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高阶多元 Euler 多项式和高阶多元 Bernoulli 多项式

刘国栋[①]

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摘　　要

本文给出了高阶多元 Euler 数和多项式与高阶多元 Bernoulli 数和多项式的定义, 讨论了它们的一些重要性质, 得到了高阶多元 Euler 多项式(数) 和高阶多元 Bernoulli 多项式(数) 的关系式.

关键词　高阶多元 Euler 数　高阶多元 Euler 多项式　高阶多元 Bernoulli 数　高阶多元 Bernoulli 多项式

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§ 1. 前　　言

m 阶 Euler 数和多项式, m 阶 Bernoulli 数和多项式是两类特殊函数, 它们在函数论和理论物理学中占有重要的地位, 有着广泛的应用. 一直以来人们对 Euler 数和多项式、Bernoulli 数和多项式的研究多限于一元高阶的情形. 我们在[1], [2], [3] 的基础上把 Euler 数和多项式, Bernoulli 数和多项式推广到高阶多元, 给出了 m 阶 n 元 Euler 数和多项式, m 阶 n 元 Bernoulli 数和多项式的定义, 并作了深入的研究, 得到了重要的结果. 这些结果包括高阶多元 Euler 数和多项式、高阶多元 Bernoulli 数和多项式的性质及相互关系, 它们是[1], [2], [3] 中相应问题的结果的推广和深化.

§ 2. 定 义 和 引 理

定义 1　m 阶 n 元 Euler 数 $E^{(m)}_{v_1\cdots v_n}$ 由下列展开式给出:

$$\left(\left\{2\exp\left[\sum_{i=1}^{n}t_i\right]\right\}\Big/\left\{\exp\left[2\sum_{i=1}^{n}t_i\right]+1\right\}\right)^m=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E^{(m)}_{v_1\cdots v_n}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

其中, n 是正整数, m 是整数.

定义 2　m 阶 n 元 Euler 多项式 $E^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)$ 由下列展开式给出:

$$\frac{2^m\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m}=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

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① 惠州大学数学系, 广东惠州 516015

其中, n 是正整数, m 是整数•

定义 3 m 阶 n 元 Bernoulli 数 $B^{(m)}_{v_1\cdots v_n}$ 由下列展开式给出:

$$\left(\left\{\sum_{i=1}^{n} t_i\right\}\Big/\left\{\exp\left[\sum_{i=1}^{n} t_i\right]-1\right\}\right)^m=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}B^{(m)}_{v_1\cdots v_n}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

其中, n 是正整数, m 是整数•

定义 4 m 阶 n 元 Bernoulli 多项式 $B^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)$ 由下列展开式给出:

$$\frac{\left(\sum_{i=1}^{n} t_i\right)^m\exp\left[\sum_{i=1}^{n} x_i t_i\right]}{\left(\exp\left[\sum_{i=1}^{n} t_i\right]-1\right)^m}=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}B^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

其中, n 是正整数, m 是整数•

引理

$$\left(\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}f(v_1,v_2,\cdots,v_n)\frac{t_1^{v_1}}{v_1!}\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\left(\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}g(v_1,v_2,\cdots,v_n)\frac{t_1^{v_1}}{v_1!}\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

$$=\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_1=0}^{v_1}\sum_{k_2=0}^{v_2}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}C_{v_2}^{k_2}\cdots C_{v_n}^{k_n}f(k_1,k_2,\cdots,k_n)\right.$$

$$\left.\cdot\, g(v_1-k_1,v_2-k_2,\cdots,v_n-k_n)\frac{t_1^{v_1}}{v_1!}\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

其中, $C_{v_i}^{k_i}=v_i!/\left(k_i!(v_i-k_i)!\right)\quad(i=1,2,\cdots,n)$

证明 (应用数学归纳法)

(1) 当 $n=1$ 时, 结论成立明显•

(2) 假设对一切自然数 $n-1$, 结论都成立• 令

$$\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}f(v_1,v_2,\cdots,v_n)\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}=p(v_1)$$

$$\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}g(v_1,v_2,\cdots,v_n)\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}=q(v_1)$$

则由假设, 有

$$p(k_1)q(v_1-k_1)=\left(\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}f(k_1,v_2,\cdots,v_n)\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

$$\cdot\left(\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}g(v_1-k_1,v_2,\cdots,v_n)\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

$$=\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_2=0}^{v_2}\cdots\sum_{k_n=0}^{v_n}C_{v_2}^{k_2}\cdots C_{v_n}^{k_n}f(k_1,k_2,\cdots,k_n)\right.$$

$$\left.\cdot\, g(v_1-k_1,v_2-k_2,\cdots,v_n-k_n)\right)t_2^{v_2}/v_2!\cdots t_n^{v_n}/v_n!$$

所以,

$$\left(\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}f(v_1,v_2,\cdots,v_n)\frac{t_1^{v_1}}{v_1!}\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\left(\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}g(v_1,v_2,\cdots,v_n)\frac{t_1^{v_1}}{v_1!}\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

$$= \left(\sum_{v_1=0}^{\infty} p(v_1)\frac{t_1^{v_1}}{v_1!}\right)\left(\sum_{v_1=0}^{\infty} q(v_1)\frac{t_1^{v_1}}{v_1!}\right) = \sum_{v_1=0}^{\infty}\left(\sum_{k_1=0}^{v_1} C_{v_1}^{k_1} p(k_1)\, q(v_1-k_1)\right)\frac{t_1^{v_1}}{v_1!}$$

$$= \sum_{v_1=0}^{\infty}\left(\sum_{k_1=0}^{v_1} C_{v_1}^{k_1}\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_2=0}^{v_2}\cdots\sum_{k_n=0}^{v_n} C_{v_2}^{k_2}\cdots C_{v_n}^{k_n} f(k_1,k_2,\ldots,k_n)\right.\right.$$

$$\left.\left.\cdot\, g(v_1-k_1, v_2-k_2, \ldots, v_n-k_n)\right)\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\frac{t_1^{v_1}}{v_1!}$$

$$= \sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_1=0}^{v_1}\sum_{k_2=0}^{v_2}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}C_{v_2}^{k_2}\cdots C_{v_n}^{k_n} f(k_1,k_2,\ldots,k_n)\right.$$

$$\left.\cdot\, g(v_1-k_1, v_2-k_2, \ldots, v_n-k_n)\right)\frac{t_1^{v_1}}{v_1!}\frac{t_2^{v_2}}{v_2!}\cdots\frac{t_n^{v_n}}{v_n!}$$

即结论对自然数 n 也成立. 综合(1)、(2)知引理成立.

§3. 主要结论

定理 1 (一阶多元 Euler 数和一阶多元 Bernoulli 数的递推公式)

(1) $$\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}\left[1+(-1)^{\sum_{i=1}^{n}(v_i-k_i)}\right] E^{(1)}_{k_1\cdots k_n} = \begin{cases} 2 & (v_1=\cdots=v_n=0)\\ 0 & (v_1+\cdots+v_n>0)\end{cases}$$

(2) $$B^{(1)}_{v_1\cdots v_n} = \begin{cases} 1 & (v_1=\cdots=v_n=0)\\ \sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n} B^{(1)}_{k_1\cdots k_n} & (v_1+\cdots+v_n>0)\end{cases}$$

证明 (1) 由引理和定义 1, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}\left[1+(-1)^{\sum_{i=1}^{n}(v_i-k_i)}\right] E^{(1)}_{k_1\cdots k_n}\right)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$= \left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty} E^{(1)}_{v_1\cdots v_n}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left[1+(-1)^{\sum_{i=1}^{n}v_i}\right]\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

$$= \frac{2\exp\left[\sum_{i=1}^{n}t_i\right]}{\exp\left[2\sum_{i=1}^{n}t_i\right]+1}\left(\exp\left[\sum_{i=1}^{n}t_i\right]+\exp\left[-\sum_{i=1}^{n}t_i\right]\right) = 2$$

所以, $$\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}\left[1+(-1)^{\sum_{i=1}^{n}(v_i-k_i)}\right] E^{(1)}_{k_1\cdots k_n} = \begin{cases} 2 & (v_1=\cdots=v_n=0)\\ 0 & (v_1+\cdots+v_n>0)\end{cases}$$

(2) 由引理和定义 3, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n} B^{(1)}_{k_1\cdots k_n} - B^{(1)}_{v_1\cdots v_n}\right)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$= \left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty} B^{(1)}_{v_1\cdots v_n}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right) - \sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty} B^{(1)}_{v_1\cdots v_n}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$= \left\{\sum_{i=1}^{n}t_i \Big/ \left(\exp\left[\sum_{i=1}^{n}t_i\right]-1\right)\right\}\left(\exp\left[\sum_{i=1}^{n}t_i\right]-1\right) = \sum_{i=1}^{n}t_i$$

所以，$\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}B_{k_1\cdots k_n}^{(1)}-B_{v_1\cdots v_n}^{(1)}=\begin{cases}1 & (v_1+\cdots+v_n=1)\\ 0 & (v_1+\cdots+v_n\neq 1)\end{cases}$

即 $B_{v_1\cdots v_n}^{(1)}=\begin{cases}1 & (v_1=\cdots=v_n=0)\\ \sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}B_{k_1\cdots k_n}^{(1)} & (v_1+\cdots+v_n>0)\end{cases}$

定理 2 (1) $E_{v_1\cdots v_n}^{(m)}=\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}E_{k_1\cdots k_n}^{(j)}E_{(v_1-k_1)\cdots(v_n-k_n)}^{(m-j)}$;

(2) $B_{v_1\cdots v_n}^{(m)}=\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}B_{k_1\cdots k_n}^{(j)}B_{(v_1-k_1)\cdots(v_n-k_n)}^{(m-j)}$.

其中，j 是整数.

证明 (1) 由引理和定义 1，有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}E_{k_1\cdots k_n}^{(j)}E_{(v_1-k_1)\cdots(v_n-k_n)}^{(m-j)}\right)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E_{v_1\cdots v_n}^{(j)}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E_{v_1\cdots v_n}^{(m-j)}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)=$$

$$\left[\frac{2\exp\left[\sum_{i=1}^{n}t_i\right]}{\exp\left[2\sum_{i=1}^{n}t_i\right]+1}\right]^{j}\left[\frac{2\exp\left[\sum_{i=1}^{n}t_i\right]}{\exp\left[2\sum_{i=1}^{n}t_i\right]+1}\right]^{m-j}=\left[\frac{2\exp\left[\sum_{i=1}^{n}t_i\right]}{\exp\left[2\sum_{i=1}^{n}t_i\right]+1}\right]^{m}$$

$$=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E_{v_1\cdots v_n}^{(m)}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

所以，$E_{v_1\cdots v_n}^{(m)}=\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}E_{k_1\cdots k_n}^{(j)}E_{(v_1-k_1)\cdots(v_n-k_n)}^{(m-j)}$.

(2) 证法同(1).

推论

(1) $\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}E_{k_1\cdots k_n}^{(m)}E_{(v_1-k_1)\cdots(v_n-k_n)}^{(-m)}=\begin{cases}1 & (v_1=\cdots=v_n=0)\\ 0 & (v_1+\cdots+v_n>0)\end{cases}$;

(2) $\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}B_{k_1\cdots k_n}^{(m)}B_{(v_1-k_1)\cdots(v_n-k_n)}^{(-m)}=\begin{cases}1 & (v_1=\cdots=v_n=0)\\ 0 & (v_1+\cdots+v_n>0)\end{cases}$.

注 1 由定理 1 和定理 2 可逐一求出高阶多元 Euler 数和 Bernoulli 数.

定理 3 (1) $E_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)=\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}\left(\frac{1}{2}\right)^{\sum_{i=1}^{n}k_i}\cdot\left(x_1-\frac{m}{2}\right)^{v_1-k_1}\cdots\left(x_n-\frac{m}{2}\right)^{v_n-k_n}E_{k_1\cdots k_n}^{(m)}$;

(2) $B_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)=\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}x_1^{v_1-k_1}\cdots x_n^{v_n-k_n}B_{k_1\cdots k_n}^{(m)}$.

证明 (1) 由引理和定义 1，有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}\left(\frac{1}{2}\right)^{\sum_{i=1}^{n}k_i}\left(x_1-\frac{m}{2}\right)^{v_1-k_1}\cdots\left(x_n-\frac{m}{2}\right)^{v_n-k_n}E_{k_1\cdots k_n}^{(m)}\right)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\left(\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}\left(\frac{1}{2}\right)^{\sum_{i=1}^{n}v_i}E_{v_1\cdots v_n}^{(m)}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(x_1-\frac{m}{2}\right)^{v_1}\right.$$

$$\left.\cdot\cdots\left(x_n-\frac{m}{2}\right)^{v_n}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

$$=\left(2\exp\left[\sum_{i=1}^{n}\frac{1}{2}t_i\right]\Big/\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)\right)^m\exp\left[\sum_{i=1}^{n}\left(x_i-\frac{m}{2}\right)t_i\right]$$

$$=\frac{2^m\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m}=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

所以，$E_{v_1\cdots v_n}^{(m)}(x_1,\cdots x_n)=\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}\left(\frac{1}{2}\right)^{\sum_{i=1}^{n}k_i}\left(x_1-\frac{m}{2}\right)^{v_1-k_1}\cdot\cdots\left(x_n-\frac{m}{2}\right)^{v_n-k_n}E_{k_1\cdots k_n}^{(m)}$;

(2) 证法同(1)．

注2 由定理 3 可逐一求出高阶多元 Euler 多项式和 Bernoulli 多项式．

定理4 (1) $E_{v_1\cdots v_n}^{(m)}(m-x_1,\cdots,m-x_n)=(-1)^{\sum_{i=1}^{n}v_i}E_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)$;

(2) $B_{v_1\cdots v_n}^{(m)}(m-x_1,\cdots,m-x_n)=(-1)^{\sum_{i=1}^{n}v_i}B_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)$．

证明 (1) 由定义 2，有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E_{v_1\cdots v_n}^{(m)}(m-x_1,\cdots,m-x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}=\frac{2^m\exp\left[\sum_{i=1}^{n}(m-x_i)t_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m}$$

$$=\frac{2^m\exp\left[\sum_{i=1}^{n}x_i(-t_i)\right]}{\left(\exp\left[\sum_{i=1}^{n}(-t_i)\right]+1\right)^m}=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)\frac{(-t_1)^{v_1}}{v_1!}\cdots\frac{(-t_n)^{v_n}}{v_n!}$$

$$=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}(-1)^{\sum_{i=1}^{n}v_i}E_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

所以，$E_{v_1\cdots v_n}^{(m)}(m-x_1,\cdots,m-x_n)=(-1)^{\sum_{i=1}^{n}v_i}E_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)$．

(2) 证法同(1)．

定理5 (1) $E_{v_1\cdots v_n}^{(m)}(1+x_1,\cdots,1+x_n)+E_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)=2E_{v_1\cdots v_n}^{(m-1)}(x_1,\cdots,x_n)$;

(2) $B_{v_1\cdots v_n}^{(m)}(1+x_1,\cdots,1+x_n)-B_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)=\sum_{i=1}^{n}v_iB_{v_1\cdots(v_i-1)\cdots v_n}^{(m-1)}(x_1,\cdots,x_n)$．

证明 (1) 由定义 2, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(E^{(m)}_{v_1\cdots v_n}(1+x_1,\cdots,1+x_n)+E^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)\right)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\frac{2^m\exp\left[\sum_{i=1}^{n}(1+x_i)t_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m}+\frac{2^m\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m}=\frac{2^m\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^{m-1}}$$

$$=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}2E^{(m-1)}_{v_1\cdots v_n}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

所以, $E^{(m)}_{v_1\cdots v_n}(1+x_1,\cdots,1+x_n)+E^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)=2E^{(m-1)}_{v_1\cdots v_n}(x_1,\cdots,x_n)$.

(2) 由定义 4, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(B^{(m)}_{v_1\cdots v_n}(1+x_1,\cdots,1+x_n)-B^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)\right)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\frac{\left(\sum_{i=1}^{n}t_i\right)^m\exp\left[\sum_{i=1}^{n}(1+x_i)t_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]-1\right)^m}-\frac{\left(\sum_{i=1}^{n}t_i\right)^m\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]-1\right)^m}=\frac{\left(\sum_{i=1}^{n}t_i\right)^m\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]-1\right)^{m-1}}$$

$$=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{i=1}^{n}t_i\right)B^{(m-1)}_{v_1\cdots v_n}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{i=1}^{n}v_iB^{(m-1)}_{v_1\cdots(v_i-1)\cdots v_n}(x_1,\cdots,x_n)\right)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

所以, $B^{(m)}_{v_1\cdots v_n}(1+x_1,\cdots,1+x_n)-B^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)=\sum_{i=1}^{n}v_iB^{(m-1)}_{v_1\cdots(v_i-1)\cdots v_n}(x_1,\cdots,x_n)$.

定理 6 (1) $\frac{\partial}{\partial x_i}E^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)=v_iE^{(m)}_{v_1\cdots(v_i-1)\cdots v_n}(x_1,\cdots,x_n)$;

(2) $\frac{\partial}{\partial x_i}B^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)=v_iB^{(m)}_{v_1\cdots(v_i-1)\cdots v_n}(x_1,\cdots,x_n)$.

证明 (1) 由定义 2, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\frac{\partial}{\partial x_i}E^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}=\frac{\partial}{\partial x_i}\frac{2^m\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m}$$

$$=\frac{2^mt_i\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m}=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_i^{v_i+1}}{v_i!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}v_iE^{(m)}_{v_1\cdots(v_i-1)\cdots v_n}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

所以, $\frac{\partial}{\partial x_i}E^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)=v_iE^{(m)}_{v_1\cdots(v_i-1)\cdots v_n}(x_1,\cdots,x_n)$.

(2) 证法同(1).

定理 7 (1) $E^{(m+p)}_{v_1\cdots v_n}(x_1+y_1,\cdots,x_n+y_n)$

$$= \sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n} E_{k_1\cdots k_n}^{(m)}(x_1,\cdots,x_n)\,E_{(v_1-k_1)\cdots(v_n-k_n)}^{(p)}(y_1,\cdots,y_n);$$

(2) $B_{v_1\cdots v_n}^{(m+p)}(x_1+y_1,\cdots,x_n+y_n)$

$$\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n} B_{k_1\cdots k_n}^{(m)}(x_1,\cdots,x_n)\,B_{(v_1-k_1)\cdots(v_n-k_n)}^{(p)}(y_1,\cdots,y_n).$$

证明 (1) 由引理和定义 2, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n} E_{k_1\cdots k_n}^{(m)}(x_1,\cdots,x_n)\,E_{(v_1-k_1)\cdots(v_n-k_n)}^{(p)}(y_1,\cdots,y_n)\right)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty} E_{v_1\cdots v_n}^{(m)}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty} E_{v_1\cdots v_n}^{(p)}(y_1,\cdots,y_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

$$=\frac{2^m\exp\left[\sum_{i=1}^n x_it_i\right]}{\left(\exp\left[\sum_{i=1}^n t_i\right]+1\right)^m}\cdot\frac{2^p\exp\left[\sum_{i=1}^n y_it_i\right]}{\left(\exp\left[\sum_{i=1}^n t_i\right]+1\right)^p}=\frac{2^{m+p}\exp\left[\sum_{i=1}^n (x_i+y_i)t_i\right]}{\left(\exp\left[\sum_{i=1}^n t_i\right]+1\right)^{m+p}}$$

$$=\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty} E_{v_1\cdots v_n}^{(m+p)}(x_1+y_1,\cdots,x_n+y_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

所以, $E_{v_1\cdots v_n}^{(m+p)}(x_1+y_1,\cdots,x_n+y_n)=\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n} E_{k_1\cdots k_n}^{(m)}(x_1,\cdots,x_n)$

$\cdot\, E_{(v_1-k_1)\cdots(v_n-k_n)}^{(p)}(y_1,\cdots,y_n)$

(2) 证法同(1).

推论

(1) $$\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n} E_{k_1\cdots k_n}^{(m)}(x_1,\cdots,x_n)\,E_{(v_1-k_1)\cdots(v_n-k_n)}^{(-m)}(x_1,\cdots,x_n)=2^{\sum_{i=1}^n v_i}x_1^{v_1}\cdots x_n^{v_n};$$

(2) $$\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n} B_{k_1\cdots k_n}^{(m)}(x_1,\cdots,x_n)\,B_{(v_1-k_1)\cdots(v_n-k_n)}^{(-m)}(x_1,\cdots,x_n)=2^{\sum_{i=1}^n v_i}x_1^{v_1}\cdots x_n^{v_n}.$$

定理 8 (高阶多元 Euler 数和高阶多元 Bernoulli 数的关系)

$$\sum_{i=1}^n v_i\left(\sum_{k_1=0}^{v_1}\cdots\sum_{k_i=0}^{v_i-1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_i-1}^{k_i}\cdots C_{v_n}^{k_n}(2-m)^{\sum_{i=1}^n(v_i-k_i)-1}E_{k_1\cdots k_n}^{(m)}\right.$$

$$= 2^{\left(\sum_{i=1}^n v_i-1\right.}\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}\left[\sum_{j_1=0}^{k_1}\cdots\sum_{j_n=0}^{k_n}\left(2^{1\sum_{i=1}^n k_i}-2^{\sum_{i=1}^n j_i}\right.\right. C_{k_1}^{j_1}\cdots C_{k_n}^{j_n}B_{j_1\cdots j_n}^{(m)}\ B_{(v_1-k_1)\cdots(v_n-k_n)}^{(1-m)}$$

证明 由定义 1 和引理, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{i=1}^n v_i\left(\sum_{k_1=0}^{v_1}\cdots\sum_{k_i=0}^{v_i-1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_i-1}^{k_i}\cdots C_{v_n}^{k_n}(2-m)^{\sum_{i=1}^n(v_i-k_i)-1}E_{k_1\cdots k_n}^{(m)}\right.\right.\ \frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\sum_{i=1}^n t_i\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n} C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}(2-m)^{\sum_{i=1}^n(v_i-k_i)}E_{k_1\cdots k_n}^{(m)}\ \frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right.$$

$$= \sum_{i=1}^{n} t_i \left(\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} E^{(m)}_{v_1 \cdots v_n} \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!} \right) \left(\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} (2-m)^{\sum_{i=1}^{n} v_i} \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!} \right)$$

$$= \sum_{i=1}^{n} t_i \left(\frac{2\exp\left[\sum_{i=1}^{n} t_i\right]}{\exp\left[2\sum_{i=1}^{n} t_i\right] + 1} \right)^m \exp\left[(2-m)\sum_{i=1}^{n} t_i\right] = \sum_{i=1}^{n} t_i \frac{2^m \exp\left[2\sum_{i=1}^{n} t_i\right]}{\left(\exp\left[2\sum_{i=1}^{n} t_i\right] + 1\right)^m} \qquad (*)$$

由定义3 和引理, 有

$$\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} 2^{\sum_{i=1}^{n} v_i - 1} \sum_{k_1=0}^{v_1} \cdots \sum_{k_n=0}^{v_n} C_{v_1}^{k_1} \cdots C_{v_n}^{k_n} \left[\sum_{j_1=0}^{k_1} \cdots \sum_{j_n=0}^{k_n} \left(2^{\sum_{i=1}^{n} k_i} - 2^{\sum_{i=1}^{n} j_i} \right) C_{k_1}^{j_1} \cdots C_{k_n}^{j_n} B^{(m)}_{j_1 \cdots j_n} \right.$$

$$\left. \cdot B^{(1-m)}_{(v_1-k_1) \cdots (v_n-k_n)} \right] \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!}$$

$$= \frac{1}{2} \left(\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} \left[\sum_{j_1=0}^{v_1} \cdots \sum_{j_n=0}^{v_n} \left(2^{\sum_{i=1}^{n} v_i} - 2^{\sum_{i=1}^{n} j_i} \right) C_{v_1}^{j_1} \cdots C_{v_n}^{j_n} B^{(m)}_{j_1 \cdots j_n} \right] \frac{(2t_1)^{v_1}}{v_1!} \cdots \frac{(2t_n)^{v_n}}{v_n!} \right)$$

$$\cdot \left(\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} B^{(1-m)}_{v_1 \cdots v_n} \frac{(2t_1)^{v_1}}{v_1!} \cdots \frac{(2t_n)^{v_n}}{v_n!} \right)$$

$$= \frac{1}{2} \left(\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} 2^{\sum_{i=1}^{n} v_i} B^{(m)}_{v_1 \cdots v_n} \frac{(2t_1)^{v_1}}{v_1!} \cdots \frac{(2t_n)^{v_n}}{v_n!} \right) \left(\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} \left(2^{\sum_{i=1}^{n} v_i} - 1 \right) \frac{(2t_1)^{v_1}}{v_1!} \cdots \frac{(2t_n)^{v_n}}{v_n!} \right)$$

$$\cdot \left(\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} B^{(1-m)}_{v_1 \cdots v_n} \frac{(2t_1)^{v_1}}{v_1!} \cdots \frac{(2t_n)^{v_n}}{v_n!} \right)$$

$$= \frac{1}{2} \left(\frac{4\sum_{i=1}^{n} t_i}{\exp\left[4\sum_{i=1}^{n} t_i\right] - 1} \right)^m \left(\exp\left[4\sum_{i=1}^{n} t_i\right] - \exp\left[2\sum_{i=1}^{n} t_i\right] \right) \left(\frac{2\sum_{i=1}^{n} t_i}{\exp\left[2\sum_{i=1}^{n} t_i\right] - 1} \right)^{1-m}$$

$$= \sum_{i=1}^{n} t_i \left\{ 2^m \exp\left[2\sum_{i=1}^{n} t_i\right] \Big/ \left(\exp\left[2\sum_{i=1}^{n} t_i\right] + 1 \right)^m \right\} \qquad (**)$$

比较(*)、(* *)知结论成立.

推论 (1) (Euler 数与 Bernoulli 数的关系)

$$v \sum_{k=0}^{v-1} C_{v-1}^{k} E_k = 2^{v-1} \sum_{k=0}^{v} (2^v - 2^k) C_v^k B_k;$$

(2) (高阶 Euler 数与高阶 Bernoulli 数的关系)

$$v \sum_{k=0}^{v-1} C_{v-1}^{k} (2-m)^{v-k-1} E_k^{(m)} = 2^{v-1} \sum_{k=0}^{v} C_v^k \left[\sum_{j=0}^{k} (2^k - 2^j) C_k^j B_j^{(m)} \right] B^{(1-m)}_{(v-k)}$$

定理9 (高阶多元 Euler 多项式与高阶多元 Bernoulli 多项式的关系)

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} E^{(m)}_{v_1 \cdots v_n}(x_1, \cdots, x_n) = \sum_{k_1=0}^{v_1} \cdots \sum_{k_n=0}^{v_n} C_{v_1}^{k_1} \cdots C_{v_n}^{k_n} 2^{\sum_{i=1}^{n} k_i} \left[B^{(m)}_{k_1 \cdots k_n} \left(\frac{1}{2} x_1 + \frac{1}{2}, \cdots, \frac{1}{2} x_n + \frac{1}{2} \right) \right.$$

$$\left. - B^{(m)}_{k_1 \cdots k_n} \left(\frac{1}{2} x_1, \cdots, \frac{1}{2} x_n \right) \right] \cdot B^{(1-m)}_{(v_1-k_1) \cdots (v_n-k_n)}$$

证明 由定义 2, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}E^{(m)}_{v_1\cdots v_n}(x_1,\cdots,x_n)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left\{2^m\exp\left[\sum_{i=1}^{n}x_it_i\right]/\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m\right\}=\sum_{i=1}^{n}t_i\frac{2^m\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m}\qquad(***)$$

由引理和定义 4, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left\{\sum_{k_1=0}^{v_1}\cdots\sum_{k_n=0}^{v_n}C_{v_1}^{k_1}\cdots C_{v_n}^{k_n}2^{\sum_{i=1}^{n}k_i}\left[B^{(m)}_{k_1\cdots k_n}\left(\frac{1}{2}x_1+\frac{1}{2},\cdots,\frac{1}{2}x_n+\frac{1}{2}\right)\right.\right.$$

$$\left.\left.-B^{(m)}_{k_1\cdots k_n}\left(\frac{1}{2}x_1,\cdots,\frac{1}{2}x_n\right)\right]\cdot B^{(1-m)}_{(v_1-k_1)\cdots(v_n-k_n)}(0)\right\}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}$$

$$=\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}2^{\sum_{i=1}^{n}v_i}\left[B^{(m)}_{v_1\cdots v_n}\left(\frac{1}{2}x_1+\frac{1}{2},\cdots,\frac{1}{2}x_n+\frac{1}{2}\right)-B^{(m)}_{v_1\cdots v_n}\left(\frac{1}{2}x_1,\cdots,\frac{1}{2}x_n\right)\right]\right.$$

$$\left.\cdot\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\cdot\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}B^{(1-m)}_{v_1\cdots v_n}\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

$$=\frac{\left(2\sum_{i=1}^{n}t_i\right)^m\exp\left[\sum_{i=1}^{n}x_it_i\right]}{\left(\exp\left[2\sum_{i=1}^{n}t_i\right]-1\right)^m}\left(\exp\left[\sum_{i=1}^{n}t_i\right]-1\right)\left(\frac{\sum_{i=1}^{n}t_i}{\exp\left[\sum_{i=1}^{n}t_i\right]-1}\right)^{(1-m)}$$

$$=\sum_{i=1}^{n}t_i\left\{2^m\exp\left[\sum_{i=1}^{n}x_it_i\right]/\left(\exp\left[\sum_{i=1}^{n}t_i\right]+1\right)^m\right\}\qquad(****)$$

比较(***)、(****)知结论成立.

推论 (1) (Euler 多项式和 Bernoulli 多项式的关系)

$$vE_{v-1}(x)=2^v\left[B_v\left(\frac{1}{2}x+\frac{1}{2}\right)-B_v\left(\frac{1}{2}x\right)\right]$$

(2) (高阶 Euler 多项式和高阶 Bernoulli 多项式的关系)

$$vE^{(m)}_{v-1}(x)=\sum_{k=0}^{v}C_v^k2^k\left[B_k^{(m)}\left(\frac{1}{2}x+\frac{1}{2}\right)-B_k^{(m)}\left(\frac{1}{2}x\right)\right]B^{(1-m)}_{(v-k)}$$

(3) 多元 Euler 多项式(一阶) 和多元 Bernoulli 多项式(一阶) 的关系

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}E^{(1)}_{v_1\cdots v_n}(x_1,\cdots,x_n)=2\left[B^{(1)}_{v_1\cdots v_n}(x_1,\cdots,x_n)-2^{\sum_{i=1}^{n}v_i}B^{(1)}_{v_1\cdots v_n}\left(\frac{1}{2}x_1,\cdots,\frac{1}{2}x_n\right)\right]$$

应用举例

例 $f(x_1,x_2)=v_1x_1^{v_1-1}x_2^{v_2}+v_2x_1^{v_1}x_2^{v_2-1}$, $g(x_1,x_2)=x_1^{v_1}x_2^{v_2}$, 其中 v_1, v_2 是非负整数, 则

(a) $$\sum_{i=1}^{n}f(x_1+i,x_2+i)=B^{(1)}_{v_1v_2}(x_1+n+1,x_2+n+1)-B^{(1)}_{v_1v_2}(x_1+1,x_2+1)$$

(b) $$\sum_{i=1}^{n}(-1)^ig(x_1+i,x_2+i)=\frac{1}{2}\left[(-1)^nE^{(1)}_{v_1v_2}(x_1+n+1,x_2+n+1)-E^{(1)}_{v_1v_2}(x_1+1,x_2+1)\right]$$

1° 在定理 5 中令 $m=1$, $n=2$, 得 $E^{(1)}_{v_1v_2}(1+x_1,1+x_2)+E^{(1)}_{v_1v_2}(x_1,x_2)=2E^{(0)}_{v_1v_2}(x_1,x_2)$

$B^{(1)}_{v_1v_2}(1+x_1,1+x_2)-B^{(1)}_{v_1v_2}(x_1,x_2)=v_1B^{(0)}_{(v_1-1)v_2}(x_1,x_2)+v_2B^{(0)}_{v_1(v_2-1)}(x_1,x_2)$

2° 在定义2、定义4中令 $m=1, n=2$, 得

$$e^{x_1t_1+x_2t_2}=\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}E_{v_1v_2}^{(0)}(x_1,x_2)\frac{t_1^{v_1}}{v_1!}\frac{t_2^{v_2}}{v_2!},\quad e^{x_1t_1+x_2t_2}=\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}E_{v_1v_2}^{(0)}(x_1,x_2)\frac{t_1^{v_1}}{v_1!}\frac{t_2^{v_2}}{v_2!}$$

所以 $E_{v_1v_2}^{(0)}(x_1,x_2)=x_1^{v_1}x_2^{v_2}$, $B_{v_1v_2}^{(0)}(x_1,x_2)=x_1^{v_1}x_2^{v_2}$

由1°和2°, 得

$$E_{v_1v_2}^{(1)}(1+x_1,1+x_2)+E_{v_1v_2}^{(1)}(x_1,x_2)=2x_1^{v_1}x_2^{v_2}$$

$$B_{v_1v_2}^{(1)}(1+x_1,1+x_2)-B_{v_1v_2}^{(1)}(x_1x_2)=v_1x_1^{v_1-1}x_2^{v_2}+v_2x_1^{v_1}x_2^{v_2-1}$$

所以有(a)和(b)•

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Higher_Order Multivariable Euler's Polynomial and Higher_Order Multivariable Bernoulli's Polynomial

Liu Guodong

(Departmet of Mathematics, Huizhou University, Huizhou, Guangdong 516015, P. R. China)

Abstract

In this paper, the definitions of both higher_order multivariable Euler's numbers and polynomial, higher_order multivariable Bernoulli's numbers and polynomial are given and some of their important properties are expounded. As a resut, the mathematical relationship between higher_order multivariable Euler's polynomial (numbers) and higher_order multivariable Bernoulli's polynomial (numbers) are thus obtained.

Key words higher_order multivariable Euler's numbers, highe_order multivariable Euler's polynomial, higher_order multivariable Bernoulli's numbers, higher_order multivariable Bernoulli's polynomial