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关于层合板层间的连续条件*

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摘要: 给出了保证满足复合材料层合板层间位移和应力连续条件的各种形式混合能量原理的泛涵, 并通过算例证明了泛函的可靠性•

关 键 词: 层合板; 层间连续条件; 混合能量; 泛函
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前 言

目前, 在层合板的理论研究和实际计算中, 人们所关注的一个重要问题之一, 是层间界面的位移和应力的连续条件• 本文从这一关注出发, 给出了保证满足复合材料层合板层间位移和应力连续条件的各种形式混合能量原理的泛函, 并运用其中之一, 通过有限元数值计算证实了泛函的可靠性•

1 混合应力应变关系

设层合板的各层分别由均匀材料所组成、应力应变关系记为

$$\sigma_i = Q_{ij}\varepsilon_j \quad (i, j = 1, 2, \cdots, 6), \tag{1}$$

或

$$\varepsilon_i = q_{ij}\varepsilon_j \quad (i, j = 1, 2, \cdots, 6), \tag{2}$$

式中

$$\left.\begin{aligned} &\sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sigma_{23}, \quad \sigma_5 = \sigma_{13}, \quad \sigma_6 = \sigma_{12}, \\ &\varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \varepsilon_3 = \varepsilon_{33}, \quad \varepsilon_4 = 2\varepsilon_{23}, \quad \varepsilon_5 = 2\varepsilon_{13}, \quad \varepsilon_6 = 2\varepsilon_{12}, \end{aligned}\right\} \tag{3a, b}$$

$\sigma_{\lambda\mu}$, $\varepsilon_{\lambda\mu}$($\lambda, \mu = 1, 2, 3$) 是应力和应变张量在直角笛卡儿坐标系(X_λ) 下的分量•

将(3) 的左边的次序重排, 记为

$$\left.\begin{aligned} &\Sigma = [\sigma_p, \cdots, \sigma_q; \sigma_r, \cdots, \sigma_t]^{\mathrm{T}} = \begin{Bmatrix} \Sigma_K \\ \boldsymbol{E}_J \end{Bmatrix} \\ &\boldsymbol{E} = [\varepsilon_p, \cdots, \varepsilon_q; \varepsilon_r, \cdots, \varepsilon_t]^{\mathrm{T}} = \begin{Bmatrix} \Sigma_K \\ \boldsymbol{E}_J \end{Bmatrix} \end{aligned}\right\} \tag{4}$$

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其中 $\Sigma_K = [\sigma_p, \cdots, \sigma_q]^T$, $\Sigma_J = [\sigma_r, \cdots, \sigma_t]^T$ 分别含 K, J 个分量, $\boldsymbol{E}_K = [\varepsilon_p, \cdots, \varepsilon_q]^T$, $\boldsymbol{E}_J = [\varepsilon_r, \cdots, \varepsilon_t]^T$ 分别含 K, J 个分量, 且 $K + J = 6$,

如果我们写

$$\Sigma = \boldsymbol{RE}, \quad \boldsymbol{E} = \boldsymbol{r}\Sigma, \tag{5}$$

则

$$\boldsymbol{R} = \begin{bmatrix} Q_{pp} & \cdots Q_{pq} & Q_{pr} & \cdots Q_{pt} \\ \cdots & \cdots & \cdots & \cdots \\ Q_{qp} & \cdots Q_{qq} & Q_{qr} & \cdots Q_{qt} \\ Q_{rp} & \cdots Q_{rq} & Q_{rr} & \cdots Q_{rt} \\ \cdots & \cdots & \cdots & \cdots \\ Q_{tp} & \cdots Q_{tq} & Q_{tr} & \cdots Q_{tt} \end{bmatrix}, \quad \boldsymbol{r} = \begin{bmatrix} q_{pp} & \cdots q_{pq} & q_{pr} & \cdots q_{pt} \\ \cdots & \cdots & \cdots & \cdots \\ q_{qp} & \cdots q_{qq} & q_{qr} & \cdots q_{qt} \\ q_{rp} & \cdots q_{rq} & q_{rr} & \cdots q_{rt} \\ \cdots & \cdots & \cdots & \cdots \\ q_{tp} & \cdots q_{tq} & q_{tr} & \cdots q_{tt} \end{bmatrix}, \tag{6}$$

是 6×6 矩阵, $\boldsymbol{Rr} = \boldsymbol{I}$ ($\boldsymbol{I}$ 是单位矩阵)•

如果将(5) 写成

$$\left.\begin{aligned} \begin{pmatrix} \Sigma_K \\ \Sigma_J \end{pmatrix} &= \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_K \\ \boldsymbol{E}_J \end{pmatrix}, \\ \begin{pmatrix} \boldsymbol{E}_K \\ \boldsymbol{E}_J \end{pmatrix} &= \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} \Sigma_K \\ \Sigma_J \end{pmatrix}, \end{aligned}\right\} \tag{7}$$

则

$$\begin{aligned} \begin{pmatrix} \Sigma_K \\ \Sigma_J \end{pmatrix} &= \begin{pmatrix} R_{11}^{-1} & R_{11}^{-1}R_{12} \\ R_{21}R_{11}^{-1} & R_{21}R_{11}^{-1}R_{12} - R_{22} \end{pmatrix} \begin{pmatrix} \Sigma_K \\ -\boldsymbol{E}_J \end{pmatrix} = \\ &\begin{pmatrix} r_{11} - r_{12}r_{22}^{-1}r_{21} & -r_{12}r_{22}^{-1} \\ -r_{22}^{-1}r_{21} & -r_{22}^{-1} \end{pmatrix} \begin{pmatrix} \Sigma_K \\ -\boldsymbol{E}_J \end{pmatrix}, \end{aligned} \tag{8}$$

$$\begin{aligned} \begin{pmatrix} \Sigma_K \\ -\boldsymbol{E}_J \end{pmatrix} &= \begin{pmatrix} R_{11} - R_{12}R_{22}^{-1}R_{21} & R_{12}R_{22}^{-1} \\ R_{22}^{-1}R_{21} & -R_{22}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_K \\ \Sigma_J \end{pmatrix} = \\ &\begin{pmatrix} r_{11}^{-1} & -r_{11}^{-1}r_{12} \\ -r_{21}r_{11}^{-1} & r_{21}r_{11}^{-1}r_{12} - r_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_K \\ \Sigma_J \end{pmatrix}• \end{aligned} \tag{9}$$

对于(8), (9) 我们进一步用分量形式分别记成

$$s_i = C_{ij} r_j \tag{10}$$

$$r_i = H_{ij} s_j \tag{11}$$

其中 $C_{ij} = C_{ji}, H_{ij} = H_{ji}$ 和

$$\left.\begin{aligned} (s_1, s_2, s_3, s_4, s_5, s_6) &= (\varepsilon_p, \cdots, \varepsilon_q; \sigma_r, \cdots, \sigma_t), \\ (r_1, r_2, r_3, r_4, r_5, r_6) &= (\sigma_p, \cdots, \sigma_q; -\varepsilon_r, \cdots, \varepsilon_t), \end{aligned}\right\} \tag{12}$$

我们定义混合能量密度

$$\overset{c}{W}_{KJ} = \frac{1}{2}C_{ij}r_ir_j, \quad W_{KJ} = \frac{1}{2}H_{ij}s_is_j, \tag{13}$$

根据(10), (11), (13), 我们得

$$\overset{c}{W}_{KJ, r_i} = s_i, \quad W_{KJ, s_i} = r_i \tag{14}$$

其中 $\overset{c}{W}_{KJ, r_i}$ 和 W_{KJ, s_i} 表示 $\overset{c}{W}_{KJ}$ 和 W_{KJ} 分别关于 r_i 和 s_i 的偏导数, 由此, 我们找到两类应力应变关

系·

$$\left.\begin{aligned} \overset{c}{W}_{KJ,\sigma_\alpha} &= \varepsilon_\alpha \quad (\alpha = p, \cdots, q), \\ \overset{c}{W}_{KJ,\varepsilon_\beta} &= -\sigma_\beta \quad (\beta = r, \cdots, t), \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} W_{KJ,\varepsilon_\alpha} &= \sigma_\alpha \quad (\alpha = p, \cdots, q), \\ W_{KJ,\sigma_\beta} &= -\varepsilon_\beta \quad (\beta = r, \cdots, t)\cdot \end{aligned}\right\} \tag{16}$$

对任意变分 $\delta\sigma_\alpha(\alpha = p, \cdots, q), \delta\varepsilon_\beta(\beta = r, \cdots, t)$
可以得到

$$\delta\overset{c}{W}_{KJ} = \sum_{\alpha=p,\cdots,q} \varepsilon_\alpha \delta\sigma_\alpha - \sum_{\beta=r,\cdots,t} \sigma_\beta \delta\varepsilon_\beta \tag{17}$$

对任意变分 $\delta\varepsilon_\alpha(\alpha = p, \cdots, q), \delta\sigma_\beta(\beta = r, \cdots, t)$
可以得到

$$\delta W_{KJ} = \sum_{\alpha=p,\cdots,q} \sigma_\alpha \delta\varepsilon_\alpha - \sum_{\beta=r,\cdots,t} \varepsilon_\beta \delta\sigma_\beta \tag{18}$$

当应力应变关系被满足时,我们有

$$\overset{c}{W}_{KJ} = W_{KJ} \tag{19}$$

如果 $\alpha = 1, 2, 3, 6$; $\beta = 4, 5$ 则正交各向异性和横观各向同性材料矩阵 $[C_{ij}]$, $[q_{ij}]$ 分别是

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/S_{55} \end{bmatrix}$$

和

$$\begin{bmatrix} 1/E_{11} & -\nu_{21}/E_{22} & -\nu_{21}/E_{22} & 0 & 0 & 0 \\ -\nu_{21}/E_{22} & 1/E_{11} & -\nu_{23}/E_{22} & 0 & 0 & 0 \\ -\nu_{21}/E_{22} & -\nu_{23}/E_{22} & 1/E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mu_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\mu_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\mu_{12} \end{bmatrix}$$

这里 S_{ij} 是弹性柔度系数, $E_{11}, \cdots, \mu_{12}$ 是工程弹性常数[1]·

2 关于层合板的混合能量原理

设

$$\Gamma(u_\lambda, \sigma_i, \varepsilon_r, \cdots, \varepsilon_t) =$$
$$\sum_{m=1-M}^{N} \int_{V_m} \Big[-\overset{c(m)}{W}_{KJ} - \sum_{\beta=r,\cdots,t} \overset{(m)}{\varepsilon}_\beta \overset{(m)}{\sigma}_\beta + \overset{(m)}{\varepsilon}_i(u) \overset{(m)}{\sigma}_i -$$
$$\overset{(m)}{u}_\lambda \overset{(m)}{f}_\lambda \Big] \mathrm{d}V - \frac{1}{2} \sum_m \int_{S_m} \Big[\overset{(m)}{u}_{d\lambda} - \overset{(m+1)}{u}_{u\lambda} \Big] \Big[\overset{(m)}{p}_{d\lambda} - \overset{(m+1)}{p}_{u\lambda} \Big] \mathrm{d}s -$$

$$\int_{S_\sigma} \bar{p}_\lambda u_\lambda \mathrm{d}s - \int_{S_u} (u_\lambda - \bar{u}_\lambda) p_\lambda \mathrm{d}s, \tag{20}$$

$$\Pi(u_\lambda, \sigma_i, \varepsilon_p, \cdots, \varepsilon_q) =$$

$$\sum_{m=1-M}^{N} \int_{V_m} \Big\{ \sum_{\alpha=p,\cdots,q} \overset{(m)}{\sigma}_\alpha \overset{(m)}{\varepsilon}_\alpha - \overset{(m)}{w}_{KL} + \overset{(m)}{u}_\lambda [(\overset{(m)}{F}_{\lambda\lambda'} \overset{(m)}{\sigma}_{\lambda'\mu})_{,\mu}] +$$

$$\overset{(m)}{f}_\lambda] \Big\} \mathrm{d}V - \frac{1}{2} \sum_m \int_{S_m} [\overset{(m)}{p}_{d\lambda} + \overset{(m+1)}{p}_{u\lambda}][\overset{(m)}{u}_{d\lambda} + \overset{(m+1)}{u}_{u\lambda}] \mathrm{d}s -$$

$$\int_{S_u} u_\lambda \bar{p}_\lambda \mathrm{d}s - \int_{S_\sigma} (p_\lambda - \bar{p}_\lambda) u_\lambda \mathrm{d}s, \tag{21}$$

其中 $\bar{u}_\lambda$ 和f_λ 分别是位移和体力分量,

$$p_\lambda = F_{\lambda\lambda'} \sigma_{\lambda'\mu} \boldsymbol{n}_\mu, \quad F_{\lambda\lambda'} = \delta_{\lambda\lambda'} + u_{\lambda,\lambda'}, \tag{22}$$

$\boldsymbol{n}_\mu$ 是层合板表面向外单位法向量; $\overset{c(m)}{W}_{KJ}$, $\overset{(m)}{\varepsilon}_\beta$ 等是第 m 层在直角笛卡儿坐标系$\{X_\lambda\}$ (X_s 轴向下) 中所占区域 V_m 上 $\overset{c}{W}_{KJ}$ 和 ε_β 等的取值; S_{1-M} 和 S_N 分别是层合板的上下表面, S_m 是 V_m 和 V_{m+1} 的分界面, S_{um} 是 V_m 的上表面, S_{dm} 是 V_m 的下表面, 即 $S_m = S_{dm} = S_{u(m+1)}$; $\overset{(m)}{u}_{d\lambda}$, $\overset{(m+1)}{u}_{u\lambda}$ 等的定义域是 $S_{dm} = S_{u(m+1)}$, 层合板共有 $M+N$ 层, 中面($X_s = 0$) 以上有 M 层; $\bar{u}_\lambda, \bar{p}_\lambda$, 等表示已知量

$$\varepsilon_{\lambda\lambda'}(u) = \frac{1}{2}(u_{\lambda,\lambda'} + u_{\lambda',\lambda} + u_{\mu,\lambda} u_{\mu,\lambda'}), \tag{23}$$

其中( ), λ表示( ) 关于 X_λ 的偏导数·

有时, 为了简化记号, 我们略去上标 (m), 例如: 式(1), (8) 等即是·

为了方便, 我们引进如下记号:

$$\overset{(m)}{T}_J = (\overset{(m)}{u}_\lambda, \overset{(m)}{\sigma}_i, \overset{(m)}{\varepsilon}_r, \cdots, \overset{(m)}{\varepsilon}_t) \in \overset{(m)}{L}, \text{ 当且仅当 } \overset{(m)}{u}_\lambda \in C^1(V_m),$$
$$\overset{(m)}{\sigma}_i \in C^1(V_m), \overset{(m)}{\varepsilon}_r, \cdots, \overset{(m)}{\varepsilon}_q \in C(V_m), \overset{(m)}{U}_K = (\overset{(m)}{u}_\lambda, \overset{(m)}{\sigma}_i,$$
$$\overset{(m)}{\varepsilon}_p, \cdots, \overset{(m)}{\varepsilon}_q) \in \overset{(m)}{H}_K \text{ 当且仅当 } \overset{(m)}{u}_\lambda \in C^1(V_m), \overset{(m)}{\sigma}_i \in C^1(V_m),$$
$$\overset{(m)}{\varepsilon}, \cdots, \overset{(m)}{\varepsilon}_q \in C(V_m)\cdot$$

当 $\overset{(m)}{T}_J$, $\overset{(m)}{T}_J + \delta\overset{(m)}{T}_J \in \overset{(m)}{L}_J$ 时, 我们得

$$\delta\Gamma = \sum_{m=1-M}^{N} \int_{V_m} \Big\{ \sum_{\alpha=p,\cdots,q} [\varepsilon_\alpha(u) - \overset{c}{W}_{KJ,\sigma^\alpha}] \delta\sigma_\alpha -$$
$$\sum_{\beta=r,\cdots,t} [(\overset{c}{W}_{KJ,\varepsilon_\beta} + \sigma_\beta)\delta\varepsilon_\beta + (\varepsilon_\beta - \varepsilon_\beta(u))\delta\sigma_\beta] -$$
$$[(F_{\lambda\lambda'}\sigma_{\lambda'\mu})_{,\mu} + f_\lambda]\delta u_\lambda \Big\} \mathrm{d}V + \frac{1}{2}\sum_m \int_{S_m} \Big\{ [\overset{(m)}{p}_{d\lambda} + \overset{(m+1)}{p}_{u\lambda}] \times$$
$$\delta[\overset{(m)}{u}_{d\lambda} + \overset{(m+1)}{u}_{u\lambda}] - [\overset{(m)}{u}_{d\lambda} - \overset{(m+1)}{u}_{u\lambda}]\delta[\overset{(m)}{p}_{d\lambda} - \overset{(m+1)}{p}_{u\lambda}] \Big\} \mathrm{d}S +$$
$$\int_{S_\sigma} (p_\lambda - \bar{p}_\lambda)\delta u_\lambda \mathrm{d}S - \int_{S_u} (u_\lambda - \bar{u}_\lambda)\delta p_\lambda \mathrm{d}S \tag{24}$$

当 $\overset{(m)}{U}_K$, $\overset{(m)}{U}_K + \delta\overset{(m)}{K}_K \in \overset{(m)}{H}_K$ 时, 我们得

$$\delta\Pi = \sum_{m=1-M}^{N} \int_{V_m} \Big\{ \sum_{\alpha=p,\cdots,q} (\sigma_\alpha - W_{KJ,\varepsilon_\alpha})\delta\varepsilon_\alpha -$$

$$\sum_{\beta=r,\cdots,t}[(W_{KJ},\sigma_\beta+\varepsilon_\beta(u)\delta\sigma_\beta+\sum_{\alpha=p,\cdots,q}[\varepsilon_\alpha-\varepsilon_\alpha(u)]\delta\sigma_\alpha+[(F_{\lambda\lambda},\sigma_{\lambda,\mu}),_\mu+f_\lambda]\delta u_\lambda\}\mathrm{d}V-\frac{1}{2}\sum_m\int_{S_m}\{[\overset{(m)}{p}_{d\lambda}+\overset{(m+1)}{p}_{u\lambda}]\delta[\overset{(m)}{u}_{d\lambda}+\overset{(m+1)}{u}_{u\lambda}]+[\overset{(m+1)}{u}_{u\lambda}-\overset{(m)}{u}_{d\lambda}]\delta[\overset{(m)}{p}_{d\lambda}-\overset{(m+1)}{p}_{u\lambda}]\}\mathrm{d}S-\int_{S_\sigma}(p_\lambda-p_\lambda)\delta u_\lambda\mathrm{d}S+\int_{S_u}(u_\lambda-u_\lambda)\delta p_\lambda\mathrm{d}S, \tag{25}$$

由(24)得

$$\delta\Gamma=0, \tag{26}$$

当且仅当

1) $\overset{c}{W}_{KJ},\sigma_\alpha=\varepsilon_\alpha(u)\quad(\alpha=p,\cdots,q)$ (在 V_m 内), (27)

$\overset{c}{W}_{KJ},\varepsilon_\beta=-\sigma_\beta\quad(\beta=r,\cdots,t)$ (在 V_m 内), (28)

$\varepsilon_\beta=\varepsilon_\beta(u)\quad(\beta=r,\cdots,t)$ (在 V_m 内) (29)

$(F_{\lambda\lambda},\sigma_{\lambda,\mu}),+f_\lambda=0$ (在 V_m 内)• (30)

2) $\overset{(m)}{p}_{d\lambda}=-\overset{(m+1)}{p}_{u\lambda},\ \overset{(m)}{u}_{u\lambda}=\overset{(m+1)}{u}_{u\lambda}$ (在 S_m 上)• (31)

3) $u_\lambda=u_\lambda$ (在 S_u 上), (32)

4) $p_\lambda=p_\lambda$ (在 S_σ 上), (33)

其中(27)是部分应力应变关系和应变位移关系的混合方程,(28),(29)是其余的应力应变关系和应变位移关系;(30)是关于 V_m 的平衡方程•

类似地,由(25)可得

$$\delta\Pi=0 \tag{34}$$

当且仅当(30)~(33)被满足,并且

1) $W_{KJ},\sigma_\beta=-\varepsilon_\beta(u)\quad(\beta=r,\cdots,t)$(在 V_m 内), (27)′

$W_{KJ},\varepsilon_\alpha=\sigma_\alpha\quad(\alpha=p,\cdots,q)$(在 V_m 内), (28)′

$\varepsilon_\alpha=\varepsilon_\alpha(u)\quad(\alpha=p,\cdots,q)$(在 V_m 内)• (29)′

注意到(22), $\overset{(m)}{\boldsymbol{n}}_{d\lambda}=-\overset{(m+1)}{\boldsymbol{n}}_{u\lambda}$, 从(31a), 我们得

$$[\delta_{\lambda\lambda},+\overset{(m)}{u}_{\lambda,\chi}]\overset{(m)}{\sigma}_{\chi3}=[\delta_{\lambda,\chi}+\overset{(m+1)}{u}_{\lambda,\chi}]\overset{(m+1)}{\sigma}_{\chi3}\text{(在 } S_m \text{ 上)}, \tag{35}$$

由此,对于小变形,(35)成为

$$\overset{(m)}{\sigma}_{\chi3}=\overset{(m+1)}{\sigma}_{\chi3}\quad\text{(在 } S_m \text{ 上)}• \tag{36}$$

应力和位移的连续条件(31)在层合板的理论和应用中是很重要的[2~5]•

注意到$[C_{ij}][H_{ij}]=I$, $Rr=I$;(7)分别等价于(8)和(9),易见,在(5),(7)~(11),(14)~(16)中只要有一个被满足,则其余的全满足,进而推出

$$\Pi+\Gamma=0• \tag{37}$$

3 算　　例

本文对铺层为$[0°/90°/0°]$($b/a=3$)和$[0°/90°/90°/0°]$($b/a=1$, b 和, a 分别为板面的长

和宽)、四边简支矩形层合板(各层厚度均相等)在上表面载荷 $p_3 = p_0\sin(\pi x_1/a)\sin(\pi x_2/b)$ 作用下,利用变分式 $\delta\Gamma(u_\lambda, \sigma_i, \varepsilon_4, \varepsilon_5) = 0$,通过4×4的网格的有限元法进行了数值计算,并求得了板的中点处的挠度,(利用了对称性只取板的四分之一进行了计算)• 并将 $a/h = 100$,(h 为板的厚度)的计算结果与已见到的结果比较如下:

i) $[0°/90°/0°]$ $(b/a = 3)$

本文: $W = 0.513$, 文[6]: $W = 0.505$

其中 $W = \left\{100E_2h^3/(p_0a^4)\right\}W_0$, W_0 为中点处的挠度•

ii) $[0°/90°/90°/0°]$ $(b/a = 1)$

本文: $W = 1.025$, 文[6]: $W = 1.003$,

其中 $W = \left\{100Eh^3/(p_0a^4)\right\}W_0$, W_0 为中点处的挠度•

$$E = (\pi^4/12)\left\{4G_{12} + [E_1 + (1 + 2\nu)E_2]/(1 - \nu_{12}\nu_{21})\right\}$$

在计算中,每层材料的弹性常数为

$$E_1/E_2 = 25, \quad G_{12}/E_2 = 0.5, \quad \nu_{12} = 0.25 (从而求得\ \nu_{21} = 0.01)$$

数值结果表明,本文提供的泛函是可靠的•

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Continual Conditions Between Layers for Laminated Plates

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Abstract: In this paper, various forms of functional on blending energy principles of composite laminated plates are given, which guarantee satisfied continual conditions of displacements and stress between layers, and then the reliability of the functional are proved by the computing example.

Key words: laminated plates; continual conditions between layers; blending energy; functional