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Whitham_Broer_Kaup 浅水波方程的 Backlund 变换和精确解*

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摘　　要

用一种新的方法并借助 Mathematica, 求出了 Whitham_Broer_Kaup(简记 WBK)方程的一种 Backlund 变换, 并建立了 WBK 方程与热传导方程及 Burgers 方程的联系• 利用这种关系得到了 WBK 方程的三组精确解, 其中一组为孤波解•

关键词　WBK 方程　Backlund 变换　精确解　孤波解
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§1. 引　　言

为描述浅水波运动, 人们得到了许多完全可积模型, 如 KdV 方程, Boussinesq 方程, K_P 方程, WBK 方程等, 在 Boussinesq 近似下, Whitham, Broer 和 Kaup[1,2,3] 得到了非线性 WBK 方程

$$\begin{cases} u_t + uu_x + v_x + \beta u_{xx} = 0, & (1.1) \\ v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0 & (1.2) \end{cases}$$

其中 $u = u(x, t)$ 为水平速度场; $v = v(x, t)$ 为偏离液面平衡位置的高度; α, β 为表征不同色散程度的常数• 这个方程可以很好地描述色散波• 当 $\alpha = 0$, $\beta \neq 0$ 时, 方程(1.1)、(1.2)正是描述浅水波无色散的经典长波方程[4]; 当 $\alpha = 1$, $\beta = 0$ 时, 方程(1.1)、(1.2) 简化为修正 Boussinesq 方程[5]• Kaup[3], Ablowitz[5] 研究了方程(1.1)、(1.2) 特殊情形的反散射变换解, Kapershmidt[4] 讨论了该方程的对称和守恒律• 本文从另一角度出发, 采用一种新的方法并借助符号运算, 导出了 WBK 方程的一种 Backlund 变换, 同时发现 WBK 方程与热传导方程及 Burgers 方程的密切联系, 由这种关系得到了 WBK 方程的三组精确解, 其中一组为孤波解• 这种方法具有一般性和算法性, 可应用于众多的非线性演化方程, 如 KdV 方程, MKdV 方程, K_P 方程, Burgers 方程, Caudrey_Dodd_Gibbon 方程等•

§2. WBK 方程的 Backlund 变换和精确解

设方程(1.1)、(1.2) 具有如下形式的解:

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$$
\begin{cases}
u = f'(w)w_x + u_1(x,t) & (2.1)\\
v = g''(w)w_x^2 + g'(w)w_{xx} + v_1(x,t) & (2.2)
\end{cases}
$$

其中 f, g, w 为待定函数, u_1, v_1 为(1.1)、(1.2)的某一组解。

将(2.1)、(2.2)代入方程(1.1)、(1.2),使用 Mathematica,经整理可得到

$$
\begin{aligned}
u_t + uu_x + v_x + \beta u_{xx} = {} & (f'f'' + \beta f''' + g''')w_x^3 + (f''w_xw_t + f'^2w_xw_{xx} \\
& + f''u_1w_x^2 + 3g''w_xw_{xx} + 3\beta f''w_xw_{xx}) + (w_{xt}f' + u_{1x}w_xf' \\
& + u_1w_{xx}f' + w_{xxx}g' + \beta w_{xxx}f') + (u_{1t} + u_1u_{1x} + v_{1x} + \beta u_{1xx}) = 0 \quad (2.3)
\end{aligned}
$$

$$
\begin{aligned}
v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = {} & (f''g'' + f'g''' + \alpha f^{(4)} - \beta g^{(4)})w_x^4 + (g'''w_tw_x^2 \\
& + f''g'w_{xx}w_x^2 + 4f'g''w_{xx}w_x^2 + g'''u_1w_x^3 + 6\alpha f'''w_{xx}w_x^2 - 6\beta g'''w_{xx}w_x^2) \\
& + (2g''w_xw_{xt} + g''w_tw_{xx} + f''v_1w_x^2 + f'g'w_{xx}^2 + g''u_{1x}w_x^2 + f'g'w_xw_{xxx} \\
& + 3g''u_1w_xw_{xx} + 3\alpha f''w_{xx}^2 + 4\alpha f''w_xw_{xxx} - 3\beta g''w_{xx}^2 - 4\beta g''w_xw_{xxx}) \\
& + (g'w_{xxt} + f'v_1w_{xx} + g'u_1w_{xxx} + f'v_{1x}w_x + \alpha f'w_{xxxx} - \beta g'w_{xxxx}) \\
& + (v_{1t} + (u_1v_1)_x + \alpha u_{1xxx} - \beta v_{1xx}) = 0 \quad (2.4)
\end{aligned}
$$

令(2.3)中 w_x^3 的系数,(2.4)中 w_x^4 的系数分别为0,得到如下常微分方程组

$$
\begin{cases}
f'f'' + \beta f''' + g''' = 0 & (2.5)\\
f''g'' + f'g''' + \alpha f^{(4)} - \beta g^{(4)} = 0 & (2.6)
\end{cases}
$$

当 $\alpha + \beta^2 > 0$,解之,可得到一组特解

$$
\begin{cases}
f = -2\sqrt{\alpha + \beta^2}\ln w & (2.7)\\
g = 2\sqrt{\alpha + \beta^2}(\beta + \sqrt{\alpha + \beta^2})\ln w & (2.8)
\end{cases}
$$

令 $\gamma = \beta + \sqrt{\alpha + \beta^2}$,则(2.7)、(2.8)可表示为

$$
\begin{cases}
f = -2(\gamma - \beta)\ln w\\
g = 2\gamma(\gamma - \beta)\ln w = -\gamma f
\end{cases}
$$

从而

$$
\begin{aligned}
& f'^2 = 2(\gamma - \beta)f'', \quad g' = -\gamma f', \quad g'' = -\gamma f'', \quad g''' = -\gamma f''' \\
& f'g' = -2\gamma(\gamma - \beta)f'', \quad f'g'' = f''g' = -\gamma(\gamma - \beta)f'''
\end{aligned}
$$

将以上各式代入(2.3)、(2.4),并利用(2.5)、(2.6)可得

$$
\begin{aligned}
u_t + uu_x + v_x + \beta u_{xx} = {} & [w_tw_x - (\gamma - \beta)w_xw_{xx} + u_1w_x^2]f'' + [w_{xt} + u_{1x}w_x \\
& + u_1w_{xx} - (\gamma - \beta)w_{xxx}]f' + (u_{1t} + u_1u_{1x} + v_{1x} + \beta u_{1xx}) = 0
\end{aligned}
$$

$$
\begin{aligned}
v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = {} & (-\gamma w_tw_x^2 - \gamma(\gamma - \beta)w_x^2w_{xx} - 4\gamma(\gamma - \beta)w_x^2w_{xx} \\
& - \gamma u_1w_x^3 + 6\alpha w_x^2w_{xx} + 6\gamma\beta w_x^2w_{xx})f''' + (-2\gamma w_xw_{xt} - \gamma w_tw_{xx} + v_1w_x^2 \\
& - 2\gamma(\gamma - \beta)w_{xx}^2 - \gamma u_{1x}w_x^2 - 2\gamma(\gamma - \beta)w_xw_{xxx} - 3\gamma u_1w_xw_{xx} + 3\alpha w_{xx}^2 \\
& + 4\alpha w_xw_{xxx} + 3\gamma w_{xx}^2 + 4\beta\gamma w_xw_{xxx})f'' + (-\gamma w_{xxt} + v_1w_{xx} - \gamma u_{1x}w_{xx} + v_{1x}w_x \\
& - \gamma u_1w_{xxx} + \alpha w_{xxxx} + \beta\gamma w_{xxxx})f' + (v_{1t} + (u_1v_1)_x + \alpha u_{1xxx} - \beta v_{1xx}) = 0
\end{aligned}
$$

分别令 f''', f'', f', f^0 的各系数为0,得

$$\begin{cases} w_x[w_t-(\gamma-\beta)w_{xx}+u_1w_x]=0 \\ (\partial/\partial x)[w_t-(\gamma-\beta)w_{xx}+u_1w_x]=0 \\ -\gamma w_x^2[w_t-(\gamma-\beta)w_{xx}+u_1w_x]=0 \\ -\gamma(2w_x\partial/\partial x+w_{xx})[w_t-(\gamma-\beta)w_{xx}+u_1w_x]-w_x^2(\gamma u_{1x}+v_1)=0 \\ -\gamma(\partial^2/\partial x^2)[w_t-(\gamma-\beta)w_{xx}+u_1w_x]-(\partial/\partial x)[w_x(\gamma u_{1x}+v_1)]=0 \\ u_{1t}+u_1u_{1x}+v_{1x}+\beta u_{1xx}=0 \\ v_{1t}+(u_1v_1)_x+\alpha u_{1xxx}-\beta v_{1xx}=0 \end{cases}$$

上述各式成立, 只须 u_1, w 满足

$$\begin{cases} w_t-\sqrt{\alpha+\beta^2}w_{xx}+u_1w_x=0 & (2.9) \\ (\beta+\sqrt{\alpha+\beta^2})u_{1x}+v_1=0 & (2.10) \\ u_{1t}+u_1u_{1x}+v_{1x}+\beta u_{1xx}=0 & (2.11) \\ v_{1t}+(u_1v_1)_x+\alpha u_{1xxx}-\beta v_{1xx}=0 \quad \mathrm{w} & (2.12) \end{cases}$$

于是将(2.7)、(2.8)代入(2.1)、(2.2), 得到 WBK 方程(1.1)、(1.2)的一种 Backlund 变换

$$\begin{cases} u=-2\sqrt{\alpha+\beta^2}\partial\ln w/\partial x+u_1 & (2.13) \\ v=2\sqrt{\alpha+\beta^2}(\beta+\sqrt{\alpha+\beta^2})\partial^2\ln w/\partial x^2+v_1 & (2.14) \end{cases}$$

其中 u_1, v_1, w 满足(2.9)~(2.12)•

i) 令 $u_1=v_1=0$, 由(2.9)~(2.14)可得 WBK 方程与热传导方程

$$w_t-\sqrt{\alpha+\beta^2}w_{xx}=0 \tag{2.15}$$

之间的变换

$$\begin{cases} u=-2\sqrt{\alpha+\beta^2}\partial\ln w/\partial x & (2.16) \\ v=2\sqrt{\alpha+\beta^2}(\beta+\sqrt{\alpha+\beta^2})\partial^2\ln w/\partial x^2 & (2.17) \end{cases}$$

ii) 令 $u_1=w$, $v_1=-(\beta+\sqrt{\alpha+\beta^2})w_x$, 由(2.9)~(2.14)可得 WBK 方程与 Burgers 方程

$$w_t+ww_x-\sqrt{\alpha+\beta^2}w_{xx}=0 \tag{2.18}$$

之间的变换

$$\begin{cases} u=-2\sqrt{\alpha+\beta^2}\partial\ln w/\partial x+w \\ v=2\sqrt{\alpha+\beta^2}(\beta+\sqrt{\alpha+\beta^2})\partial^2\ln w/\partial x^2-(\beta+\sqrt{\alpha+\beta^2})w_x \end{cases}$$

实际上, 由 Burgers 方程(2.18)与热传导方程(2.15)间著名的 Hopf_Cole 变换[7], 可得 WBK 方程与 Burgers 方程间更简单的一种变换关系

$$\begin{cases} u=w \\ v=-(\beta+\sqrt{\alpha+\beta^2})w_x \end{cases}$$

iii) 由热传导方程(2.15)及变换(2.16)、(2.17)构造 WBK 方程的精确解• 可以验证

$$w=1+A\exp(-\lambda^2\sqrt{\alpha+\beta^2}t)\sin\lambda x,\quad w=1+A\exp(-\lambda^2\sqrt{\alpha+\beta^2}t)\cos\lambda x$$

$$w=1+\exp(kx+\sqrt{\alpha+\beta^2}k^2t+c)$$

均为热传导方程(2.15)的解, 将其分别代入(2.16)、(2.17)便得到 WBK 方程的三组精确解

$$\begin{cases} u = -2\lambda A\sqrt{\alpha+\beta^2}\,\dfrac{\exp(-\lambda^2\sqrt{\alpha+\beta^2}t)\cos\lambda x}{1+A\exp(-\lambda^2\sqrt{\alpha+\beta^2}t)\sin\lambda x} \\ u = -2\lambda^2 A\sqrt{\alpha+\beta^2}(\beta+\sqrt{\alpha+\beta^2})\,\dfrac{\sin\lambda x + A\exp(-2\lambda^2\sqrt{\alpha+\beta^2}t)}{(1+A\exp(-\lambda^2\sqrt{\alpha+\beta^2}t)\sin\lambda x)^2} \end{cases}$$

$$\begin{cases} u = -2\lambda A\sqrt{\alpha+\beta^2}\,\dfrac{\exp(-\lambda^2\sqrt{\alpha+\beta^2}t)\sin\lambda x}{1+A\exp(-\lambda^2\sqrt{\alpha+\beta^2}t)\cos\lambda x} \\ u = -2\lambda^2 A\sqrt{\alpha+\beta^2}(\beta+\sqrt{\alpha+\beta^2})\,\dfrac{\cos\lambda x + A\exp(-\lambda^2\sqrt{\alpha+\beta^2}t)}{(1+A\exp(-\lambda^2\sqrt{\alpha+\beta^2}t)\cos\lambda x)^2} \end{cases}$$

和一组孤波解

$$\begin{cases} u = -k\sqrt{\alpha+\beta^2}\tanh\dfrac{k}{2}(x+\sqrt{\alpha+\beta^2}kt+c) - k\sqrt{\alpha+\beta^2} \\ u = -\dfrac{k^2}{2}\sqrt{\alpha+\beta^2}(\beta+\sqrt{\alpha+\beta^2})\operatorname{sech}^2\dfrac{k}{2}(x+\sqrt{\alpha+\beta^2}kt+c) \end{cases}$$

其中可 k, c, A, λ 为任意常数•

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Backlund Transformation and Exact Solutions for Whitham_Broer_Kaup Equations in Shallow Water

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Abstract

By using a new method and Mathematica, the Backlund transformations for Whitham_Broer_Kaup equations(WBK) are derived. The connections between WBK equation, heat equation and Burgers equation are found, which are used to obtain three families of solutions for WBK equations, one of which is the family of solitary wave solutions.

Key words WBK equation, Backlund transformation, exact solution, solitary wave solution