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# 获得非线性演化方程 Backlund 变换的一种新的途径*

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摘　要

本文给出一种求非线性演化方程 Backlund 变换的方法, 应用于非线性演化方程时, 得到了与 WTC 方法一致的 Backlund 变换, 避开了 WTC 方法涉及到的递推关系和截尾的讨论•

关键词　非线性演化方程　Backlund 变换　Lax 表示
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## §1. 引　　言

Backlund变换和 Lax 表示在孤立子理论中占有非常重要的地位, 由 Backlund 变换引出的非线性叠加原则将非线性方程的求解问题归结为纯代数运算, 从 Backlund 变换入手又可揭示孤子方程的许多内在特性, 如 Backlund 变换与无穷守恒律, 反散射求解等之间的联系[4, 6]• 而 Lax 表示则将非线性方程写成一对线性问题•

Weiss, Tabor 和 Carnevale[1, 5] 对偏微分方程给出了 Painleve 性质的自然推广(简称 WTC 方法), 他们的方法引出内涵丰富的形式, 由此人们可以导出 Backlund 变换和 Lax 表示等•

WTC 方法是设非线性演化方程

$$u_t = K(u, u_x, \cdots) \tag{1.1}$$

的解 $u(x, t)$ 在流形 $\phi(x, t) = 0$ 上奇异, 在 $\phi = 0$ 的邻域内构造(1.1)的解

$$u(x, t) = \frac{1}{\phi^\alpha}\sum_{j=0}^{\infty} u_j(x, t)\phi^j \tag{1.2}$$

其中, α 为正整数, $\phi(x, t)$, $u_j(x, t)$ 为解析函数, 将(1.2)代入(1.1), 对 $j = 0, 1, 2, \cdots$确定 α 和 $u_j$ 的第推关系, 通过截尾可得 Backlund 变换•

本文方法的思想是: 设(1.1)的解具有如下形式

$$u = \frac{\partial^\alpha}{\partial x^\alpha} f(\phi) + u_1 \tag{1.3}$$

其中, $u_1$ 为(1.1)的解, $f$, $\phi$ 为待定函数, α 为正整数•

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1. 通过比较使 $\phi_x$ 出现在有关 $u$ 的最高导数项中与出现在有关 $u$ 的非线性项中的最高次幂相同, 可确定出 α•

2. 将(1. 3) 代入(1. 1), 令 $\phi_x$ 最高次幂的系数为 0, 可确定 $f(\phi)$•

3. 令 $f$ 的各阶导数的系数为 0, 可得到 $\phi$ 应满足的相容条件•

经过上述步骤即可确定 Backlund 变换, 这种方法非常直观、自然, 绕开 Painleve 质性进行递推关系和截尾讨论• 下面通过应用于几个著名的方程加以说明•

## §2. Burgers 方程

对 Burgers 方程

$$u_t + uu_x - \sigma u_{xx} = 0 \tag{2.1}$$

设其解具有如下形式

$$u = \frac{\partial^\alpha}{\partial x^\alpha} f(\phi) + u_1(x, t)$$

比较使 $\phi_x$ 出现在 $uu_x$ 与 $-\sigma u_{xx}$ 中的最高次幂相同, 可得 $\alpha = 1$, 故

$$u = f(\phi)\phi_x + u_1(x, t) \tag{2.2}$$

将(2. 2) 代入(2. 1), 整理可得

$$u_t + uu_x - \sigma u_{xx} = (f'f'' - \sigma f''')\phi_x^3 + (f''\phi_t\phi_x + f'^2\phi_x\phi_{xx} + u_1 f''\phi_x^2 - 3\sigma f''\phi_x\phi_{xx}) + (\phi_{xx} + u_{1x}\phi_x + u_1\phi_{xx} - \sigma\phi_{xxx})f' + (u_{1t} + u_1u_{1x} - \sigma u_{1xx}) = 0 \tag{2.3}$$

令

$$f'f'' - \sigma f''' = 0 \tag{2.4}$$

解之得

$$f = -2\sigma\ln\phi \tag{2.5}$$

从而

$$f'^2 = 2\sigma f'' \tag{2.6}$$

将(2. 6) 代入(2. 3) 并利用(2. 4), 得

$$u_t + uu_x - \sigma u_{xx} = (\phi_t\phi_x + u_1\phi_x^2 - \sigma\phi_x\phi_{xx})f'' + \frac{\partial}{\partial x}(\phi_t + u_1\phi_x - \sigma\phi_{xx})f' + u_{1t} + u_1u_{1x} - \sigma u_{1xx} = 0$$

令 $f''$, $f'$ 的系数及 $u_1$ 的函数组合项为 0, 得

$$\begin{cases} \phi_x(\phi_t + u_1\phi_x - \sigma\phi_{xx}) = 0 & (2.7) \\ \dfrac{\partial}{\partial x}(\phi_t + u_1\phi_x - \sigma\phi_{xx}) = 0 & (2.8) \\ u_{1t} + u_1u_{1x} - \sigma u_{1xx} = 0 & (2.9) \end{cases}$$

由于 $\phi_x \neq 0$, 否则由(2. 2) 知 $u = u_1$, 并非期望的变换, 故应有

$$\phi_t + u_1\phi_x - \sigma\phi_{xx} = 0$$

从而(2. 8) 满足, 于是将(2. 5) 代入(2. 2), 得到 Backlund 变换

$$u = -2\sigma\frac{\phi_x}{\phi} + u_1 \tag{2.10}$$

其中, $\phi$ 满足(2. 10), $u_1$ 满足(2. 9)•

当 $u_1 = 0$ 时, 得到 Cole– Hopf 变换 $u = -2\sigma\dfrac{\phi_x}{\phi}$.

当 $u_1 = \phi$ 时, 得到自 Backlund 变换 $u = -2\sigma\dfrac{\phi_x}{\phi} + \phi$.

## § 3. KdV 方程

对 KdV 方程

$$u_t + uu_x + \sigma u_{xxx} = 0 \tag{3.1}$$

设其解具有如下形式

$$u = f''\phi_x^2 + f'\phi_{xx} + u_1 \tag{3.2}$$

将(3.2)代入(3.1)并整理, 可得

$$\begin{aligned} u_t + uu_x + \sigma u_{xxx} = {} & (f''f''' + \sigma f^{(5)})\phi_x^5 + (3f''^2\phi_x^3\phi_{xx} + f'f'''\phi_x^3\phi_{xx} + 10f^{(4)}\phi_x^3\phi_{xx}) \\ & + (f'''\phi_x^2\phi_t + f'f''\phi_x^2\phi_{xxx} + 3f'f''\phi_x\phi_{xx}^2 + f'''u_1\phi_x^3 + 15\sigma f'''\phi_x\phi_{xx}^2 + 10\sigma f'''\phi_x^2\phi_{xxx}) \\ & + (2f''\phi_x\phi_{xt} + f''\phi_t\phi_{xx} + f''u_{1x}\phi_x^2 + f'^2\phi_{xx}\phi_{xxx} + 3f''u_1\phi_x\phi_{xx} + 10\sigma f''\phi_{xx}\phi_{xxx}) \\ & 5\sigma f''\phi_x\phi_{xxxx}) + (\phi_{xxt} + u_{1x}\phi_{xx} + u_1\phi_{xxx} + \phi_{xxxxx})f' + (u_{1t} + u_1u_{1x} + \sigma u_{1xxx}) = \\ 0 & \end{aligned} \tag{3.3}$$

令

$$f''f''' + \sigma f^{(5)} = 0 \tag{3.4}$$

解之得

$$f = 12\sigma\ln\phi \tag{3.5}$$

从而

$$\begin{aligned} & f''^2 = -2\sigma f^{(4)}, \quad f'f''' = -4\sigma f^{(4)} \\ & f'f'' = -6\sigma f''', \quad f'^2 = -12\sigma f'' \end{aligned} \tag{3.6}$$

将(3.6)代入(3.3)并利用(3.4), 得

$$\begin{aligned} u_t + uu_x + \sigma u_{xxx} = {} & (\phi_x^2\phi_t + 4\sigma\phi_x^2\phi_{xxx} - 3\sigma\phi_x\phi_{xx}^2 + u_1\phi_x^3)f''' \\ & (2\phi_x\phi_{xt} + \phi_t\phi_{xx} + u_{1x}\phi_x^2 - 12\sigma\phi_{xx}\phi_{xxx} + 3u_1\phi_x\phi_{xx} + 10\sigma\phi_{xx}\phi_{xxx} \\ & + 5\sigma\phi_x\phi_{xxx})f'' + (\phi_{xxt} + u_{1x}\phi_{xx} + u_1\phi_{xxx} + \sigma\phi_{xxxx})f' \\ & + (u_{1t} + u_1u_{1x} + \sigma u_{1xxx}) = 0 \end{aligned} \tag{3.7}$$

令 $f'''$, $f''$, $f'$ 的系数及最后括号为 0, 得到

$$\begin{cases} \phi_x(\phi_x\phi_t + 4\sigma\phi_x\phi_{xxx} - 3\sigma\phi_{xx}^2 + u_1\phi_x^2) = 0 \\ \phi_x(\phi_{xt} + u_1\phi_{xx} + \sigma\phi_{xxxx}) + \dfrac{\partial}{\partial x}(\phi_x\phi_t + 4\sigma\phi_x\phi_{xxx} - 3\sigma\phi_{xx}^2 + u_1\phi_x^2) = 0 \\ \dfrac{\partial}{\partial x}(\phi_{xt} + u_1\phi_{xx} + \sigma\phi_{xxxx}) = 0 \\ u_{1t} + u_1u_{1x} + \sigma u_{1xxx} = 0 \end{cases}$$

欲使上述条件成立, 只须

$$\begin{cases} \phi_x\phi_t + 4\sigma\phi_x\phi_{xxx} - 3\sigma\phi_{xx}^2 + u_1\phi_x^2 = 0 & (3.8) \\ \phi_{xt} + u_1\phi_{xx} + \sigma\phi_{xxx} = 0 & (3.9) \\ u_{1t} + u_1u_{1x} + \sigma u_{1xxx} = 0 & (3.10) \end{cases}$$

使得

将(3.5)代入(3.2)，得 Backlund 变换

$$u = 12\sigma\frac{\partial^2}{\partial x^2}\ln\phi + u_1$$

其中，$\phi$, $u_1$ 满足(3.8)～(3.10)•

从(3.8)解出 $\phi_t$，再对 $x$ 微分，利用(3.9)并令 $\phi_x = v^2$，可得 Lax 表示

$$\begin{cases} 6\sigma v_{xx} + u_1 v = \lambda v \\ 2v_t + u_1 v \text{整理} + \lambda v_x + 2\sigma v_{xxx} = 0 \end{cases}$$

## §4. Boussinesq 方程组

对 Boussinesq 方程组[7]

$$\begin{cases} u_t + (uv)_x + v_{xxx} = 0 \\ v_t + u_x + vv_x = 0 \end{cases} \tag{4.1}$$

设

$$\begin{cases} u = f''\phi_x^2 + f'\phi_{xx} + u_1 \\ v = f'\phi_x + v_1 \end{cases} \tag{4.2}$$

将(4.2)代入(4.1)，经整理可得

$$\begin{aligned} u_t + (uv)_x + v_{xxx} = {} & (f'f''' + f''^2 + f^{(4)})\phi_x^4 + (f'''\phi_t\phi_x^2 + 5f'f''\phi_x^2\phi_{xx} + f'''v_1\phi_x^3 \\ & + 6f'''\phi_x^2\phi_{xx}) + (2f''\phi_x\phi_{xt} + f''\phi_t\phi_{xx} + f'^2\phi_x\phi_{xxx} + 3f''v_1\phi_x\phi_{xx} + f''v_{1x}\phi_x^2 + f'^2\phi_{xx}^2 \\ & + f''u_1\phi_x^2 + 4f''\phi_x\phi_{xxx}) + (\phi_{xxt} + u_{1x}\phi_x + u_1\phi_{xx} + v_1\phi_{xxx} + v_{1x}\phi_{xx} + \phi_{xxx})f' + (u_{1t} \\ & + (u_1v_1)_x + v_{1xxx}) = 0 \end{aligned} \tag{4.3}$$

$$\begin{aligned} v_t + u_x + vv_x = {} & (f''' + f'f'')\phi_x^3 + (f''\phi_t\phi_x + 3f''\phi_x\phi_{xx} + f'^2\phi_x\phi_{xx} + f''v_1\phi_x^2) \\ & + (\phi_{xt} + v_{1x}\phi_x + v_1\phi_{xx} + \phi_{xxx})f' + (v_{1t} + u_{1x} + v_1v_{1x}) = 0 \end{aligned} \tag{4.4}$$

令(4.3)中 $\phi_x^4$ 的系数为0，(4.4)中 $\phi_x^3$ 的系数为0，可得如下方程组

$$\begin{cases} f^{(4)} + f'f''' + f'' = 0 \\ f''' + f'f'' = 0 \end{cases} \tag{4.5}$$

解之得

$$f = 2\ln\phi \tag{4.6}$$

从而

$$f'f'' = -f''', \quad f'^2 = -2f'' \tag{4.7}$$

将(4.7)代入(4.3)和(4.4)并利用(4.5)，整理后再令 $f'''$, $f''$, $f'$ 及最后函数组合项为0，可得

$$\begin{cases} \phi_x^2(\phi_t + v_1\phi_x + \phi_{xx}) = 0 \\ \phi_{xx}(\phi_t + v_1\phi_x + \phi_{xx}) + \phi_x\left[\frac{\partial}{\partial x}(\phi_t + v_1\phi_x + \phi_{xx}) + (\phi_{xt} + v_1\phi_{xx} + u_1\phi_x + \phi_{xxx})\right] = 0 \\ \frac{\partial}{\partial x}(\phi_{xt} + v_1\phi_{xx} + u_1\phi_x + \phi_{xxx}) = 0 \\ u_{1t} + (u_1v_1)_x + v_{1xxx} = 0 \end{cases}$$

$$\begin{cases} \phi_x(\phi_t + v_1\phi_x + \phi_{xx}) = 0 \\ \dfrac{\partial}{\partial x}(\phi_t + v_1\phi_x + \phi_{xx}) = 0 \\ v_{1t} + u_{1x} + v_1 v_{1x} = 0 \end{cases}$$

易见上述方程组的适定条件为

$$\begin{cases} \phi_t + v_1\phi_x + \phi_{xx} = 0 \\ \phi_{xt} + v_1\phi_{xx} + u_1\phi_x + \phi_{xxx} = 0 \\ v_{1t} + (u_1 v_1)_x + v_{1xxx} = 0 \\ u_{1t} + u_{1x} + v_1 v_{1x} = 0 \end{cases} \tag{4.8}$$

将(4.6)代入(4.2),得 Backlund 变换

$$\begin{cases} u = 2\dfrac{\partial^2}{\partial x^2}\ln\phi + u_1 \\ v = 2\dfrac{\partial}{\partial x}\ln\phi + v_1 \end{cases} \tag{4.9}$$

其中,$\phi$, $u_1$, $v_1$ 满足(4.8)

1) 令 $u_1 = v_1 = 0$, (4.9)变为

$$\begin{cases} u = 2\dfrac{\partial^2}{\partial x^2}\ln\phi \\ v = 2\dfrac{\partial}{\partial x}\ln\phi \end{cases} \tag{4.10}$$

(4.10)变为

$$\phi_t + \phi_{xx} = 0 \tag{4.11}$$

这说明在变换(4.10)下, Boussinesq 方程组可化为线性方程(4.11)

2) 令 $v_1 = \phi$, $u_1 = \phi_x$, 由(4.9)可得 Backlund 变换

$$\begin{cases} u = 2\dfrac{\partial^2}{\partial x^2}\ln v_1 + u_1 \\ v = 2\dfrac{\partial}{\partial x}\ln v_1 + v_1 \end{cases}$$

其中,$u_1$, $v_1$ 满足(4.1)

使用本文方法,还可求许多著名方程的 Backlund 变换,如 KP 方程, Sine- Gordon 方程, Caudrey- Dodd- Gibbon 方程等•

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# A New Approach to Backlund Transformations of Nonlinear Evolution Equations

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**Abstract**

In this paper, a new approach to Backlund transformations of nonlinear evolution equations is presented. The results obtained by this procedure are completely the same as that by Painleve truncating expansion.

**Key words** nonlinear evolution equation, Backlund transformation, Lax pair