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解三维热传导方程的一种高精度的显格式*

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摘要: 对解三维热传导方程利用待定参数方法构造出一种精度 $O(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4)$ 的高精度易于计算的显式差分格式, 并给出了其稳定性, 通过数值例子可见其精度较其它方法提高 2~3 位有效数字·

关 键 词: 三维热传导方程; 隐式差分格式; 显式差分格式; 局部截断误差; 绝对稳定性; 条件稳定性

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引　言

从实际问题中, 可以归纳出许多求解热传导方程问题· 目前, 关于一维和二维热传导方程的数值解法较多, 如文[1]和[2]等, 但对三维的情形研究较少, 因此我们有必要给出三维热传导方程的一种好的算法, 因为利用隐式方法求解二维以上的问题时, 计算起来相当繁琐, 所以我们必须研制好的显式差分格式· 文[3]给出了求解此问题的误差为 $O(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2)$ 的绝对稳定的显格式, 精度较低, 本文构造出误差为 $O(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4)$ 的便于计算的三层显格式, 通过数值实例, 可见该格式较[3]的计算结果提高二位以上数字·

1　差分格式的构造

考虑区域 $D:\{0\leqslant x,y,z\leqslant\pi, 0\leqslant t\leqslant T\}$ 上的初边值问题:

$$\frac{\partial u}{\partial t}=\sigma\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\qquad(\sigma>0),\tag{1}$$

$$u(x,y,z,0)=g(x,y,z),\tag{2}$$

$$u(0,y,z,t)=f_1(y,z,t),\tag{3}$$

$$u(\pi,y,z,t)=f_2(y,z,t),\tag{4}$$

$$u(x,0,z,t)=f_3(x,z,t),\tag{5}$$

$$u(x,\pi,z,t)=f_4(x,z,t),\tag{6}$$

$$u(x,y,0,t)=f_5(x,y,t),\tag{7}$$

$$u(x,y,\pi,t)=f_6(x,y,t)\cdot\tag{8}$$

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我们取时间步长 $\Delta t = \tau$, 空间步长 $\Delta x = \Delta y = \Delta z = h$, 剖分区域 D, 节点为 (x_m, y_n, z_p, t_k), 简记为 (m, n, p, k) 在 (m, n, p, k) 点从Hermite方法出发构造三层对称含参数的差分格式如下:

$$\begin{aligned}&\Delta_t u_{mnp}^k + \alpha_1 \nabla_t u_{mnp}^k + \alpha_2 \nabla_t (u_{m+1,n,p}^k + u_{m-1,n,p}^k + u_{m,n+1,p}^k + u_{m,n-1,p}^k + u_{m,n,p+1}^k + \\ &u_{m,n,p-1}^k) + \alpha_3 \nabla_t (u_{m+1,n-1,p+1}^k + u_{m+1,n+1,p+1}^k + u_{m-1,n-1,p+1}^k + u_{m-1,n+1,p+1}^k + \\ &u_{m+1,n-1,p-1}^k + u_{m+1,n+1,p-1}^k + u_{m-1,n-1,p-1}^k + u_{m-1,n+1,p-1}^k) = \sigma\alpha_4 (\delta_x^2 + \delta_y^2 + \delta_z^2) u_{mnp}^k + \\ &\sigma\alpha_5 (\delta_x^2 + \delta_y^2 + \delta_z^2) u_{mnp}^{k-1} + \sigma\alpha_6 [\delta_x^2 (u_{m,n-1,p}^k + u_{m,n+1,p}^k + u_{m,n,p+1}^k + u_{m,n,p-1}^k + \\ &u_{m,n-1,p+1}^k + u_{m,n+1,p+1}^k + u_{m,n-1,p-1}^k + u_{m,n+1,p-1}^k) + \delta_y^2 (u_{m+1,n,p}^k + u_{m-1,n,p}^k + u_{m,n,p+1}^k + \\ &u_{m,n,p-1}^k + u_{m+1,n,p+1}^k + u_{m+1,n,p-1}^k + u_{m-1,n,p+1}^k + u_{m-1,n,p-1}^k) + \delta_z^2 (u_{m,n-1,p}^k + u_{m,n+1,p}^k + \\ &u_{m+1,n,p}^k + u_{m-1,n,p}^k + u_{m+1,n-1,p}^k + u_{m+1,n+1,p}^k + u_{m-1,n+1,p}^k + u_{m-1,n-1,p}^k)],\end{aligned} \tag{9}$$

其中 Δ_t, ∇_t 是关于 t 的一阶向前, 向后差商, δ_x^2, δ_y^2, δ_z^2 是关于 x, y, z 的二阶中心差商, α_j $(j = 1, 2, \dots, 6)$ 是待定参数, 适当选择这些参数, 可使差分方程(9) 具有尽可能高阶的截断误差, 同时还有较好的稳定性•

将(9) 中各节点上的 u 在节点 (m, n, p, k) 处展开的Taylor 级数代入, 并利用方程式(1), 经整理可得:

$$\begin{aligned}&(1 + \alpha_1 + 6\alpha_2 + 8\alpha_3) \frac{\partial u}{\partial t} + \left(\frac{1}{2} - \frac{\alpha_1}{2} - 3\alpha_2 - 4\alpha_3\right) \tau \frac{\partial^2 u}{\partial t^2} + (\alpha_2 + 4\alpha_3) \sigma h^2 \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \right. \\ &\left.\frac{\partial^4 u}{\partial z^4}\right) + 2(\alpha_2 + 4\alpha_3) \sigma h^2 \left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2} + \frac{\partial^4 u}{\partial y^2 \partial z^2}\right) = (\alpha_4 + \alpha_5 + 8\alpha_6) \frac{\partial u}{\partial t} + \\ &\left(\frac{\alpha_4}{12} + \frac{\alpha_5}{12} + \frac{2\alpha_6}{3}\right) \sigma h^2 \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial z^4}\right) - \alpha_5 \tau \frac{\partial^2 u}{\partial t^2} + 6\alpha_6 \sigma h^2 \left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2} + \right. \\ &\left.\frac{\partial^4 u}{\partial y^2 \partial z^2}\right) + O(\tau^2 + \tau h^2 + h^4)\bullet\end{aligned}$$

为使(9) 误差达到 $O(\tau^2 + \tau h^2 + h^4)$, 只需下列方程成立:

$$\left.\begin{aligned}&1 + \alpha_1 + 6\alpha_2 + 8\alpha_3 = \alpha_4 + \alpha_5 + 8\alpha_6, \\ &\frac{1}{2} - \frac{\alpha_2}{2} - 3\alpha_2 - 4\alpha_3 = -\alpha_5, \\ &\alpha_2 + 4\alpha_3 = \frac{\alpha_4}{12} + \frac{\alpha_5}{12} + \frac{2\alpha_6}{3}, \\ &2(\alpha_2 + 4\alpha_3) = 6\alpha_6\bullet\end{aligned}\right\}$$

解得

$$\left.\begin{aligned}&\alpha_1 = -1 + 16\alpha_3 + 18\alpha_6, \\ &\alpha_2 = 3\alpha_6 - 4\alpha_3, \\ &\alpha_4 = 10\alpha_6 + 1, \\ &\alpha_5 = 18\alpha_6 - 1\bullet\end{aligned}\right\}$$

令 $\alpha_3 = \xi$, $\alpha_6 = \eta$, 且将 α_1, α_2, α_4, α_5 的表达式代入(9), 利用 Δ_t, ∇_t, δ_x^2, δ_y^2, δ_z^2 的定义, 可写成截断误差为 $O(\tau^2 + \tau h^2 + h^4)$ 的含有参数 ξ, η 的三层显格式:

$$u_{mnp}^{k+1} = W_1(u_{mnp}^k) + W_2(u_{mnp}^{k-1}), \tag{10}$$

其中

$$W_1(u_{mnp}^k) = (2 - 6r - 16\xi - 18\eta - 60r\eta) u_{mnp}^k + (r + 4\xi - 3\eta + 6r\eta)(u_{m+1,n,p}^k +$$

$$u^k_{m-1,n,p}+u^k_{m,n+1,p}+u^k_{m,n-1,p}+u^k_{m,n,p+1}+u^k_{m,n,p-1})+$$
$$(3r\eta-\xi)(u^k_{m+1,n-1,p+1}+u^k_{m-1,n-1,p+1}+u^k_{m+1,n+1,p+1}+u^k_{m-1,n+1,p+1}+$$
$$u^k_{m+1,n-1,p-1}+u^k_{m-1,n-1,p-1}+u^k_{m+1,n+1,p-1}+u^k_{m-1,n+1,p-1}),$$
$$W_2(u^{k-1}_{mnp})=(-1+6r+16\xi+18\eta-108r\eta)u^{k-1}_{mnp}+(-r+$$
$$18r\eta+3\eta-4\xi)(u^{k-1}_{m+1,n,p}+u^{k-1}_{m-1,n,p}+u^{k-1}_{m,n+1,p}+u^{k-1}_{m,n-1,p}+$$
$$u^{k-1}_{m,n,p+1}+u^{k-1}_{m,n,p-1})+\xi(u^{k-1}_{m+1,n-1,p+1}+u^{k-1}_{m-1,n-1,p+1}+$$
$$u^{k-1}_{m+1,n+1,p+1}+u^{k-1}_{m-1,n+1,p+1}+u^{k-1}_{m+1,n-1,p-1}+u^{k-1}_{m-1,n-1,p-1}+$$
$$u^{k-1}_{m+1,n+1,p-1}+u^{k-1}_{m-1,n+1,p-1}),$$

其中网比 $r=\sigma\tau/h^2$, $m,\ n,\ p=1,\ 2,\ \dots,\ N-1$, $k=1,\ 2,\ 3,\ \dots$, 又由于 $O(\tau h^2)\leqslant \max\left\{O(\tau^2),O(h^4)\right\}$, 所以 $O(\tau^2+\tau h^2+h^4)=O(\tau^2+h^4)$, 则(10) 为截断误差 $O(\tau^2+h^4)$ 的三层显格式•

2 稳定性分析

为了讨论稳定性, 把(10) 改写成等价的差分方程组:

$$\left.\begin{aligned}u^{k+1}_{mnp}&=W_1(u^k_{mnp})+W_2(u^{k-1}_{mnp}),\\ u^k_{mnp}&=u^k_{mnp}\bullet\end{aligned}\right\}\tag{11}$$

令 $u^k_{mnp}=U^k\mathrm{e}^{\mathrm{i}(m\varphi+n\psi-p\theta)}\qquad(\mathrm{i}=\sqrt{-1})$,

代入(11), 约去 $\mathrm{e}^{\mathrm{i}(m\varphi+n\psi+p\theta)}$, 得

$$\begin{pmatrix}U^{k+1}\\U^k\end{pmatrix}=\begin{pmatrix}M_{11}&M_{12}\\M_{21}&M_{22}\end{pmatrix}\begin{pmatrix}U^k\\U^{k-1}\end{pmatrix},$$

其中 $M_{21}=1, M_{22}=0$,

$$M_{11}=(2-6r-16\xi-18\eta-60r\eta)+(r+4\xi-3\eta+6r\eta)\times$$
$$(6-4S_1)+(3r\eta-\xi)(8-4S_2),$$
$$M_{12}=(-1+6r+16\xi+18\eta-108r\eta)+(3\eta-4\xi+$$
$$18r\eta-r)(6-4S_1)+\xi(8-4S_2),$$
$$S_1=\sin^2\frac{\phi}{2}+\sin^2\frac{\psi}{2}+\sin^2\frac{\theta}{2}\in[0,3],$$
$$S_2=\sin^2\frac{\varphi+\psi+\theta}{2}+\sin^2\frac{\varphi+\psi-\theta}{2}+\sin^2\frac{\varphi-\psi+\theta}{2}+\sin^2\frac{\varphi-\psi-\theta}{2}\in[0,4]\bullet$$

由传播矩阵 $\boldsymbol{M}(S_1,S_2)=\begin{pmatrix}M_{11}&M_{12}\\M_{21}&M_{22}\end{pmatrix}$,

得特征方程

$$\lambda^2-M_{11}\lambda-M_{12}=0\bullet\tag{12}$$

引理 1[4] 特征方程(12) 的根满足 $|\lambda_{1,2}|\leqslant 1$ 的充要条件是

$$|M_{11}|\leqslant 1-M_{12}\leqslant 2\bullet\tag{13}$$

引理 2[4] 差分方程(10) 稳定, 即二阶矩阵族 $M^n(S_1,S_2)(0\leqslant S_1\leqslant 3, 0\leqslant S_2\leqslant 4, n=1, 2,\dots)$ 一致有界的充要条件是

$$1)\ |\lambda_{1,2}|\leqslant 1,\tag{14}$$

$$2)\ 使\ 1-\frac{1}{4}M_{11}^2=M_{11}^2+4M_{12}=0,\tag{15}$$

成立的 (S_1, S_2) 或不存在, 或不属于区域 $[0,3]\times[0,4]$•

定理 当网比 $r<\frac{1}{2}$, 且参数满足

$$0<\eta\leqslant\min\left\{\frac{1-2r}{12+48r}, \frac{r}{3(1+6r)}\right\},$$

$$\min\left\{0, \frac{1-6r+24r\eta}{16}\right\}\geqslant\xi\geqslant\frac{3(1+6r)\eta-r}{4},$$

时, 差分格式(10)稳定•

证明 由引理 1 和引理 2 知, 当 $M_{12}\neq 1$ 时, (15) 式对所有的 S_1 和 S_2 都不成立, 则由(13) 知差分格式(10) 稳定条件为

$$-1+M_{12}\leqslant M_{11}\leqslant 1-M_{12}<2• \tag{16}$$

由 $M_{11}\leqslant 1-M_{12}$, 得 $(8S_1+S_2)\eta\geqslant 0$,

则得 $\eta\geqslant 0$• (17)

又由 $1-M_{12}<2$, 得

$$9\eta>S_1[3(1+6r)\eta-4\xi-r]+\xi S_2• \tag{18}$$

则当

$$\begin{cases}\eta>0, & (19)\\ 3(1+6r)\eta-4\xi-r\leqslant 0, & (20)\\ \xi\leqslant 0, & (21)\end{cases}$$

满足时, (18) 式恒成立•

再由 $-1+M_{12}\leqslant M_{11}$ 可得

$$18\eta+2S_1[r-3(2r+1)\eta+4\xi]+S_2[3r\eta-2\xi]\leqslant 1, \tag{22}$$

则由 $S_1\leqslant 3$, $S_2\leqslant 4$ 得

$$18\eta+6[r-3(2r+1)\eta+4\xi]+4[3r\eta-2\xi]\leqslant 1,$$

得 $6r-24r\eta+16\xi\leqslant 1$• (23)

下面解不等式组:

$$\begin{cases}3(1+6r)\eta-4\xi-r\leqslant 0, & (20)\\ 6r-24r\eta+16\xi\leqslant 1, & (23)\\ \eta>0, & (19)\\ \xi\leqslant 0• & (21)\end{cases}$$

解得

$$r<\frac{1}{2},$$

$$0<\eta\leqslant\min\left\{\frac{1-2r}{12+48r}, \frac{r}{3(1+6r)}\right\},$$

$$\frac{3(1+6r)\eta-r}{4}\leqslant\xi\leqslant\min\left\{0, \frac{1-6r+24r\eta}{16}\right\}•$$

即上述条件满足时, 差分格式(10)稳定•

推论 若三维热传导方程为

$$\frac{\partial u}{\partial t}=\sigma\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+f(x,y,z,t)\qquad(\sigma>0),$$

则差分格式为

$$u_{mnp}^{k+1} = W_1(u_{mnp}^{k}) + W_2(u_{mnp}^{k-1}) + \tau \cdot f_{mnp}^{k}, \tag{24}$$

其精度与稳定性讨论不变·

3 数值例子

例1 对初边值问题

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{1}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (0 < x, y, z < \pi, t > 0),\\
&u(x, y, z, 0) = \sin x \cdot \sin y \cdot \sin z,\\
&u(0, y, z, t) = u(\pi, y, z, t) = 0,\\
&u(x, 0, z, t) = u(x, \pi, z, t) = 0,\\
&u(x, y, 0, t) = u(x, y, \pi, t) = 0\cdot
\end{aligned}\right\} \tag{25}$$

利用(10)格式与文[3]格式分别计算并与真解比较·

如果取 $h = \frac{\pi}{16}$, $\tau = \frac{3\pi^2}{1024}$, $r = \frac{1}{4}$, 取参数 $\xi = -\frac{1}{32}$, $\eta = \frac{1}{48}$, 则差分格式(10)满足稳定条件, 计算到第4层时比较结果如表1·

表1

(m, n, p)	(1, 1, 1)	(2, 2, 2)	(3, 3, 3)	(4, 4, 4)	(5, 5, 5)
真 值	6.614 17E-03	4.992 15E-03	1.527 51E-01	3.149 37E-01	5.120 45E-01
(10) 格 式	6.614 10E-03	4.992 09E-02	1.527 50E-01	3.149 33E-01	5.120 39E-01
文[3] 格 式	6.612 50E-03	4.997 18E-02	1.527 19E-01	3.151 42E-01	5.124 55E-01

例2 考虑非齐方程

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{1}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + t \qquad (0 < x, y, z < \pi, t > 0),\\
&u(x, y, z, 0) = \sin x \cdot \sin y \cdot \sin z,\\
&u(0, y, z, t) = u(\pi, y, z, t) = t^2/2,\\
&u(x, 0, z, t) = u(x, \pi, z, t) = t^2/2,\\
&u(x, y, 0, t) = u(x, y, \pi, t) = t^2/2\cdot
\end{aligned}\right\} \tag{26}$$

h, τ, r, ξ, η 取值与例1相同, 在第4层上利用(22)格式与文[3]比较结果如表2· :

表2

(m, n, p)	(1, 1, 1)	(2, 2, 2)	(3, 3, 3)	(4, 4, 4)	(5, 5, 5)
真 值	0.013 302 6	0.056 613 7	0.159 439 5	0.321 625 5	0.518 733 5
(12) 格 式	0.013 303 1	0.056 613 9	0.159 439 1	0.321 626 1	0.518 734 6
文[3] 格 式	0.013 320 3	0.057 216 2	0.159 367 1	0.334 728 1	0.518 690 2

从上述两例可见本文(10)格式和(22)格式都较文[3]格式确2~ 3位有效数字•

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The High Accuracy Explicit Difference Scheme for Solving Parabolic Equations 3_Dimension

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Abstract: In this paper, an explicit three_level symmetrical differencing scheme with parameters for solving parbolic partial differential equation of three_dimension will be considered. The stability condition and local truncation error for the scheme are $r < 1/2$ and $O(\Delta t^2 + \Delta x^4 + \Delta y^4 + \Delta z^4)$, respectively.

Key words: parabolic partial differential equation of three_dimension; implicit difference scheme; explicit difference scheme; local truncation error; absolutely stable; conditin stable