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用张量导数法求横观各向同性弹性材料的弹性张量的一般形式*

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(杨桂通推荐)

摘要: 用对张量函数求导的方法导出了横观各向同性材料和各向同性材料的弹性张量的一般形式与应力_应变关系式• 从推导过程可更清楚地看出为什么横观各向同性材料和各向同性材料分别有五个和两个独立的弹性常数, 即材料有几个独立的弹性常数是由其应变能函数的形式所决定的•

关键词: 张量; 横观各向同性材料; 应变能; 弹性; 张量导数
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符号约定

U: 应变能函数
$\boldsymbol{Q}$: 标准正交张量
$\boldsymbol{E}$: 应变张量
E_{ij}: 应变张量 $\boldsymbol{E}$ 的分量, $i, j = 1, 2, 3$
$\boldsymbol{n}$: 横观各向同性材料的对称轴向量
U^*, $\boldsymbol{E}^*$, $\boldsymbol{n}^*$: 分别为 $U, \boldsymbol{E}, \boldsymbol{n}$ 在另一标架下的形式
$J_i^{\boldsymbol{E}}$: 应变张量 $\boldsymbol{E}$ 的主不变量, $i = 1, 2, 3$
$J_i^{\boldsymbol{E},\boldsymbol{n}}$: 应变张量 $\boldsymbol{E}$ 与向量 $\boldsymbol{n}$ 的不变量, $i = 4, 5$
J_i: 分别为 $J_1^{\boldsymbol{E}}, J_2^{\boldsymbol{E}}, J_3^{\boldsymbol{E}}, J_4^{\boldsymbol{E},\boldsymbol{n}}, J_5^{\boldsymbol{E},\boldsymbol{n}}$ 的简写, $i = 1, 2, 3, 4, 5$
β_i: 与 $\boldsymbol{E}, \boldsymbol{n}$ 无关的独立常数, $i = 1, 2, 3, 4, 5$
$\boldsymbol{e}_l$, $\boldsymbol{e}^m$: 分别为所取标架系统的协变与逆变基矢量, $l, m = 1, 2, 3$
$\otimes$: 张量的并矢符号
$\boldsymbol{I}$: 度量张量, 形式为 $\boldsymbol{e}_l \otimes \boldsymbol{e}^l$
$\boldsymbol{l}$: 四阶张量 $\boldsymbol{I} \otimes \boldsymbol{I}$
$\boldsymbol{l}^{(1324)}$: 四阶张量 $\boldsymbol{l}$ 的同构体, 其形式为 $\boldsymbol{e}_l \otimes \boldsymbol{e}_m \otimes \boldsymbol{e}^l \otimes \boldsymbol{e}^m$
$\boldsymbol{C}$: 四阶弹性张量
$\lambda, \mu, \alpha_1, \alpha_2, \alpha_3$: 与 $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ 有关的独立常数
$C_{11}, C_{12}, C_{13}, C_{33}, C_{44}, C_{55}$: 横观各向同性材料的弹性系数
$\boldsymbol{T}$: 应力张量
T_{ij}: 应力张量 $\boldsymbol{T}$ 的分量, $i, j = 1, 2, 3$

引 言

众所周知, 各向同性线弹性材料应变与应力的关系可用两个独立的参数来表征, 横观各向同性线弹性材料应变与应力的关系可用五个独立的参数来表征• 在经典的弹性理论著作中, 上述本构关系是用坐标轴旋转和对坐标轴的反射得到的[1],[2],[3]• 本文从另一个思路, 即用张量的不变量的概念和对张量函数求导的方法[4], 重新导出了上述关系•

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1 横观各向同性材料的应变能函数和对应的张量不变量

对横观各向同性材料其应变能 U 可表示为应变张量 $\boldsymbol{E}$ 和各向同性轴的方向矢量 $\boldsymbol{n}$ 的函数, 即

$$U = u(\boldsymbol{E}, \boldsymbol{n}), \tag{1}$$

这里 $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{T}}$ 是对称二阶张量, $\boldsymbol{n}$ 为单位矢量, U 是各向同性标量函数, 因此在正交变换 $\boldsymbol{Q}$ 下 U 的值保持不变:

$$U^* = u(\boldsymbol{Q}\cdot\boldsymbol{E}\cdot\boldsymbol{Q}, \boldsymbol{Q}\cdot\boldsymbol{n}) = u(\boldsymbol{E}^*, \boldsymbol{n}^*) = U,$$

这里, $\boldsymbol{Q}$ 是标准正交张量, U^*, $\boldsymbol{E}^*$, $\boldsymbol{n}^*$ 分别为 U, $\boldsymbol{E}$, $\boldsymbol{n}$ 在变换后的标架下的值.

由(1)式知 U 应为 $\boldsymbol{E}$ 与 $\boldsymbol{n}$ 的不变量的函数, 在这些不变量中独立的只有五个, 如:

$$J_1^{\boldsymbol{E}} = \mathrm{tr}\boldsymbol{E}, \tag{2}$$

$$J_2^{\boldsymbol{E}} = \frac{1}{2}[(\mathrm{tr}\boldsymbol{E})^2 - \mathrm{tr}(\boldsymbol{E}\cdot\boldsymbol{E})], \tag{3}$$

$$J_3^{\boldsymbol{E}} = \det\boldsymbol{E}, \tag{4}$$

$$J_4^{\boldsymbol{E},\boldsymbol{n}} = \boldsymbol{n}\cdot\boldsymbol{E}\cdot\boldsymbol{n}, \tag{5}$$

$$J_5^{\boldsymbol{E},\boldsymbol{n}} = \boldsymbol{n}\cdot\boldsymbol{E}^2\cdot\boldsymbol{n}. \tag{6}$$

以下我们省去 J 的上标不写只写 J 的下标. 如 J_5 代表 $J_5^{\boldsymbol{E},\boldsymbol{n}}$, 这样,

$$U = u(\boldsymbol{E}, \boldsymbol{n}) = f(J_1, J_2, J_3, J_4, J_5), \tag{7}$$

对于线弹性材料来说, 由于 U 是应变分量的二次函数, 所以必然有:

$$U = \beta_1 J_1^2 + \beta_2 J_2 + \beta_3 J_1 J_4 + \beta_4 J_4^2 + \beta_5 J_5, \tag{8}$$

式中 β_1, β_2, β_3, β_4, β_5 为与 $\boldsymbol{E}$, $\boldsymbol{n}$ 无关的常数.

2 横观各向同性材料的弹性张量

下面我们来求 $\dfrac{\partial U}{\partial \boldsymbol{E}}$. 由(8) 式对 $\boldsymbol{E}$ 求导得:

$$\frac{\partial U}{\partial \boldsymbol{E}} = 2\beta_1 J_1 \frac{\partial J_1}{\partial \boldsymbol{E}} + \beta_2 \frac{\partial J_2}{\partial \boldsymbol{E}} + \beta_3 J_1 \frac{\partial J_4}{\partial \boldsymbol{E}} + \beta_3 J_4 \frac{\partial J_1}{\partial \boldsymbol{E}} + 2\beta_4 J_4 \frac{\partial J_4}{\partial \boldsymbol{E}} + \beta_5 \frac{\partial J_5}{\partial \boldsymbol{E}}. \tag{9}$$

这是因为设 $\dfrac{\partial(J_1 J_4)}{\partial \boldsymbol{E}} = \boldsymbol{B}$, $\dfrac{\partial J_1}{\partial \boldsymbol{E}} = \boldsymbol{B}_1$, $\dfrac{\partial J_4}{\partial \boldsymbol{E}} = \boldsymbol{B}_4$,

则由二阶张量的标量函数的导数定义:

$$\lim_{\tau\to 0}\frac{\mathrm{d}J_1(\boldsymbol{E}+\tau\boldsymbol{A})}{\mathrm{d}\tau} = \boldsymbol{B}_1:\boldsymbol{A}, \quad \lim_{\tau\to 0}\frac{\mathrm{d}J_4(\boldsymbol{E}+\tau\boldsymbol{A})}{\mathrm{d}\tau} = \boldsymbol{B}_4:\boldsymbol{A},$$

$$\lim_{\tau\to 0}\frac{\mathrm{d}[J_1(\boldsymbol{E}+\tau\boldsymbol{A})\cdot J_4(\boldsymbol{E}+\tau\boldsymbol{A})]}{\mathrm{d}\tau} = \boldsymbol{B}:\boldsymbol{A}.$$

因为

$$\lim_{\tau\to 0}\frac{\mathrm{d}[J_1(\boldsymbol{E}+\tau\boldsymbol{A})\cdot J_4(\boldsymbol{E}+\tau\boldsymbol{A})]}{\mathrm{d}\tau} = J_1\lim_{\tau\to 0}\frac{\mathrm{d}J_4(\boldsymbol{E}+\tau\boldsymbol{A})}{\mathrm{d}\tau} + J_4\lim_{\tau\to 0}\frac{\mathrm{d}J_1(\boldsymbol{E}+\tau\boldsymbol{A})}{\mathrm{d}\tau} = J_1\boldsymbol{B}_4:\boldsymbol{A} + J_4\boldsymbol{B}_1:\boldsymbol{A} = \boldsymbol{B}:\boldsymbol{A},$$

比较上式两端得:

$$\boldsymbol{B} = J_1\boldsymbol{B}_4 + J_4\boldsymbol{B}_1 = J_1\frac{\partial J_4}{\partial \boldsymbol{E}} + J_4\frac{\partial J_1}{\partial \boldsymbol{E}}.$$

不难求得各个不变量对 $\boldsymbol{E}$ 的导数为:

$$\frac{\partial J_1}{\partial \boldsymbol{E}} = \boldsymbol{I}, \tag{a}$$

$$\frac{\partial J_2}{\partial \boldsymbol{E}} = J_1\boldsymbol{I} - E^T = J_1\boldsymbol{I} - \boldsymbol{E}, \tag{b}$$

$$\frac{\partial J_4}{\partial \boldsymbol{E}} = \boldsymbol{n}\otimes\boldsymbol{n}, \tag{c}$$

$$\frac{\partial J_5}{\partial \boldsymbol{E}} = \boldsymbol{n}\cdot\boldsymbol{E}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{n}\cdot\boldsymbol{E}. \tag{d}$$

下面我们只给出(c),(d)两式的证明.

$$\lim_{\tau\to 0}\frac{\mathrm{d}J_4(\boldsymbol{E}+\tau\boldsymbol{A})}{\mathrm{d}\tau} = \lim_{\tau\to 0}\frac{\mathrm{d}(\boldsymbol{n}\cdot(\boldsymbol{E}+\tau\boldsymbol{A})\cdot\boldsymbol{n})}{\mathrm{d}\tau} =$$
$$\boldsymbol{n}\cdot\boldsymbol{A}\cdot\boldsymbol{n} = \boldsymbol{B}_4:\boldsymbol{A} = \boldsymbol{n}\otimes\boldsymbol{n}:\boldsymbol{A},$$

因此: $\boldsymbol{B}_4 = \dfrac{\partial J_4}{\partial \boldsymbol{E}} = \boldsymbol{n}\otimes\boldsymbol{n}$,

同样 $$\lim_{\tau\to 0}\frac{\mathrm{d}J_5(\boldsymbol{E}+\tau\boldsymbol{A})}{\mathrm{d}\tau} = \lim_{\tau\to 0}\frac{\mathrm{d}[\boldsymbol{n}\cdot(\boldsymbol{E}+\tau\boldsymbol{A})\cdot(\boldsymbol{E}+\tau\boldsymbol{A})\cdot\boldsymbol{n}]}{\mathrm{d}\tau} =$$
$$\boldsymbol{n}\cdot\boldsymbol{E}\cdot\boldsymbol{A}\cdot\boldsymbol{n} + \boldsymbol{n}\cdot\boldsymbol{A}\cdot\boldsymbol{E}\cdot\boldsymbol{n} = \boldsymbol{B}_5:\boldsymbol{A},$$

因为

$$\boldsymbol{n}\cdot\boldsymbol{E}\cdot\boldsymbol{A}\cdot\boldsymbol{n} = (\boldsymbol{n}\cdot\boldsymbol{E})\otimes\boldsymbol{n}:\boldsymbol{A},\quad \boldsymbol{n}\cdot\boldsymbol{A}\cdot\boldsymbol{E}\cdot\boldsymbol{n} = \boldsymbol{n}\otimes(\boldsymbol{E}\cdot\boldsymbol{n}):\boldsymbol{A},$$

所以

$$\boldsymbol{B}_5 = (\boldsymbol{n}\cdot\boldsymbol{E})\otimes\boldsymbol{n} + \boldsymbol{n}\otimes(\boldsymbol{E}\cdot\boldsymbol{n}),$$

所以我们有

$$\begin{aligned}\frac{\partial U}{\partial \boldsymbol{E}} = {} & 2\beta_1 J_1\boldsymbol{I} + \beta_2 J_1\boldsymbol{I} - \beta_2\boldsymbol{E} + \beta_3 J_1\boldsymbol{n}\otimes\boldsymbol{n} + \\ & \beta_3 J_4\boldsymbol{I} + 2\beta_4 J_4\boldsymbol{n}\otimes\boldsymbol{n} + \beta_5((\boldsymbol{n}\cdot\boldsymbol{E})\otimes\boldsymbol{n} + \\ & \boldsymbol{n}\otimes(\boldsymbol{E}\cdot\boldsymbol{n})),\end{aligned} \tag{10}$$

从等式(10)可看出 $\dfrac{\partial U}{\partial \boldsymbol{E}}$ 是对称二阶张量.

下面我们来求 $\dfrac{\partial^2 U}{\partial \boldsymbol{E}^2}$ 即弹性张量 $\boldsymbol{C}$.

$$\frac{\partial[J_1\boldsymbol{I}]}{\partial \boldsymbol{E}} = \boldsymbol{I}\otimes\boldsymbol{I} = \boldsymbol{l}, \tag{e}$$

$$\frac{\partial[\boldsymbol{E}]}{\partial \boldsymbol{E}} = \boldsymbol{l}^{(1324)} = \boldsymbol{e}_l\otimes\boldsymbol{e}_m\otimes\boldsymbol{e}^l\otimes\boldsymbol{e}^m, \tag{f}$$

$$\frac{\partial[J_4\boldsymbol{n}\otimes\boldsymbol{n}]}{\partial \boldsymbol{E}} = \boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}, \tag{g}$$

$$\frac{\partial[\boldsymbol{I}J_4]}{\partial \boldsymbol{E}} = \boldsymbol{I}\otimes\boldsymbol{n}\otimes\boldsymbol{n}, \tag{h}$$

$$\frac{\partial[J_1\boldsymbol{n}\otimes\boldsymbol{n}]}{\partial \boldsymbol{E}} = \boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{I}, \tag{i}$$

$$\frac{\partial[\boldsymbol{n}\cdot(\boldsymbol{E}\otimes\boldsymbol{n})]}{\partial \boldsymbol{E}} = \boldsymbol{e}^m\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{e}_m, \tag{j}$$

$$\frac{\partial[\boldsymbol{n}\cdot(\boldsymbol{E}^{\mathrm{T}}\otimes\boldsymbol{n})]}{\partial\boldsymbol{E}} = \boldsymbol{e}_l\otimes\boldsymbol{n}\otimes\boldsymbol{e}^l\otimes\boldsymbol{n}, \tag{k}$$

$$\frac{\partial[\boldsymbol{n}\otimes\boldsymbol{n}\cdot\boldsymbol{E}]}{\partial\boldsymbol{E}} = \boldsymbol{n}\otimes\boldsymbol{e}^k\otimes\boldsymbol{n}\otimes\boldsymbol{e}_k, \tag{l}$$

$$\frac{\partial[\boldsymbol{n}\otimes\boldsymbol{n}\cdot\boldsymbol{E}^{\mathrm{T}}]}{\partial\boldsymbol{E}} = \boldsymbol{n}\otimes\boldsymbol{e}_s\otimes\boldsymbol{e}^s\otimes\boldsymbol{n}, \tag{m}$$

以上各式很容易证明• 例如对等式(m), 因为

$$\lim_{\tau\to 0}\frac{\mathrm{d}[\boldsymbol{n}\cdot(\boldsymbol{E}^{\mathrm{T}}+\tau\boldsymbol{A}^{\mathrm{T}})\otimes\boldsymbol{n}]}{\mathrm{d}\tau} = \boldsymbol{n}\cdot(\boldsymbol{A}^{\mathrm{T}}\otimes\boldsymbol{n}) = \boldsymbol{D}:\boldsymbol{A},$$

上式的分量形式为: $A^{ij}n_j n^l\boldsymbol{e}_i\boldsymbol{e}_l = D^{il}_{\cdot\cdot st}A^{st}\boldsymbol{e}_i\boldsymbol{e}_l,\ n_j n^l\delta^i_s\delta^j_t A^{st}\boldsymbol{e}_i\boldsymbol{e}_l = D^{il}_{\cdot\cdot st}\boldsymbol{e}_i\boldsymbol{e}_l A^{st}$,

所以

$$D^{il}_{\cdot\cdot st} = n_t n^l\delta^i_s,$$

即

$$\boldsymbol{D} = \boldsymbol{e}_l\otimes\boldsymbol{n}\otimes\boldsymbol{e}^l\otimes\boldsymbol{n}\cdot$$

由以上等式可得

$$\begin{aligned}\boldsymbol{C} = {} & (2\beta_1+\beta_2)\boldsymbol{I}\otimes\boldsymbol{I}-\beta_2\boldsymbol{I}^{(1324)}+\beta_3(\boldsymbol{I}\otimes\boldsymbol{n}\otimes\boldsymbol{n}+\\ & \boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{I})+2\beta_4\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}+\\ & \frac{1}{2}\beta_5(\boldsymbol{e}^m\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{e}_m+\boldsymbol{e}_l\otimes\boldsymbol{n}\otimes\boldsymbol{e}^l\otimes\boldsymbol{n}+\\ & \boldsymbol{n}\otimes\boldsymbol{e}^k\otimes\boldsymbol{n}\otimes\boldsymbol{e}_k+\boldsymbol{n}\otimes\boldsymbol{e}_s\otimes\boldsymbol{e}^s\otimes\boldsymbol{n}),\end{aligned} \tag{11}$$

且有: $\boldsymbol{I}\otimes\boldsymbol{I} = \boldsymbol{e}_l\otimes\boldsymbol{e}^l\otimes\boldsymbol{e}_m\otimes\boldsymbol{e}^m$,

$$\begin{aligned}& \boldsymbol{e}^m\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{e}_m+\boldsymbol{e}_l\otimes\boldsymbol{n}\otimes\boldsymbol{e}^l\otimes\boldsymbol{n}+\\ & \quad\boldsymbol{n}\otimes\boldsymbol{e}^k\otimes\boldsymbol{n}\otimes\boldsymbol{e}_k+\boldsymbol{n}\otimes\boldsymbol{e}_s\otimes\boldsymbol{e}^s\otimes\boldsymbol{n} =\\ & \quad(\boldsymbol{e}^m\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{e}^m)\otimes(\boldsymbol{e}_m\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{e}_m),\end{aligned}$$

如果取: $\lambda = 2\beta_1+\beta_2,\ 2\mu = -\beta_2,\ \alpha_1 = 2\beta_4,\ \alpha_2 = \beta_3,\ \alpha_3 = \frac{1}{2}\beta_5$,

则最后可得:

$$\begin{aligned}\boldsymbol{C} = {} & \lambda\boldsymbol{I}\otimes\boldsymbol{I}+2\mu\boldsymbol{I}^{(1324)}+\alpha_1\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}+\\ & \alpha_2(\boldsymbol{I}\otimes\boldsymbol{n}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{I})+\\ & \alpha_3(\boldsymbol{e}^m\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{e}^m)\otimes(\boldsymbol{e}_m\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{e}_m)\cdot\end{aligned} \tag{12}$$

可以验证 $C^{(1234)} = C^{(2134)} = C^{(1243)} = C^{(3412)}$•

对于各向同性弹性材料, 其应变能函数只要在(8)式中取 $\beta_3 = \beta_4 = \beta_5 = 0$(即应变能与方向矢量 $\boldsymbol{n}$ 无关), 用相同的方法可得弹性张量 $\boldsymbol{C}$ 为:

$$\boldsymbol{C} = \lambda\boldsymbol{I}\otimes\boldsymbol{I}+2\mu\boldsymbol{I}^{(1324)}\cdot$$

如果我们采用笛卡尔坐标系, 并选 $\boldsymbol{n} = \boldsymbol{e}_3$, 则由(10)式得应力张量 $\boldsymbol{T}$ 为:

$$\begin{aligned}\boldsymbol{T} = \frac{\partial U}{\partial\boldsymbol{E}} = {} & \lambda J_1\boldsymbol{I}+2\mu\boldsymbol{E}+\alpha_1(\boldsymbol{e}_3\cdot\boldsymbol{E}\cdot\boldsymbol{e}_3)\boldsymbol{e}_3\otimes\boldsymbol{e}_3+\alpha_2\boldsymbol{I}(\boldsymbol{e}_3\cdot\boldsymbol{E}\cdot\boldsymbol{e}_3)+\\ & \alpha_2 J_1\boldsymbol{e}_3\otimes\boldsymbol{e}_3+2\alpha_3[\boldsymbol{e}_3\cdot(\boldsymbol{E}\otimes\boldsymbol{e}_3)+\boldsymbol{e}_3\otimes\boldsymbol{E}\cdot\boldsymbol{e}_3] =\\ & \lambda J_1\boldsymbol{I}+2\mu\boldsymbol{E}+\alpha_1 E_{33}\boldsymbol{e}_3\otimes\boldsymbol{e}_3+\alpha_2 E_{33}\boldsymbol{I}+\alpha_2 J_1\boldsymbol{e}_3\otimes\boldsymbol{e}_3+\\ & 2\alpha_3(E_{3i}\boldsymbol{e}_i\otimes\boldsymbol{e}_3+E_{i3}\boldsymbol{e}_3\otimes\boldsymbol{e}_i),\end{aligned} \tag{13}$$

其分量为:

$$
\begin{aligned}
T_{11} &= \lambda J_1 + 2\mu E_{11} + \alpha_2 E_{33} = (\lambda + 2\mu) E_{11} + \lambda(E_{22} + E_{33}) + \alpha_2 E_{33}, \\
T_{22} &= \lambda J_1 + 2\mu E_{22} + \alpha_2 E_{33} = (\lambda + 2\mu) E_{22} + \lambda(E_{11} + E_{33}) + \alpha_2 E_{33}, \\
T_{33} &= (\lambda + 2\mu) E_{33} + (\alpha_1 + 2\alpha_2 + 4\alpha_3) E_{33} + (\lambda + \alpha_2)(E_{11} + E_{22}), \\
T_{12} &= T_{21} = 2\mu E_{12}, \\
T_{13} &= T_{31} = 2\mu E_{13} + 2\alpha_3 E_{13}, \\
T_{23} &= T_{32} = 2\mu E_{23} + 2\alpha_3 E_{23},
\end{aligned}
$$

$$
\boldsymbol{T} = \begin{pmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{13} \\ T_{23} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda + \alpha_2 & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda + \alpha_2 & 0 & 0 & 0 \\ \lambda + \alpha_2 & \lambda + \alpha_2 & \lambda + 2\mu + \alpha_1 + 2\alpha_2 + 4\alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(\lambda + \alpha_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(\mu + \alpha_3) \end{pmatrix} \begin{pmatrix} E_{11} \\ E_{22} \\ E_{33} \\ E_{12} \\ E_{13} \\ E_{23} \end{pmatrix},
$$

取: $C_{11} = \lambda + 2\mu$, $C_{12} = \lambda$, $C_{13} = \lambda + \alpha_2$, $C_{33} = \lambda + 2\mu + \alpha_1 + 2\alpha_2 + 4\alpha_3$, $C_{55} = 2(\lambda + \alpha_3)$ 则

$$
\boldsymbol{T} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{pmatrix} \begin{pmatrix} E_{11} \\ E_{22} \\ E_{33} \\ E_{12} \\ E_{13} \\ E_{23} \end{pmatrix},
$$

其中 $C_{44} = \frac{1}{2}(C_{11} - C_{12})$•

3 讨　论

从上面的推导可以看出一种材料有几个独立的弹性常数是由其象(8) 式一样的应变能方程里有几个张量不变量和有几个独立系数(β_i, $i = 1, 2, 3 \dots$) 所决定的• 因此可以更清楚地看出为什么横观各向同性材料与各向同性材料分别有五个和两个独立的弹性常数•

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参 考 文 献

[1] 钱伟长, 叶开沅. 弹性力学[M]. 北京: 科学出版社, 1956

[2] 冯元桢. 连续介质力学[M]. 北京: 科学出版社, 1992, 205~ 217

[3] 武际可, 王敏中. 弹性力学引论[M]. 北京: 北京大学出版社, 1981, 71~ 85

[4] 黄克智, 薛明德, 陆明万. 张量分析[M]. 北京: 清华大学出版社, 1986, 74~ 78, 172~ 193

Derivation of The General Form of Elasticity Tensor of The Transverse Isotropic Material by Tensor Derivate

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Abstract: In the paper the elasticity tensor and the relation between stress and strain of transverse isotropic material and isotropic material are deduced by tensor derivate. From the derivation the reason why there are two independent elasticity coefficients for isotropic elastic material and five for transverse isotropic elastic material can be seen more clearly.

Key words: tensor; transverse isotropic material; strain energy; elasticity; tensor derivate