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文克尔地基上阶梯式矩形薄板的振动*

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摘要: 用奇异函数建立文克尔地基上阶梯式矩形薄板自由振动和强迫振动的微分方程并求得其通解,用 W 算子给出振型函数的表达式及常见支承条件下板的频率方程,用广义函数的富里叶展开,讨论板在不同载荷作用下的强迫振动响应•

关 键 词: 文克尔地基; 阶梯式矩形薄板; 自由振动; 强迫振动响应

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引 言

弹性地基板在工程中极为常见, 如: 土建工程中的基础底板, 公路路面, 挡土墙, 地下结构、水闸、船坞以及溢流坝的底板等• 工程中常常采用阶梯式结构, 以满足强度、刚度及稳定性的要求并节约人力、物力• 本文讨论文克尔地基上两对边简支的阶梯式变厚度矩形薄板的振动•

1 自由振动

1.1 自由振动微分方程及其通解

设有文克尔地基上的矩形薄板, 边长分别为 a 和 b, 板厚沿 y 方向呈 n 级阶梯式变化, 各级阶梯衔接处坐标为 y_i, 在各级阶梯内, 板的单位面积质量为 m_i、厚度为 h_i、抗弯刚度为 $D_i = \dfrac{Eh_i^3}{12(1-\mu^2)}$, $i = 1, 2, \cdots, n$, E 为材料的弹性模量, μ 为泊松系数• 设 k 为地基系数, $w(x, y, t)$ 为振动的任一瞬时从平衡位置量起的板的挠度•

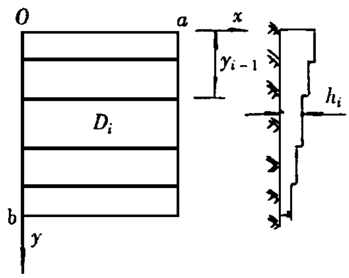

图 1 文克尔地基上阶梯式矩形薄板

当 D, m 分别都是 x, y 的函数时, 文克尔地基上变厚度矩形薄板自由振动的微分方程为[1]

$$D\nabla^4 w + 2\frac{\partial D}{\partial x}\frac{\partial}{\partial x}\nabla^2 w + 2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}\nabla^2 w + \nabla^2 D\,\nabla^2 w - (1-\mu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) + kw + m\frac{\partial^2 w}{\partial t^2} = 0• \quad (1)$$

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若板厚沿 y 方向呈阶梯式变化(图 1), 则可将板的抗弯刚度与单位面积质量分别表为

$$D(y)=\sum_{i=1}^{n}D_i(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0), \tag{2}$$

$$m(y)=\sum_{i=1}^{n}m_i(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0), \tag{3}$$

式中, n 为板厚沿 y 方向变化的阶梯数; D_i, m_i 分别为第 i 级板的抗弯刚度和单位面积质量; $y_0=0, y_n=b$; $\langle y-y_0\rangle^0=1$, $\langle y-y_n\rangle^0=0$。

将式(2)~(3)代入方程(1), 有

$$\begin{aligned}&\sum_{i=1}^{n}D_i(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0)\nabla^4 w+2\Big\{-D_1\delta(y-y_1)+\sum_{i=2}^{n}D_i[\delta(y-y_{i-1})-\\&\quad\delta(y-y_i)]\Big\}\frac{\partial}{\partial y}\nabla^2 w+\Big\{-D_1\delta'(y-y_1)+\sum_{i=2}^{n}D_i[\delta'(y-y_{i-1})-\\&\quad\delta'(y-y_i)]\Big\}\nabla^2 w+(1-\mu)\Big\{D_1\delta'(y-y_1)-\sum_{i=2}^{n}D_i[\delta'(y-y_{i-1})-\\&\quad\delta'(y-y_i)]\Big\}\frac{\partial^2 w}{\partial x^2}+kw+\sum_{i=1}^{n}m_i(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0)\frac{\partial^2 w}{\partial t^2}=0\text{。}\end{aligned}\tag{4}$$

根据 δ 函数的性质[1], 对函数 $f(y)$, 有

$$\left.\begin{aligned}&\delta(y-y_i)f(y)=f(y_i)\delta(y-y_i),\\&\delta'(y-y_i)f(y)=-f'(y_i)\delta(y-y_i)+f(y_i)\delta'(y-y_i),\\&\qquad\qquad(i=1,2,\cdots,n-1)\text{。}\end{aligned}\right\}\tag{5}$$

将式(5)代入式(4), 有

$$\begin{aligned}&\sum_{i=1}^{n}D_i(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0)\nabla^4 w-D_1\delta(y-y_1)\Big[\frac{\partial^3}{\partial y^3}+\\&\quad(2-\mu)\frac{\partial^3}{\partial x^2\partial y}\Big]w_1(x,y,t)\Big|_{y=y_1}+\sum_{i=2}^{n}D_i\Big\{\delta(y-y_{i-1})\Big[\frac{\partial^3}{\partial y^3}+\\&\quad(2-\mu)\frac{\partial^3}{\partial x^2\partial y}\Big]w_i(x,y,t)\Big|_{y=y_{i-1}}-\delta(y-y_i)\Big[\frac{\partial^3}{\partial y^3}+\\&\quad(2-\mu)\frac{\partial^3}{\partial x^2\partial y}\Big]w_i(x,y,t)\Big|_{y=y_i}\Big\}-D_1\delta'(y-y_1)\Big(\frac{\partial^2}{\partial y^2}+\\&\quad\mu\frac{\partial^2}{\partial x^2}\Big)w_1(x,y,t)\Big|_{y=y_1}+\sum_{i=2}^{n}D_i\Big\{\delta'(y-y_{i-1})\Big(\frac{\partial^2}{\partial y^2}+\\&\quad\mu\frac{\partial^2}{\partial x^2}\Big)w_i(x,y,t)\Big|_{y=y_{i-1}}-\delta'(y-y_i)\Big(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}\Big)w_i(x,y,t)\Big|_{y=y_i}\Big\}+\\&\quad kw+\sum_{i=1}^{n}m_i(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0)\frac{\partial^2 w}{\partial t^2}=0\text{。}\end{aligned}\tag{6}$$

若板的各阶梯衔接处 $y=y_i$ 边上的总弯矩和总剪力连续变化, 即若下式成立

$$\left.\begin{aligned}
&D_i\left[\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}\right)w_i(x,y,t)\right]_{y=y_i}=D_{i+1}\left[\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}\right)w_{i+1}(x,y,t)\right]_{y=y_i},\\
&D_i\left[\frac{\partial^3}{\partial y^3}+(2-\mu)\frac{\partial^3}{\partial x^2\partial y}w_i(x,y,t)\right]_{y=y_i}=\\
&\quad D_{i+1}\left[\frac{\partial^3}{\partial y^3}+(2-\mu)\frac{\partial^3}{\partial x^2\partial y}w_{i+1}(x,y,t)\right]_{y=y_i},\\
&\qquad\qquad (i=1,2,\cdots,n-1),
\end{aligned}\right\}\tag{7}$$

式(6)便简化为

$$D(y)\nabla^4 w(x,y,t)+kw(x,y,t)+m(y)\partial^2 w(x,y,t)/\partial t^2=0\cdot\tag{8}$$

此即文克尔地基上阶梯式矩形薄板自由振动的微分方程• 式中, $D(y)$、$m(y)$ 分别由式(2)、(3) 给出•

讨论板的 $x=0$, $x=a$ 两边简支, 另两边为任意支承的情况•

将方程(8)的通解表为

$$w(x,y,t)=\sum_{j=1}^{\infty}W_j(x,y)T_j(t),\tag{9}$$

其中 $$W_j(x,y)=Y_j(y)\sin(j\pi x/a),\tag{10}$$

为板的振型函数• 将式(9)~(10)代入式(8),可得

$$\ddot{T}_j(t)+\omega_j^2T_j(t)=0,\tag{11}$$

$$Y_j^{(4)}(y)-2\left(\frac{j\pi}{a}\right)^2Y''_j(y)+\left\{\left(\frac{j\pi}{a}\right)^4+\frac{1}{D(y)}[k-m(y)\omega_j^2]\right\}Y_j(y)=0\cdot\tag{12}$$

式中, ω_j 为板的第j 阶固有频率•

方程(11)之通解为

$$T_j(t)=T_j(0)\cos\omega_jt+\frac{\dot{T}_j(0)}{\omega_j}\sin\omega_jt,\tag{13}$$

式中, $T_j(0)$、$\dot{T}_j(0)$ 为函数 $T_j(t)$ 的初参数, 可由振动的初始条件确定•

方程(12)之通解为

$$\begin{aligned}
Y_j(y)=\ &Y_j(0)\sum_{i=1}^{n}(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0)W^{(i-1)}\phi_1(\alpha_1,\beta_1,y)+\\
&Y'_j(0)\sum_{i=1}^{n}(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0)W^{(i-1)}\phi_2(\alpha_1,\beta_1,y)+\\
&Y''_j(0)\sum_{i=1}^{n}(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0)W^{(i-1)}\phi_3(\alpha_1,\beta_1,y)+\\
&Y'''_j(0)\sum_{i=1}^{n}(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0)W^{(i-1)}\phi_4(\alpha_1,\beta_1,y),
\end{aligned}\tag{14}$$

式中, $Y_j(0)$、$Y'_j(0)$、$Y''_j(0)$、$Y'''_j(0)$ 为函数 $Y_j(y)$ 的初参数, 可由板的边界条件确定; $\phi_1\sim\phi_4$ 为影响函数, 其定义与分段常数 α_i, β_i 及 $2(j\pi/a)^2$, $(j\pi/a)^4+(k-m_i\omega_j^2)/D_i$, $i=1,2,\cdots,n$, 有关• 在下列影响函数及其微分关系的讨论中, $i=1,2,\cdots,n$•

1. $(j\pi/a)^4+(k-m_i\omega_j^2)/D_i>0$

1) $m_i\omega_j^2>k$

令 $\alpha_i=(j\pi/a)^2$, $\beta_i=\sqrt{(m_i\omega_j^2-k)/D_i}$, 则有

$$\left.\begin{aligned}
\phi_1(\alpha_i,\beta_i,y-y_{i-1}) &= [(\beta_i-\alpha_i)\,\mathrm{ch}\sqrt{\alpha_i+\beta_i}(y-y_{i-1})+\\
&\quad(\alpha_i+\beta_i)\,\mathrm{ch}\sqrt{\alpha_i-\beta_i}(y-y_{i-1})]/2\beta_i,\\
\phi_2(\alpha_i,\beta_i,y-y_{i-1}) &= \frac{1}{2\beta_i}\left[\frac{\beta_i-\alpha_i}{\sqrt{\alpha_i+\beta_i}}\mathrm{sh}\sqrt{\alpha_i+\beta_i}(y-y_{i-1})+\right.\\
&\quad\left.\frac{\alpha_i+\beta_i}{\sqrt{\alpha_i-\beta_i}}\mathrm{sh}\sqrt{\alpha_i-\beta_i}(y-y_{i-1})\right],\\
\phi_3(\alpha_i,\beta_i,y-y_{i-1}) &= [\mathrm{ch}\sqrt{\alpha_i+\beta_i}(y-y_{i-1})-\\
&\quad\mathrm{ch}\sqrt{\alpha_i-\beta_i}(y-y_{i-1})]/2\beta_i,\\
\phi_4(\alpha_i,\beta_i,y-y_{i-1}) &= \frac{1}{2\beta_i}\left[\frac{1}{\sqrt{\alpha_i+\beta_i}}\mathrm{sh}\sqrt{\alpha_i+\beta_i}(y-y_{i-1})-\right.\\
&\quad\left.\frac{1}{\sqrt{\alpha_i-\beta_i}}\mathrm{sh}\sqrt{\alpha_i-\beta_i}(y-y_{i-1})\right]\cdot
\end{aligned}\right\}\qquad(15)$$

2) $m_i\omega_j^2 = k$

令 $\alpha_i^2-\beta_i^2 = 2(j\pi/a)^2,\quad(\alpha_i^2+\beta_i^2)^2=(j\pi/a)^4+(k-m_i\omega_j^2)/D_i$,
由于 $m_i\omega_j^2=k$, 故 $\alpha_i=j\pi/a$, $\beta_i=0$, 因而有

$$\left.\begin{aligned}
\phi_1(\alpha_i,\beta_i,y-y_{i-1}) &= -\frac{1}{2}[\alpha_i(y-y_{i-1})\mathrm{sh}\,\alpha_i(y-y_{i-1})-2\mathrm{ch}\,\alpha_i(y-y_{i-1})],\\
\phi_2(\alpha_i,\beta_i,y-y_{i-1}) &= \frac{1}{2\alpha_i}[3\mathrm{sh}\,\alpha_i(y-y_{i-1})-\alpha_i(y-y_{i-1})\mathrm{ch}\,\alpha_i(y-y_{i-1})],\\
\phi_3(\alpha_i,\beta_i,y-y_{i-1}) &= \frac{1}{2\alpha_i}(y-y_{i-1})\mathrm{sh}\,\alpha_i(y-y_{i-1}),\\
\phi_4(\alpha_i,\beta_i,y-y_{i-1}) &= -\frac{1}{2\alpha_i^3}[\mathrm{sh}\,\alpha_i(y-y_{i-1})-\alpha_i(y-y_{i-1})\mathrm{ch}\,\alpha_i(y-y_{i-1})]\cdot
\end{aligned}\right\}\qquad(16)$$

3) $m_i\omega_j^2 < k$

令 $\alpha_i^2-\beta_i^2=(j\pi/a)^2,\ (\alpha_i^2+\beta_i^2)^2=(j\pi/a)^4+(k-m_i\omega_j^2)/D_i$,

即 $$\alpha_i=\sqrt{\frac{1}{2}\left[\left(\frac{j\pi}{a}\right)^2+\sqrt{\left(\frac{j\pi}{a}\right)^4+\frac{1}{D_i}(k-m_i\omega_j^2)}\right]},$$

$$\beta_i=\sqrt{\frac{1}{2}\left[-\left(\frac{j\pi}{a}\right)^2+\sqrt{\left(\frac{j\pi}{a}\right)^4+\frac{1}{D_i}(k-m_i\omega_j^2)}\right]},$$ 则有

$$\left.\begin{aligned}
\phi_1(\alpha_i,\beta_i,y-y_{i-1}) &= ((\beta_i^2-\alpha_i^2)/2\alpha_i\beta_i)\mathrm{sh}\,\alpha_i(y-y_{i-1})\sin\beta_i(y-y_{i-1})+\\
&\quad\mathrm{ch}\,\alpha_i(y-y_{i-1})\cos\beta_i(y-y_{i-1}),\\
\phi_2(\alpha_i,\beta_i,y-y_{i-1}) &= \frac{1}{2(\alpha_i^2+\beta_i^2)}\left[\frac{3\alpha_i-\beta_i}{\alpha_i}\mathrm{sh}\,\alpha_i(y-y_{i-1})\cos\beta_i(y-y_{i-1})+\right.\\
&\quad\left.((3\beta_i^2-\alpha_i^2)/\beta_i)\mathrm{ch}\,\alpha_i(y-y_{i-1})\sin\beta_i(y-y_{i-1})\right],\\
\phi_3(\alpha_i,\beta_i,y-y_{i-1}) &= (1/2\alpha_i\beta_i)\mathrm{sh}\,\alpha_i(y-y_{i-1})\sin\beta_i(y-y_{i-1}),\\
\phi_4(\alpha_i,\beta_i,y-y_{i-1}) &= -\frac{1}{2(\alpha_i^2+\beta_i^2)}\left[\frac{1}{\alpha_i}\mathrm{sh}\,\alpha_i(y-y_{i-1})\cos\beta_i(y-y_{i-1})-\right.\\
&\quad\left.\frac{1}{\beta_i}\mathrm{ch}\,\alpha_i(y-y_{i-1})\sin\beta_i(y-y_{i-1})\right]\cdot
\end{aligned}\right\}\qquad(17)$$

2. $\left(\frac{j\pi}{a}\right)^4 + \frac{1}{D_i}(k - m_i\omega_j^2) < 0$

此时，因 $\left[2\left(\frac{j\pi}{a}\right)^2\right]^2 + 4\left[\frac{1}{D_i}(m_i\omega_j^2 - k) - \left(\frac{j\pi}{a}\right)^4\right] > 0$，故影响函数只有一种形式．

令 $\alpha_i = \left(\frac{j\pi}{a}\right)^2$, $\beta_i = \sqrt{\frac{1}{D_i}(m_i\omega_j^2 - k)}$，则有

$$
\left.\begin{aligned}
\phi_1(\alpha_i, \beta_i, y - y_{i-1}) &= \frac{1}{2\beta_i}\Big[(\beta_i - \alpha_i)\,\mathrm{ch}\sqrt{\alpha_i + \beta_i}(y - y_{i-1}) + \\
&\qquad (\alpha_i + \beta_i)\cos\sqrt{\beta_i - \alpha_i}(y - y_{i-1})\Big], \\
\phi_2(\alpha_i, \beta_i, y - y_{i-1}) &= \frac{1}{2\beta_i}\left[\frac{\beta_i - \alpha_i}{\sqrt{\alpha_i + \beta_i}}\,\mathrm{sh}\sqrt{\alpha_i + \beta_i}(y - y_{i-1}) + \right. \\
&\qquad \left.\frac{\alpha_i + \beta_i}{\sqrt{\beta_i - \alpha_i}}\sin\sqrt{\beta_i - \alpha_i}(y - y_{i-1})\right], \\
\phi_3(\alpha_i, \beta_i, y - y_{i-1}) &= \frac{1}{2\beta_i}\Big[\mathrm{ch}\sqrt{\alpha_i + \beta_i}(y - y_{i-1}) - \\
&\qquad \cos\sqrt{\beta_i - \alpha_i}(y - y_{i-1})\Big], \\
\phi_4(\alpha_i, \beta_i, y - y_{i-1}) &= \frac{1}{2\beta_i}\left[\frac{1}{\sqrt{\alpha_i + \beta_i}}\,\mathrm{sh}\sqrt{\alpha_i + \beta_i}(y - y_{i-1}) - \right. \\
&\qquad \left.\frac{1}{\sqrt{\beta_i - \alpha_i}}\sin\sqrt{\beta_i - \alpha_i}(y - y_{i-1})\right].
\end{aligned}\right\} \tag{18}
$$

以上四种情况下的影响函数之间的微分关系，可统一用下式表示

$$
\left.\begin{aligned}
\phi_1'(\alpha_i, \beta_i, y - y_{i-1}) &\equiv -\left[\left(\frac{j\pi}{a}\right)^4 + \frac{1}{D_i}(k - m_i\omega_j^2)\right]\phi_4(\alpha_i, \beta_i, y - y_{i-1}), \\
\phi_2'(\alpha_i, \beta_i, y - y_{i-1}) &= \phi_1(\alpha_i, \beta_i, y - y_{i-1}), \\
\phi_3'(\alpha_i, \beta_i, y - y_{i-1}) &= \phi_2(\alpha_i, \beta_i, y - y_{i-1}) + 2\left(\frac{j\pi}{a}\right)^2\phi_4(\alpha_i, \beta_i, y - y_{i-1}), \\
\phi_4'(\alpha_i, \beta_i, y - y_{i-1}) &= \phi_3(\alpha_i, \beta_i, y - y_{i-1}).
\end{aligned}\right\} \tag{19}
$$

W 算子的定义为

$$
W\phi_q(\alpha_i, \beta_i, y - y_{i-1}) = \left[\phi_q(\alpha_i, \beta_i, b_i)\quad \phi_q'(\alpha_i, \beta_i, b_i)\quad \frac{D_i}{D_{i+1}}\phi_q''(\alpha_i, \beta_i, b_i)\quad \frac{D_i}{D_{i+1}}\phi_q'''(\alpha_i, \beta_i, b_i)\right]\cdot
\begin{bmatrix}
\phi_1(\alpha_{i+1}, \beta_{i+1}, y - y_i) \\
\phi_2(\alpha_{i+1}, \beta_{i+1}, y - y_i) \\
\phi_3(\alpha_{i+1}, \beta_{i+1}, y - y_i) \\
\phi_4(\alpha_{i+1}, \beta_{i+1}, y - y_i)
\end{bmatrix}, \tag{20}
$$

式中，$q = 1, 2, 3, 4$; $b_i = y_i - y_{i-1}$, $i = 1, 2, \dots, n$．

W 算子的运算规则为：$W^{(n)}\phi_q = WW^{(n-1)}\phi_q$, $W^{(1)}\phi_q = W\phi_q$, $W^{(0)}\phi_q = \phi_q$．

将式(14)代入式(10)，得振型函数；将振型函数与式(13)一起代入式(9)，便得方程(8)之通解，亦即板自由振动的挠度

$$
w(x, y, t) = \sum_{j=1}^{\infty} W_j(x, y)\left[T_j(0)\cos\omega_j t + \frac{\dot{T}_j(0)}{\omega_j}\sin\omega_j t\right]. \tag{21}
$$

1.2 固有频率的求法

由式(12)可得板的固有频率表达式

$$\omega_j^2 = \left\{ \frac{1}{Y_j(y)} \left[Y_j^{(4)}(y) - 2\left(\frac{j\pi}{a}\right)^2 Y_j''(y) \right] + \left(\frac{j\pi}{a}\right)^4 \right\} \sum_{i=1}^{n} \frac{D_i}{m_i} (\langle y - y_{i-1}\rangle^0 - \langle y - y_i\rangle^0) + k \sum_{i=1}^{n} \frac{1}{m_i} (\langle y - y_{i-1}\rangle^0 - \langle y - y_i\rangle^0) \tag{22}$$

具体求频率时，可用表1给出的频率方程[2]。

表 1 文克尔地基上阶梯式矩形薄板的频率方程

$y=0, y=b$ 两边支承	固定	简支	自由	弹性支承	弹性嵌固
频率方程	$\begin{vmatrix} A_3 & A_4 \\ A_3' & A_4' \end{vmatrix} = 0$	$\begin{vmatrix} A_2 & A_4 \\ A_2'' & A_4'' \end{vmatrix} = 0$	$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0$	$\begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix} = 0$	$\begin{vmatrix} A_2 & A_3 & A_4 \\ C_2 & C_3 & C_4 \\ d_2 & d_3 & d_4 \end{vmatrix} = 0$

表中

$$A_q = [W^{(n-1)} \psi_q(\alpha_1, \beta_1, y)]_{y=b},$$

$$a_{1r} = b_{1r} = [\psi_r(\alpha_1, \beta_1, y)]''_{y=0} - \mu\left(\frac{j\pi}{a}\right)^2 [\psi_r(\alpha_1, \beta_1, y)]_{y=0},$$

$$a_{2r} = [\psi_r(\alpha_1, \beta_1, y)]'''_{y=0} - (2-\mu)\left(\frac{j\pi}{a}\right)^2 [\psi_r(\alpha_1, \beta_1, y)]'_{y=0},$$

$$a_{3r} = b_{3r} = [W^{(n-1)} \psi_r(\alpha_1, \beta_1, y)]''_{y=b} - \mu\left(\frac{j\pi}{a}\right)^2 [W^{(n-1)} \psi_r(\alpha_1, \beta_1, y)]_{y=b},$$

$$a_{4r} = [W^{(n-1)} \psi_r(\alpha_1, \beta_1, y)]'''_{y=b} - (2-\mu)\left(\frac{j\pi}{a}\right)^2 [W^{(n-1)} \psi_r(\alpha_1, \beta_1, y)]'_{y=b},$$

$$b_{2r} = k_w[\psi_r(\alpha_1, \beta_1, y)]_{y=0} - D_1[\psi_r(\alpha_1, \beta_1, y)]'''_{y=0} + D_1(2-\mu)\left(\frac{j\pi}{a}\right)^2 [\psi_r(\alpha_1, \beta_1, y)]'_{y=0},$$

$$b_{4r} = k_w[W^{(n-1)} \psi_r(\alpha_1, \beta_1, y)]_{y=b} - D_n[W^{(n-1)} \psi_r(\alpha_1, \beta_1, y)]'''_{y=b} + D_n(2-\mu)\left(\frac{j\pi}{a}\right)^2 [W^{(n-1)} \psi_r(\alpha_1, \beta_1, y)]'_{y=b},$$

$$C_q = k_\phi[\psi_q(\alpha_1, \beta_1, y)]'_{y=0} + D_1[\psi_q(\alpha_1, \beta_1, y)]''_{y=0} - \mu D_1\left(\frac{j\pi}{a}\right)^2 [\psi_q(\alpha_1, \beta_1, y)]_{y=0},$$

$$d_q = k_\phi[W^{(n-1)} \psi_q(\alpha_1, \beta_1, y)]'_{y=b} + D_n[W^{(n-1)} \psi_q(\alpha_1, \beta_1, y)]''_{y=b} - \mu D_n\left(\frac{j\pi}{a}\right)^2 [W^{(n-1)} \psi_q(\alpha_1, \beta_1, y)]_{y=b},$$

其中：$q = 2, 3, 4$; $r = 1, 2, 3, 4$; “$'$”、“$''$”、“$'''$”分别表示相应项对 y 的1, 2, 3阶导数; k_w 与 k_ϕ 分别为垂直线弹簧和螺旋弹簧的弹簧常数[2]。

频率方程一般为超越方程，用数值解法可求得板的各阶固有频率。

1.3 算例

图2所示文克尔地基上的矩形薄板，四边简支，边长分别为 $a = 3$m, $b = 2$m。板沿 y 方向

分为二级阶梯, 阶梯衔接处坐标为 $y = b/2$。 $m_1 = 381\text{kg/m}^2$, $m_2 = 480\text{kg/m}^2$, $D_2/2 = D_1 = 1.37\times 10^7\text{N.m}^2/\text{m}$, $k = 2\times 10^7\text{N/m}^3$。 求板自由振动的挠度和一、二阶固有频率。

解 由板的边界条件与式(14), 有

$$Y_j(y) = Y_j'(0)[(1-\langle y-1\rangle^0)\phi_2(\alpha_1,\beta_1,y) + \langle y-1\rangle^0 W\phi_2(\alpha_1,\beta_1,y)] + Y_j^{\circleddash}(0)[(1-\langle y-1\rangle^0)\phi_4(\alpha_1,\beta_1,y) + \langle y-1\rangle^0 W\phi_4(\alpha_1,\beta_1,y)],$$

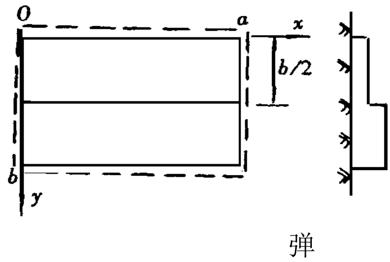

图 2 文克尔地基上二级阶梯式矩形薄板

由题意, 有 $\left(\frac{j\pi}{a}\right)^4 + \frac{1}{D_i}(k - m_i\omega_j^2) < 0$, $i = 1, 2$, 故 $\phi_1 \sim \phi_4$ 由式(18) 给出, 其中

$$\alpha_1 = \left(\frac{j\pi}{a}\right)^2, \quad \beta_1 = \sqrt{\frac{1}{D_1}(m_1\omega_j^2 - k)};$$

$$\alpha_2 = \left(\frac{j\pi}{a}\right)^2, \quad \beta_2 = \sqrt{\frac{1}{D_2}(m_2\omega_j^2 - k)}.$$

频率方程为

$$\begin{vmatrix} [W\phi_2(\alpha_1,\beta_1,y)]_{y=b} & [W\phi_4(\alpha_1,\beta_1,y)]_{y=b} \\ [W\phi_2(\alpha_1,\beta_1,y)]''_{y=b} & [W\phi_4(\alpha_1,\beta_1,y)]''_{y=b} \end{vmatrix} = 0.$$

将各影响函数分别展开为泰勒级数, 取前两项代入上式, 可得板的一、二阶固有频率

$$\omega_1 = 734.5108 = \frac{5.9043^2}{a^2}\sqrt{\frac{D_1}{m_1}}, \quad \omega_2 = 1540.2048 = \frac{8.5499^2}{a^2}\sqrt{\frac{D_1}{m_1}}.$$

其中, a, D_1, m_1 已由题目给定、相应于一阶固有频率 ω_1, 有 $\beta_1 = 3.6802m^{-2}$, $\beta_2 = 2.9532m^{-2}$; 相应于二阶固有频率 ω_2, 有 $\beta_1 = 8.0320m^{-2}$, $\beta_2 = 6.3896m^{-2}$。 又可求得 $\alpha_1 = \alpha_2 = 1.0955m^{-2}$。 由此便可求出 $Y_j(y)$, 进而求得振型函数和板自由振动的挠度。 此处, $Y_j'(0)$、$Y_j^{\circleddash}(0)$ 为任意常数。

2 强迫振动

2.1 动力响应

在动载荷 $q(x, y, t)$ 作用下, 板强迫振动的微分方程为

$$D(y)\nabla^4 w(x,y,t) + kw(x,y,t) + m(y)\frac{\partial^2 w(x,y,t)}{\partial t^2} = q(x,y,t). \tag{23}$$

设方程(23)之通解为

$$w(x,y,t) = \sum_{j=1}^{\infty} W_j(x,y)T_j(t), \tag{24}$$

并将载荷表为

$$q(x,y,t) = \sum_{j=1}^{\infty} W_j(x,y)P_j(t), \tag{25}$$

式中, 振型函数 $W_j(x, y)$ 由式(10) 给出。 将式(24) ~ (25) 代入方程(23), 用式(12) 化简后, 可得

$$\ddot{T}_j(t) + \omega_j^2 T_j(t) = \frac{P_j(t)}{m(y)} \quad (j = 1, 2, \cdots), \tag{26}$$

方程(26)之通解为

$$T_j(t)=T_j(0)\cos\omega_j t+\frac{\dot{T}_j(0)}{\omega_j}\sin\omega_j t+T_{j0}(t)\cdot \tag{27}$$

式中的 $T_j(0)$、$\dot{T}_j(0)$ 与式(13)中的意义相同, $T_{j0}(t)$ 为方程(26)之非齐次特解

$$T_{j0}(t)=\frac{1}{m(y)\omega_j}\int_0^t P_j(\tau)\sin\omega_j(t-\tau)\,\mathrm{d}\tau\cdot \tag{28}$$

将式(27)代入式(24),便得方程(23)之通解,亦即板的动力响应

$$w(x,y,t)=\sum_{j=1}^{\infty}W_j(x,y)\left[T_j(0)\cos\omega_j t+\frac{\dot{T}_j(0)}{\omega_j}\sin\omega_j t\right]+w_0(x,y,t)\cdot \tag{29}$$

2.2 强迫振动响应

式(29)中的 $w_0(x,y,t)$ 为板的强迫振动响应

$$w_0(x,y,t)=\sum_{j=1}^{\infty}\frac{W_j(x,y)}{m(y)\omega_j}\int_0^t P_j(\tau)\sin\omega_j(t-\tau)\,\mathrm{d}\tau\cdot \tag{30}$$

在不同形式的横向载荷作用下,板的强迫振动响应,即方程(23)之特解的形式不同,分述如下[1]•

2.2.1 集中力

设在板上点 $K(x_K,y_K)(y_{r-1}\leqslant y_K<y_r)$ 处作用有一横向集中力 $P(t)$(图3),则

$$w_0(x,y,t)=\sum_{j=1}^{\infty}\frac{2}{am_r\omega_j}\delta(y-y_k)\times\sin\frac{j\pi x_k}{a}\sin\frac{j\pi x}{a}\int_0^t P(\tau)\sin\omega_j(t-\tau)\,\mathrm{d}\tau\cdot \tag{31}$$

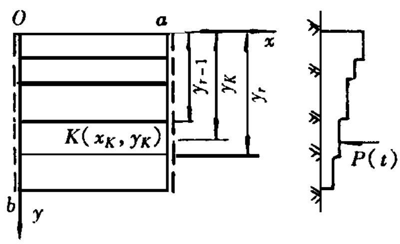

图3 受集中力作用的文克尔地基上阶梯式矩形薄板

2.2.2 与 x 轴平行的线均布载荷

设在板面的直线 $y=y_k(y_{r-1}\leqslant y_k<y_r)$ 上作用有集度为 $q(t)$ 的横向线均布载荷,则

$$w_0(x,y,t)=\sum_{j=1}^{\infty}\frac{2}{j\pi m_r\omega_j}\delta(y-y_k)(1-\cos j\pi)\times\sin\frac{j\pi x}{a}\int_0^t q(\tau)\sin\omega_j(t-\tau)\,\mathrm{d}\tau\cdot \tag{32}$$

2.2.3 与 x 轴平行的线均布力偶

设在板面的直线 $y=y_k(y_{r-1}\leqslant y_k<y_r)$ 上布满绕其旋转、集度为 $M(t)$ 的线均布力偶,则

$$w_0(x,y,t)=\sum_{j=1}^{\infty}\frac{2}{j\pi m_r\omega_j}\delta'(y-y_k)(1-\cos j\pi)\sin\frac{j\pi x}{a}\int_0^t M(\tau)\sin\omega_j(t-\tau)\,\mathrm{d}\tau\cdot \tag{33}$$

2.2.4 局部面均布载荷

设在板面的区域 $[x_k,y_k;x_{k+1},y_{k+1}](y_{r-1}\leqslant y_k<y_r,y_{s-1}\leqslant y_{k+1}<y_s)$ 内作用有垂直于板面、集度为 $q(t)$ 的面均布载荷,则

$$m_1(y)=m_r(\langle y-y_k\rangle^0-\langle y-y_r\rangle^0)+\sum_{i=r+1}^{s-1}m_i(\langle y-y_{i-1}\rangle^0-\langle y-y_i\rangle^0)+m_s(\langle y-y_{s-1}\rangle^0-\langle y-y_{k+1}\rangle^0)$$

$$w_0(x, y, t) = \sum_{j=1}^{\infty} \frac{2}{j\pi m_1(y)\omega_j}\left(\langle y - y_k\rangle^0 - \langle y - y_{k+1}\rangle^0\right)\left(\cos\frac{j\pi x_k}{a} - \cos\frac{j\pi x_{k+1}}{a}\right) \times \sin\frac{j\pi x}{a}\int_0^t q(\tau)\sin\omega_j(t-\tau)\,\mathrm{d}\tau \tag{34}$$

2.2.5 沿 y 方向的三角形面分布载荷

设在板面的区域 $[0, a; y_k, b]$ $(y_{r-1} \leqslant y_k < y_r)$ 内作用有沿 y 方向的横向三角形面分布载荷, $y = y_k$ 处集度为 0, $y = b$ 处集度为 $q(t)$, 载荷集度沿 x 方向不变, 则

$$m_2(y) = m_r(\langle y - y_k\rangle^0 - \langle y - y_r\rangle^0) + \sum_{i=r+1}^{n} m_i(\langle y - y_{i-1}\rangle^0 - \langle y - y_i\rangle^0),$$

$$w_0(x, y, t) = \sum_{j=1}^{\infty} \frac{2}{j\pi(b - y_k)m_2(y)\omega_j}\langle y - y_k\rangle^1(1 - \cos j\pi)\sin\frac{j\pi x}{a}\int_0^t q(\tau)\sin\omega_j(t-\tau)\,\mathrm{d}\tau \tag{35}$$

2.3 算例

在 $t = t_0$ 瞬时, 前例所给板面的直线 $y = \frac{2}{3}b$ 上作用有突加的线均布载荷, 集度为 q_0(图 4), 求板的动力响应•

解 将载荷用奇异函数表为

$$q(x, y, t) = q_0\langle x\rangle^0\delta\left(y - \frac{2}{3}b\right)\delta(t - t_0)•$$

强迫振动响应为

$$w_0(x, y, t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{4q_0}{j\pi bm_2\omega_j}(1 - \cos j\pi) \times \sin\frac{2k\pi}{3}\sin\frac{j\pi x}{a}\sin\frac{k\pi y}{b}\sin\omega_j(t - t_0)•$$

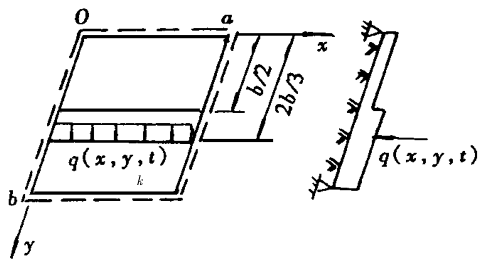

图 4 受突加线均布载荷作用的文克尔地基上二级阶梯式矩形薄板

图 5 与图 6 分别为板上沿直线 $x = a/2$ 和 $y = 2b/3$ 上的点的强迫振动响应曲线, 图中 w_0 的单位为 $\frac{4q_0}{\pi bm_2\omega_1}\sin\omega_1(t - t_0)$•

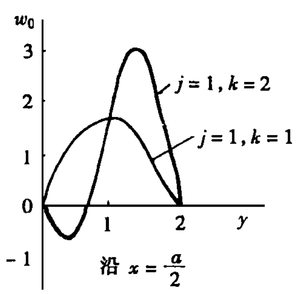

图 5 沿直线 $x = a/2$ 的板的强迫振动响应曲线

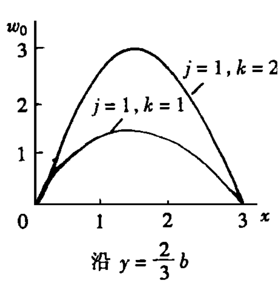

图 6 沿直线 $y = 2b/3$ 的板的强迫振动响应曲线

初始条件相同时, 将上式与前例所得板自由振动的挠度叠加, 便得其动力响应•

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Vibrations of Stepped Rectangular Thin Plates on Winkler's Foundation

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Abstract: Differential equations of free/forced vibrations of stepped rectangular thin plates on Winkler's foundation are established by using singular functions, and their general solutions are also solved for expression of vibration mode function and frequency equations on usual supports derived with W operator, as well as forced responses of such plates under different_type loads discussed with Fourier expansion of generalized functions.

Key words: Winkler's foundation; stepped rectangular thin plate; free vibration; forced response