2}}E^{*}_{16}\right)N^{T}_{x}+\left(B^{*}_{26}-\frac{4}{3t^{2}}E^{*}_{26}\right)N^{T}_{y}$$

$$
\left.
\begin{aligned}
&+\left(B_{66}^*-\frac{4}{3t^2}E_{66}^*\right)N_{xy}^T\Big]+\frac{\partial}{\partial Y}\Big[\left(B_{12}^*-\frac{4}{3t^2}E_{12}^*\right)N_x^T \\
&+\left(B_{22}^*-\frac{4}{3t^2}E_{22}^*\right)N_y^T+\left(B_{62}^*-\frac{4}{3t^2}E_{62}^*\right)N_{xy}^T\Big] \\
&L_{46}(S^T)=\frac{\partial}{\partial X}(S_{xy}^T)+\frac{\partial}{\partial Y}(S_y^T) \\
&L(\quad)=\frac{\partial^2}{\partial X^2}\frac{\partial^2}{\partial Y^2}-2\frac{\partial^2}{\partial X\partial Y}\frac{\partial^2}{\partial X\partial Y}+\frac{\partial^2}{\partial Y^2}\frac{\partial^2}{\partial X^2}
\end{aligned}
\right\}
\tag{2.19}
$$

值得指出，方程(2.15)～(2.18)不仅含有拉伸—弯曲耦合效应，同时含有热耦合效应．

三、对称正交铺设层合板热后屈曲

考虑四边简支，对称正交铺设层合板在均匀或非均匀热分布作用下的后屈曲．对于屈曲问题面外荷载q取为零．对于对称正交铺设层合板，刚度系数

$$
\left.
\begin{aligned}
&B_{ij}=E_{ij}=0,\quad A_{16}=A_{26}=D_{16}=D_{26}=F_{16}=F_{26}=H_{16}=H_{26}=0 \\
&A_{45}=D_{45}=F_{45}=0
\end{aligned}
\right\}
\tag{3.1}
$$

面内温度场假定为

$$T(X,\ Y,\ Z)=T_0+T_1[1-((2Y-b)/b)^2] \tag{3.2}$$

在(3.2)式中，当$T_1=0$为均匀温度场；$T_0=0$为抛物型温度场．

由(3.2)和(2.9a)式，热力N_{xy}^T，热弯矩M_x^T，M_y^T，M_{xy}^T和高阶热弯矩P_x^T，P_y^T，P_{xy}^T皆为零．

记单一板层的热膨胀系数为

$$\alpha_{11}=a_{11}\alpha_0,\quad \alpha_{22}=a_{22}\alpha_0 \tag{3.3}$$

其中α_0为任意参考值，并取

$$
\begin{bmatrix} A_x^T \\ A_y^T \end{bmatrix}=-\sum_{k=1}\int_{t_{k-1}}^{t_k}\begin{bmatrix} A_x \\ A_y \end{bmatrix}_k dZ
\tag{3.4}
$$

令$\lambda_T=\alpha_0 T_i$，其中对应均匀温度场时取$i=0$，其它情况即$T_i\neq 0$时取$i=1$．同时，引进无量纲参数

$$
\left.
\begin{aligned}
&x=\pi X/a,\ Y=\pi Y/b,\ \beta=a/b,\ (W^*,\ W)=(\overline{W}^*,\ \overline{W})/[D_{11}^*D_{22}^*A_{11}^*A_{22}^*]^{1/4} \\
&F=\overline{F}/[D_{11}^*D_{22}^*]^{1/2},\ (\Psi_x,\ \Psi_y)=(\overline{\Psi}_x,\ \overline{\Psi}_y)a/\pi[D_{11}^*D_{22}^*A_{11}^*A_{22}^*]^{1/4} \\
&\gamma_{14}=[D_{22}^*/D_{11}^*]^{1/2},\ \gamma_{24}=[A_{11}^*/A_{22}^*]^{1/2},\ \gamma_5=-A_{12}^*/A_{22}^* \\
&(\gamma_{110},\ \gamma_{112},\ \gamma_{114})=(4/3t^2)[F_{11}^*,\ (F_{12}^*+F_{21}^*+4F_{66}^*)/2,\ F_{22}^*]/D_{11}^* \\
&(\gamma_{120},\ \gamma_{122})=[D_{11}^*-4F_{11}^*/3t^2,\ (D_{12}^*+2D_{66}^*)-4(F_{12}^*+2F_{66}^*)/3t^2]/D_{11}^* \\
&(\gamma_{131},\ \gamma_{133})=[(D_{12}^*+2D_{66}^*)-4(F_{21}^*+2F_{66}^*)/3t^2,\ D_{22}^*-4F_{22}^*/3t^2]/D_{11}^* \\
&(\gamma_{212},\ \gamma_{214})=(A_{12}^*+A_{66}^*/2,\ A_{11}^*)/A_{22}^* \\
&(\gamma_{31},\ \gamma_{41})=(a^2/\pi^2)(A_{55}-8D_{55}/t^2+16F_{55}/t^4,\ A_{44}-8D_{44}/t^2+16F_{44}/t^4)/D_{11}^* \\
&(\gamma_{310},\ \gamma_{312})=(4/3t^2)[F_{11}^*-4H_{11}^*/3t^2,\ (F_{21}^*+2F_{66}^*)-4(H_{12}^*+2H_{66}^*)/3t^2]/D_{11}^*
\end{aligned}
\right.
$$

$$\left.\begin{aligned}
&(\gamma_{320},\ \gamma_{322})=(D^*_{11}-8F^*_{11}/3t^2+16H^*_{11}/9t^4,\ D^*_{66}-8F^*_{66}/3t^2+16H^*_{66}/9t^4)/D^*_{11}\\
&\gamma_{331}=[(D^*_{12}+D^*_{66})-4(F^*_{12}+F^*_{21}+2F^*_{66})/3t^2+16(H^*_{12}+H^*_{66})/9t^4]/D^*_{11}\\
&(\gamma_{411},\ \gamma_{413})=(4/3t^2)[(F^*_{12}+2F^*_{66})-4(H^*_{12}+2H^*_{66})/3t^2,\ F^*_{22}-4H^*_{22}/3t^2]/D^*_{11}\\
&(\gamma_{430},\ \gamma_{432})=(D^*_{66}-8F^*_{66}/3t^2+16H^*_{66}/9t^4,\ D^*_{22}-8F^*_{22}/3t^2+16H^*_{22}/9t^4)/D^*_{11}\\
&(\gamma_{T1},\ \gamma_{T2})=(A^T_x,\ A^T_y)a^2/\alpha_0\pi^2[D^*_{11}D^*_{22}]^{1/2}\\
&(M_x,\ M_y,\ P_x,\ P_y)=(\overline{M}_x,\ \overline{M}_y,\ 4\overline{P}_x/3t^2,\ 4\overline{P}_y/3t^2)a^2/\pi^2D^*_{11}[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}\\
&(\delta_x,\ \delta_y)=(\Delta_x/a,\ \Delta_y/b)b^2/4\pi^2[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/2}
\end{aligned}\right\} \tag{3.5}$$

那么，方程(2.15)～(2.18)可表为如下无量纲形式

$$L_{11}(W)-L_{12}(\Psi_x)-L_{13}(\Psi_y)=\gamma_{14}\beta^2L(W+W^*,\ F) \tag{3.6}$$

$$L_{21}(F)-C_1=-\gamma_{24}\beta^2L(W+2W^*,\ W)/2 \tag{3.7}$$

$$L_{31}(W)+L_{32}(\Psi_x)-L_{33}(\Psi_y)=0 \tag{3.8}$$

$$L_{41}(W)-L_{42}(\Psi_x)+L_{43}(\Psi_y)=0 \tag{3.9}$$

其中

$$L_{11}(\ \ )=\gamma_{110}\frac{\partial^4}{\partial x^4}+2\gamma_{112}\beta^2\frac{\partial^4}{\partial x^2\partial y^2}+\gamma_{114}\beta^4\frac{\partial^4}{\partial y^4}$$

$$L_{12}(\ \ )=\gamma_{120}\frac{\partial^3}{\partial x^3}+\gamma_{122}\beta^2\frac{\partial^3}{\partial x\partial y^2}$$

$$L_{13}(\ \ )=\gamma_{131}\beta\frac{\partial^3}{\partial x^2\partial y}+\gamma_{133}\beta^3\frac{\partial^3}{\partial y^3}$$

$$L_{21}(\ \ )=\frac{\partial^4}{\partial x^4}+2\gamma_{212}\beta^2\frac{\partial^4}{\partial x^2\partial y^2}+\gamma_{214}\beta^4\frac{\partial^4}{\partial y^4}$$

$$L_{31}(\ \ )=\gamma_{31}\frac{\partial}{\partial x}+\gamma_{310}\frac{\partial^3}{\partial x^3}+\gamma_{312}\beta^2\frac{\partial^3}{\partial x\partial y^2}$$

$$L_{32}(\ \ )=\gamma_{31}-\gamma_{320}\frac{\partial^2}{\partial x^2}-\gamma_{322}\beta^2\frac{\partial^2}{\partial y^2},\quad L_{33}(\ \ )=\gamma_{331}\beta\frac{\partial^2}{\partial x\partial y}$$

$$L_{41}(\ \ )=\gamma_{41}\beta\frac{\partial}{\partial y}+\gamma_{411}\beta\frac{\partial^3}{\partial x^2\partial y}+\gamma_{413}\beta^3\frac{\partial^3}{\partial y^3},\quad L_{42}(\ \ )=L_{33}(\ \ )$$

$$L_{43}(\ \ )=\gamma_{41}-\gamma_{430}\frac{\partial^2}{\partial x^2}-\gamma_{432}\beta^2\frac{\partial^2}{\partial y^2}$$

$$L(\ \ )=\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}-2\frac{\partial^2}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2}$$

假定板边界支承为四边不可移简支，那么边界条件为

$x=0,\ \pi$;

$$\delta_x=W=\Psi_y=0,\quad F_{,xy}=M_x=P_x=0 \tag{3.10a,b}$$

$y=0,\ \pi$;

$$\delta_y=W=\Psi_x=0,\quad F_{,xy}=M_y=P_y=0 \tag{3.10c,d}$$

板的单位端部缩短为

$$\delta_x=-\frac{1}{4\pi^2\beta^2\gamma_{24}}\int_0^\pi\int_0^\pi\left\{\left[\gamma_{24}^2\beta^2\frac{\partial^2F}{\partial y^2}-\gamma_5\frac{\partial^2F}{\partial x^2}\right]-\frac{1}{2}\gamma_{24}\left[\frac{\partial W}{\partial x}\right]^2-\gamma_{24}\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}\right.$$

$$+(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2})\lambda_T C_2\Big\}dxdy \tag{3.11a}$$

$$\delta_y=-\frac{1}{4\pi^2\beta^2\gamma_{24}}\int_0^{\pi}\int_0^{\pi}\Big\{\Big[\frac{\partial^2 F}{\partial x^2}-\gamma_5\beta^2\frac{\partial^2 F}{\partial y^2}\Big]-\frac{1}{2}\gamma_{24}\beta^2\Big[\frac{\partial W}{\partial y}\Big]^2-\gamma_{24}\beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}$$

$$+(\gamma_{T2}-\gamma_5\gamma_{T1})\lambda_T C_2\Big\}dxdy \tag{3.11b}$$

在(3.7)和(3.11)式中，对于均匀热分布作用取$C_1=0.0$，$C_2=1.0$和$\lambda_T=\alpha_0 T_0$；对于非均匀抛物型热分布作用取$C_1=8\beta^2\lambda_T(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2})/\pi^2$，$C_2=[T_0/T_1+4(y/\pi-y^2/\pi^2)]$和$\lambda_T=\alpha_0 T_1$。

方程(3.6)～(3.9)可用摄动-Galerkin 混合法求解。取方程组(3.6)～(3.9)的解为如下渐近展开式

$$\left.\begin{aligned} &W(x,\ y,\ \varepsilon)=\sum_{j=1}\varepsilon^j w_j(x,\ y),\ F(x,\ y,\ \varepsilon)=\sum_{j=0}\varepsilon^j f_j(x,\ y)\\ &\Psi_x(x,\ y,\ \varepsilon)=\sum_{j=1}\varepsilon^j\psi_{xj}(x,\ y),\ \Psi_y(x,\ y,\ \varepsilon)=\sum_{j=1}\varepsilon^j\psi_{yj}(x,\ y)\end{aligned}\right\} \tag{3.12a～d}$$

其中ε为摄动小参数。

假定$w_j(x,\ y)$的首项为小挠度经典解

$$w_1(x,\ y)=A_{11}^{(1)}\sin mx\sin ny \tag{3.13}$$

并假定板的初始几何缺陷具有相同的形式

$$W^*(x,\ y,\ \varepsilon)=\varepsilon a_{11}^*\sin mx\sin ny=\varepsilon\mu A_{11}^{(1)}\sin mx\sin ny \tag{3.14}$$

其中$\mu=a_{11}^*/A_{11}^{(1)}$为缺陷参数。

将(3.12)式代入方程(3.6)～(3.9)，得到一系列摄动方程组，并可逐级求解。在每一级摄动方程的求解过程中采用Galerkin方法确定展开式中各项$w_j(x,\ y)$，$f_j(x,\ y)$，$\psi_{xj}(x,\ y)$和$\psi_{yj}(x,\ y)$之系数。最终，我们可以导得大挠度渐近解

$$W=\varepsilon[A_{11}^{(1)}\sin mx\sin ny]+\varepsilon^3[A_{13}^{(3)}\sin mx\sin 3ny+A_{31}^{(3)}\sin 3mx\sin ny]+O(\varepsilon^5) \tag{3.15}$$

$$\begin{aligned}F=&-B_{00}^{(0)}\Big(\frac{y^2}{2}-C_4\frac{y^4}{24}\Big)-b_{00}^{(0)}\frac{x^2}{2}+\varepsilon^2\Big[-B_{00}^{(2)}\Big(\frac{y^2}{2}-C_4\frac{y^4}{24}\Big)-b_{00}^{(2)}\frac{x^2}{2}+B_{20}^{(2)}\cos 2mx\\ &+B_{02}^{(2)}\cos 2ny\Big]+\varepsilon^4\Big[-B_{00}^{(4)}\Big(\frac{y^2}{2}-C_4\frac{y^4}{24}\Big)-b_{00}^{(4)}\frac{x^2}{2}+B_{20}^{(4)}\cos 2mx+B_{02}^{(4)}\cos 2ny\\ &+B_{22}^{(4)}\cos 2mx\cos 2ny+B_{40}^{(4)}\cos 4mx+B_{04}^{(4)}\cos 4ny+B_{24}^{(4)}\cos 2mx\cos 4ny\\ &+B_{42}^{(4)}\cos 4mx\cos 2ny\Big]+O(\varepsilon^5)\end{aligned} \tag{3.16}$$

$$\Psi_x=\varepsilon[C_{11}^{(1)}\cos mx\sin ny]+\varepsilon^3[C_{13}^{(3)}\cos mx\sin 3ny+C_{31}^{(3)}\cos 3mx\sin ny]+O(\varepsilon^5) \tag{3.17}$$

$$\Psi_y=\varepsilon[D_{11}^{(1)}\sin mx\cos ny]+\varepsilon^3[D_{13}^{(3)}\sin mx\cos 3ny+D_{31}^{(3)}\sin 3mx\cos ny]+O(\varepsilon^5) \tag{3.18}$$

注意，对于均匀热分布作用情况，只需在(3.16)式中取$C_4=0$。

在(3.15)～(3.18)式中，各项系数都可表为$A_{11}^{(1)}$的形式，如附录所示。

进一步，将(3.15)和(3.16)式代入边界条件$\delta_x=0$和$\delta_y=0$，并经过摄动参数转换，我们可以导得热后屈曲平衡路径

$$\lambda_T=\lambda_T^{(0)}+\lambda_T^{(2)}W_m^2+\lambda_T^{(4)}W_m^4+\cdots \tag{3.19}$$

式中W_m为无量纲最大挠度，假定取在$(x,\ y)=(\pi/2m,\ \pi/2n)$，且$\lambda_T^{(0)}$，$\lambda_T^{(2)}$和$\lambda_T^{(4)}$的详细表达式在附录中给出。

(3.19)式可用于对称正交铺设层合板在均匀或非均匀热分布作用下的后屈曲载荷—挠度曲线计算，各向同性和正交各向异性板可作为特例处理。由附录可以看出，对于完善板(3.19)式给出精确的热屈曲载荷，取$\mu=0$（或$\overline{W}^*/t=0$），并取$W_m=0$（或$\overline{W}/t=0$）我们容易求得数值结果，其相应的屈曲模态为$(m,\ n)$，分别对应X-和Y-方向的半波数。

四、算例和讨论

作为算例我们计算了完善和非完善对称正交铺设层合板，受均匀或非均匀热分布作用的多种情况，计算结果以无量纲图示给出，其中$\lambda_T^*=12(\alpha_{11}+\nu_{21}\alpha_{22})b^2\lambda_T/\alpha_0\pi^2t^2$。对于所有算例取各层板有相同厚度，板的材料常数（除表1和图1外）取为$E_{11}/E_{22}=13.8957$，$G_{12}/E_{22}=G_{13}/E_{22}=0.4801$，$G_{23}/E_{22}=0.1838$，$\nu_{12}=0.33$，$\alpha_{11}/\alpha_0=0.139$和$\alpha_{22}/\alpha_0=9.0$。

作为比较我们计算了各向同性($\nu=0.3$)和10层0°铺设$(0)_{5s}$层合完善方板（$E_{11}/E_{22}=15$ $G_{12}/E_{22}=G_{13}/E_{22}=0.5$，$G_{23}/E_{22}=0.3356$，$\nu_{12}=0.3$，$\alpha_{11}/\alpha_0=0.015$和$\alpha_{22}/\alpha_0=1.0$）在均匀热分布作用下的热屈曲载荷（$\tilde{\lambda}_T=\alpha_0T_0\times10^3$）。计算结果列在表1中并与Noor和Burton(1992)[4]三维弹性力学计算结果，经典板理论（CPT）计算结果作了比较。显见，本文高阶理论计算结果与三维弹性力学计算结果有相当好的符合，而经典板理论对于中厚板或厚板给出过高的临界温度。另外，图1给出4层$(0/90)_s$对称正交铺设层合方板($E_{11}/E_{22}=25$，$G_{12}/E_{22}=G_{13}/E_{22}=0.5$，$G_{23}/E_{22}=0.2$，$\nu_{12}=0.25$，$\alpha_{22}/\alpha_{11}=10$及$b/t=40$）在均匀热分布作用下的后屈曲载荷—挠度曲线，并与Singh等(1994)[5]有限元计算结果进行比较。可以看出，在$\overline{W}/t\leqslant0.4$的范围内，两者有很好的符合，而在进一步的后屈曲范围，文[5]显示二次分支与模态跳跃。

图2给出对应不同宽厚比$b/t(=10.0,5.0)$ $(0/90)_s$层合方板在均匀或非均匀热分布作用下的后屈曲载荷—挠度曲线，并与经典板理论结果进行比较。计算结果表明，宽厚比$b/t=10.0$的中厚板其热屈曲载荷比经典板理论计算结果低29%。随着宽厚比b/t增加，热屈曲载荷逐渐减小，但在挠度较大的后屈曲范围，厚板比薄板有更高的后屈曲载荷。同时可以看出，层合板在非均匀热分布作用下比在均匀热分布作用下有较高的热屈曲载荷和热后屈曲强度。

图3给出不同长宽比$\beta(=1.0,\ 1.5)$对$(0/90)_s$层合板热后屈曲性态的影响。图4给出不同铺层数$(N=4,\ 8)$对对称正交铺设层合板热后屈曲性态的影响。计算结果表明，热屈曲载荷和热后屈曲强度随着长宽比β的减小而减小，

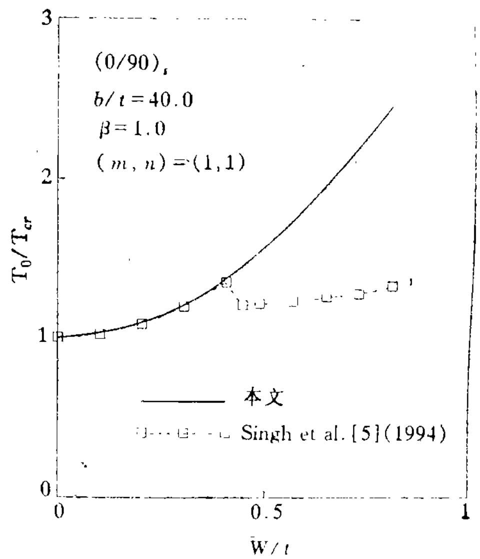


图1 $(0/90)_s$方板在均匀热分布作用下，热后屈曲载荷—挠度曲线比较

或随着铺层数N的增加而增加，且铺层数N的影响相对较小。

图5给出热荷载比T_0/T_1(=0.0，0.25，0.5)对$(0/90)_s$层合板在非均匀热分布作用下后屈曲性态的影响。可以看出，随着热荷载比T_0/T_1的增加，热屈曲载荷逐渐减小，且热荷载比T_0/T_1对后屈曲平衡路径有显著影响。

在图3～5中，板宽厚比皆取为$b/t=10.0$。

除了完善板，图2～5中还给出了非完善板的热后屈曲平衡路径。图示表明，剪切变形层合板与经典层合板一样，其热屈曲和热后屈曲行为对初始几何缺陷表现不敏感。

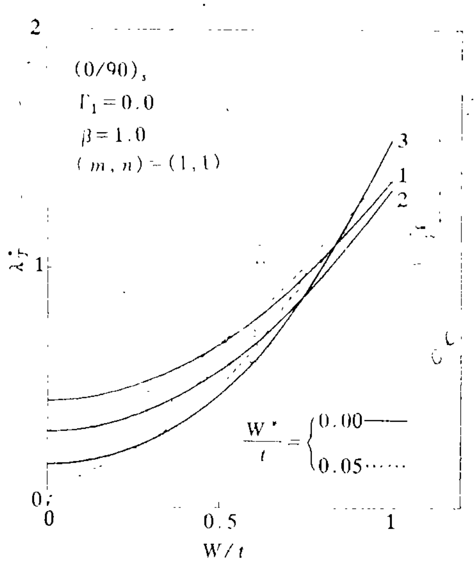
五、结　论

本文将 Reddy 高阶剪切变形板理论发展到用于非完善层合板热后屈曲分析，并给出了

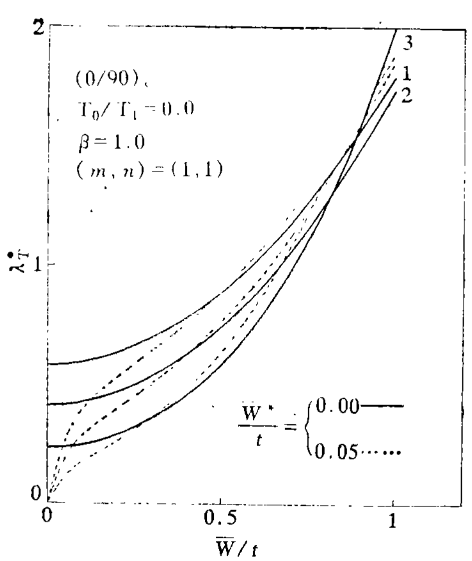
表 1　**完善方板热屈曲载荷($\bar{\lambda}_T=\alpha_0 T_0\times10^3$)比较**

铺　层	b/t	Noor & Burton[4]	本　文	经典板理论[a]
各向同性板	100	0.1264	0.1265	0.1265
	20	3.109	3.1194	3.1633
	10	11.83	11.9782	12.6533
	5	39.90	41.3175	50.6134
$(0)_{5T}$	100	0.7463	0.7466	0.7486
	20	17.39	17.5202	18.7160
	10	57.82	59.1271	74.8639
	5	143.6	149.9049	299.4555

a 按文[2]公式计算。

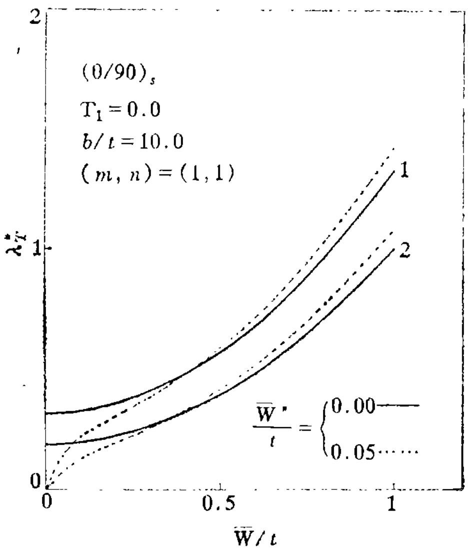


1：经典板理论　2：高阶剪切板理论($b/t=10.0$)
3：高阶剪切板理论($b/t=5.0$)
(a) 均匀热分布



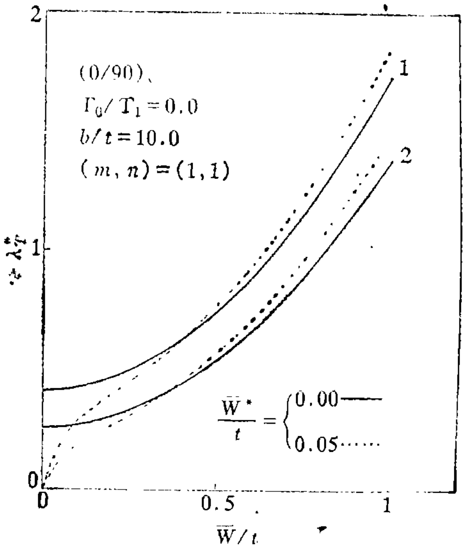
1：经典板理论　2：高阶剪切板理论($b/t=10.0$)
3：高阶剪切板理论($b/t=5.0$)
(b) 非均匀热分布

图2　宽厚比b/t对$(0/90)_s$层合板热后屈曲影响



1: β=1.0; 2: β=1.5

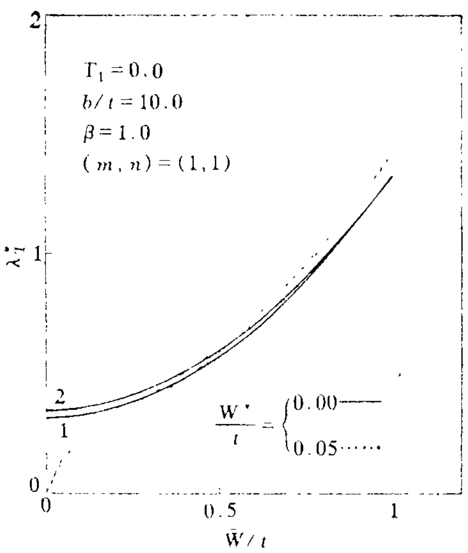
(a) 均匀热分布



1: β=1.0; 2: β=1.5

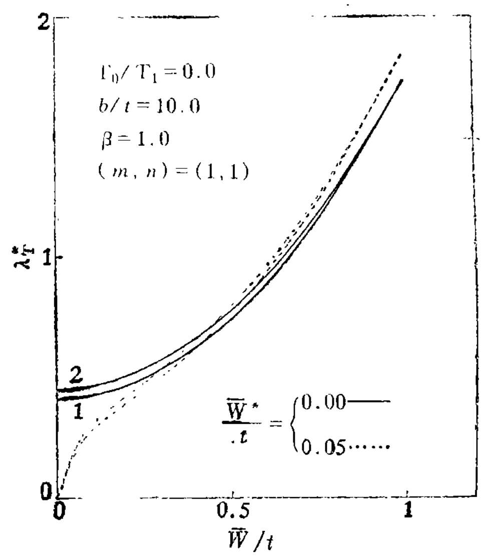
(b) 非均匀热分布

图3 长宽比β对$(0/90)_s$层合板热后屈曲影响



1: (0/90)s N=4 2: (0/90)2s N=8

(a) 均匀热分布

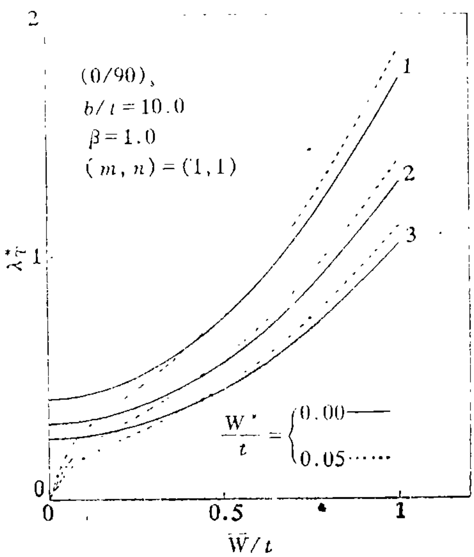


1: (0/90)s N=4 2: (0/90)2s N=8

(b) 非均匀热分布

图4 铺层数N对对称正交铺设层合板热后屈曲影响

完善和非完善对称正交铺设层合板，在均匀或非均匀抛物型热分布作用下的算例．计算结果表明，横向剪切变形，板长宽比，热荷载比以及初始几何缺陷对层合板热后屈曲性态都有显著影响，而铺层数的影响相对较小．本文数值结果虽未涉及层合板热后屈曲的所有可能情况，但作者相信可以用本文导出的公式去研究一些感兴趣的或有价值的参数影响．



1: $T_0/T_1=0.0$;　　2: $T_0/T_1=0.25$;　　3: $T_0/T_1=0.50$

图5　热荷载比T_0/T_1对$(0/90)_s$层合板在非均匀热分布作用下后屈曲影响

附　　录

(3.15)～(3.18)式中，各系数为

$$C_{11}^{(1)}=-m\frac{g_{04}}{g_{00}}A_{11}^{(1)},\quad D_{11}^{(1)}=-n\beta\frac{g_{03}}{g_{00}}A_{11}^{(1)}$$

$$B_{20}^{(2)}=\frac{1}{32}\frac{\gamma_{24}n^2\beta^2}{m^2}(1+2\mu)(A_{11}^{(1)})^2,\quad B_{02}^{(2)}=\frac{1}{32}\frac{m^2}{\gamma_{24}n^2\beta^2}(1+2\mu)(A_{11}^{(1)})^2$$

$$A_{13}^{(3)}=\frac{1}{16}\frac{\gamma_{14}}{\gamma_{24}}\frac{m^4}{J_{13}}(1+\mu)^2(1+2\mu)(A_{11}^{(1)})^3$$

$$A_{31}^{(3)}=\frac{1}{16}\frac{\gamma_{14}}{\gamma_{24}}\frac{n^4\beta^4}{J_{31}}(1+\mu)^2(1+2\mu)(A_{11}^{(1)})^3$$

$$C_{13}^{(3)}=-m\frac{g_{134}}{g_{130}}A_{13}^{(3)},\quad C_{31}^{(3)}=-3m\frac{g_{314}}{g_{310}}A_{31}^{(3)}$$

$$D_{13}^{(3)}=-3n\beta\frac{g_{133}}{g_{130}}A_{13}^{(3)},\quad D_{31}^{(3)}=-n\beta\frac{g_{313}}{g_{310}}A_{31}^{(3)}$$

$$B_{20}^{(4)}=-\frac{1}{16}\frac{\gamma_{24}n^2\beta^2}{m^2}(1+\mu)A_{11}^{(1)}A_{31}^{(3)},\quad B_{02}^{(4)}=-\frac{1}{16}\frac{m^2}{\gamma_{24}n^2\beta^2}(1+\mu)A_{11}^{(1)}A_{13}^{(3)}$$

(3.19)式中

$$(\lambda_T^{(0)},\lambda_T^{(2)},\lambda_T^{(4)})=(S_0,\ S_2,\ S_4)/\gamma_{14}C_{11}$$

其中

$$S_0=\frac{\Theta_{11}}{(1+\mu)},\quad S_2=\frac{1}{16}\frac{\gamma_{14}}{\gamma_{24}}\Theta_2(1+2\mu),\quad S_4=\frac{1}{256}\frac{\gamma_{14}^2}{\gamma_{24}^2}C_{11}(C_{24}-C_{44})+C_{33}$$

$$\Theta_{11}=(\gamma_{110}m^4+2\gamma_{112}m^2n^2\beta^2+\gamma_{114}n^4\beta^4)$$
$$+[m^2(\gamma_{120}m^2+\gamma_{122}n^2\beta^2)g_{04}+n^2\beta^2(\gamma_{131}m^2+\gamma_{133}n^2\beta^2)g_{03}]/g_{00}$$

$$\Theta_2=[(3\gamma_{24}^2-\gamma_5^2)(m^4+\gamma_{24}^2n^4\beta^4)+4\gamma_{24}^2\gamma_5m^2n^2\beta^2]/(\gamma_{24}^2-\gamma_5^2)$$

$$\Theta_{13}=(\gamma_{110}m^4+18\gamma_{112}m^2n^2\beta^2+\gamma_{114}81n^4\beta^4)$$
$$+[m^2(\gamma_{120}m^2+\gamma_{122}9n^2\beta^2)g_{134}+9n^2\beta^2(\gamma_{131}m^2+\gamma_{133}9n^2\beta^2)g_{133}]/g_{130}$$

$$\Theta_{31}=(81\gamma_{110}m^4+18\gamma_{112}m^2n^2\beta^2+\gamma_{114}n^4\beta^4)+[9m^2(\gamma_{120}9m^2+\gamma_{122}n^2\beta^2)g_{314}+n^2\beta^2(\gamma_{131}9m^2+\gamma_{133}n^2\beta^2)g_{313}]/g_{310}$$

$$C_{24}=2(1+\mu)^2(1+2\mu)^2\Theta_2\left[\frac{m^4}{J_{13}}+\frac{\gamma_{24}^2n^4\beta^4}{J_{31}}\right]$$

$$C_{44}=(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]\left[\frac{m^8}{J_{13}}+\frac{\gamma_{24}^4n^8\beta^8}{J_{31}}\right]$$

$$g_{00}=(\gamma_{31}+\gamma_{320}m^2+\gamma_{322}n^2\beta^2)(\gamma_{41}+\gamma_{430}m^2+\gamma_{432}n^2\beta^2)-\gamma_{331}^2m^2n^2\beta^2$$

$$g_{03}=(\gamma_{31}+\gamma_{320}m^2+\gamma_{322}n^2\beta^2)(\gamma_{41}-\gamma_{411}m^2-\gamma_{413}n^2\beta^2)-\gamma_{331}m^2(\gamma_{31}-\gamma_{310}m^2-\gamma_{312}n^2\beta^2)$$

$$g_{04}=(\gamma_{41}+\gamma_{430}m^2+\gamma_{432}n^2\beta^2)(\gamma_{31}-\gamma_{310}m^2-\gamma_{312}n^2\beta^2)-\gamma_{331}n^2\beta^2(\gamma_{41}-\gamma_{411}m^2-\gamma_{413}n^2\beta^2)$$

$$g_{130}=(\gamma_{31}+\gamma_{320}m^2+\gamma_{322}9n^2\beta^2)(\gamma_{41}+\gamma_{430}m^2+\gamma_{432}9n^2\beta^2)-\gamma_{331}^2 9m^2n^2\beta^2$$

$$g_{133}=(\gamma_{31}+\gamma_{320}m^2+\gamma_{322}9n^2\beta^2)(\gamma_{41}-\gamma_{411}m^2-\gamma_{413}9n^2\beta^2)-\gamma_{331}m^2(\gamma_{31}-\gamma_{310}m^2-\gamma_{312}9n^2\beta^2)$$

$$g_{134}=(\gamma_{41}+\gamma_{430}m^2+\gamma_{432}9n^2\beta^2)(\gamma_{31}-\gamma_{310}m^2-\gamma_{312}9n^2\beta^2)-\gamma_{331}9n^2\beta^2(\gamma_{41}-\gamma_{411}m^2-\gamma_{413}9n^2\beta^2)$$

$$g_{310}=(\gamma_{31}+\gamma_{320}9m^2+\gamma_{322}n^2\beta^2)(\gamma_{41}+\gamma_{430}9m^2+\gamma_{432}n^2\beta^2)-\gamma_{331}^2 9m^2n^2\beta^2$$

$$g_{313}=(\gamma_{31}+\gamma_{320}9m^2+\gamma_{322}n^2\beta^2)(\gamma_{41}-\gamma_{411}9m^2-\gamma_{413}n^2\beta^2)-\gamma_{331}9m^2(\gamma_{31}-\gamma_{310}9m^2-\gamma_{312}n^2\beta^2)$$

$$g_{314}=(\gamma_{41}+\gamma_{430}9m^2+\gamma_{432}n^2\beta^2)(\gamma_{31}-\gamma_{310}9m^2-\gamma_{312}n^2\beta^2)-\gamma_{331}n^2\beta^2(\gamma_{41}-\gamma_{411}9m^2-\gamma_{413}n^2\beta^2)$$

$$J_{13}=\Theta_{13}C_{11}(1+\mu)-\Theta_{11}C_{13},\quad J_{31}=\Theta_{31}C_{11}(1+\mu)-\Theta_{11}C_{31}$$

其中，在均匀热分布作用下

$$C_{11}=(\gamma_{T1}m^2+\gamma_{T2}n^2\beta^2),\quad C_{13}=(\gamma_{T1}m^2+9\gamma_{T2}n^2\beta^2)$$

$$C_{31}=(9\gamma_{T1}m^2+\gamma_{T2}n^2\beta^2),\quad C_{33}=0$$

在非均匀热分布作用下

$$C_{11}=(\gamma_{T1}m^2+\gamma_{T2}n^2\beta^2)\left(\frac{T_0}{T_1}+\frac{3}{2}\right)+2\gamma_8\frac{m^2}{\pi^2n^2}$$

$$C_{13}=(\gamma_{T1}m^2+9\gamma_{T2}n^2\beta^2)\left(\frac{T_0}{T_1}+\frac{3}{2}\right)+\frac{2}{9}\gamma_8\frac{m^2}{\pi^2n^2}$$

$$C_{31}=(9\gamma_{T1}m^2+\gamma_{T2}n^2\beta^2)\left(\frac{T_0}{T_1}+\frac{3}{2}\right)+18\gamma_8\frac{m^2}{\pi^2n^2}$$

$$C_{33}=\frac{1}{256}\frac{\gamma_{14}^2}{\gamma_{24}^2}(1+\mu)(1+2\mu)^2\Theta_2\left[\frac{3}{2}\gamma_8\frac{m^2}{\pi^2n^2}\frac{m^4}{J_{13}}\right]$$

$$\gamma_8=(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2})/\gamma_{24}^2$$

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Kármán-Type Equations for a Higher-Order Shear Deformation Plate Theory and Its Use in the Thermal Postbuckling Analysis

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Abstract

Kármán-type nonlinear large deflection equations are derived according to the Reddy's higher-order shear deformation plate theory and are used in the thermal postbuckling analysis. The effects of initial geometric imperfections of the plate are included in the present study which also includes the thermal effects. Simply supported, symmetric cross-ply laminated plates subjected to uniform of nonuniform parabolic temperature distribution are considered. The analysis uses a mixed Galerkin-perturbation technique to determine thermal buckling loads and postbuckling equilibrium paths. The effects played by transverse shear deformation, plate aspect ratio, total number of plies, thermal load ratio and initial geometric imperfections are also studied.

Key words composite laminated plate, higher-order shear deformation plate theory, thermal postbuckling, Galerkin-perturbation technique