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# 均布载荷下两邻边固定、一边简支、 一边自由的矩形板的弯曲\*

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(沈惠申推荐)

摘要: 用广义简支边概念和叠加法给出均布载荷下两邻边固定、一边简支、一边自由矩形板的精确解。对正方形板自由边的挠度和固定边的弯矩进行了数字计算。

关键词: 矩形板弯曲; 广义简支边; 叠加法

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## 引言

自七十年代末以来, 张福范求出了集中载荷与均布载荷作用下悬臂矩形板问题以及两邻边固定、两邻边自由的矩形板问题的精确解<sup>[1~3]</sup>。本文用上述方法求出了均布载荷下两邻边固定、一边简支、一边自由的矩形板的精确解。对正方形板自由边挠度与固定边弯矩进行了数字计算。该板能应用于贮粮仓、储水仓、传感器芯片及建筑等工程问题上。

## 1 广义简支边概念和边界条件

简支边的边界条件为沿边各点的弯矩  $M = 0$ , 挠度  $W = 0$ 。而广义简支边沿边各点  $M = 0$ , 而  $W \neq 0$ , 广义简支边与简支边都存在剪力  $V$ 。因而, 如果简支边界各点沉陷而有挠度, 这边就成为广义简支边。

用广义简支边概念和叠加法解均布载荷下两邻边  $x = 0, y = 0$  固定,  $x = a$  边为简支,  $y = b$  边为自由的矩形板(图 1)。边界条件为:

$$\begin{aligned} (W)_{x=0} &= \begin{cases} W \\ \frac{\partial W}{\partial x} \end{cases}_{x=0} = 0, \\ (W)_{y=0} &= \begin{cases} W \\ \frac{\partial W}{\partial y} \end{cases}_{y=0} = 0, \\ (W)_{x=a} &= \begin{cases} W \\ \frac{\partial^2 W}{\partial x^2} \end{cases}_{x=a} = 0, \end{aligned}$$

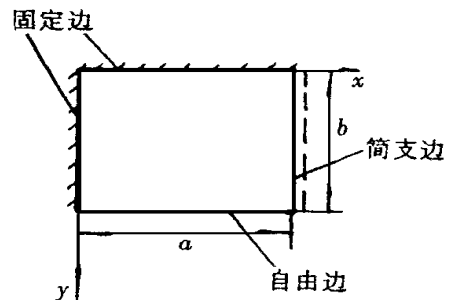


图 1

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$$\begin{aligned} \left( \frac{\partial^2 W}{\partial y^2} + \mu \frac{\partial^2 W}{\partial x^2} \right)_{y=b} &= 0, \\ \left[ \frac{\partial^3 W}{\partial y^3} + (2 - \mu) \frac{\partial^3 W}{\partial y \partial x^2} \right]_{y=b} &= 0 \end{aligned} \quad (1)$$

## 2 叠加法组成部分

1) 均布载荷  $q$  作用下的简支边矩形板, 其边界各为  $x = 0, x = a, y = 0, y = b$ , 板的挠曲面、沿边的剪力及斜度各为:

$$W = \frac{4qa^4}{D\pi^5} \sum_{m=1,3,\dots} \frac{1}{m^5} \left\{ 1 - \operatorname{ch} \frac{m\pi y}{a} + \frac{1}{2} \frac{m\pi y}{a} \operatorname{sh} \frac{m\pi y}{a} - \frac{\operatorname{ch} \alpha_m - 1}{2\operatorname{sh} \alpha_m} \times \left[ \frac{m\pi y}{a} \operatorname{ch} \frac{m\pi y}{a} - 2 \left( 1 - \frac{\alpha_m}{2\operatorname{sh} \alpha_m} \right) \operatorname{sh} \frac{m\pi y}{a} \right] \right\} \sin \frac{m\pi x}{a}, \quad (2)$$

$$\alpha_m = \frac{m\pi b}{a}.$$

$$\left( \frac{\partial W}{\partial y} \right)_{y=0} = \frac{2qa^3}{D\pi^4} \sum_{m=1,3,\dots} \frac{1}{m^4} \left[ \operatorname{th} \frac{\alpha_m}{2} - \frac{\alpha_m}{2} \frac{1}{\operatorname{ch}^2 \frac{\alpha_m}{2}} \right] \sin \frac{m\pi x}{a}. \quad (3)$$

$$\left( \frac{\partial W}{\partial x} \right)_{x=0} = \frac{2qb^3}{D\pi^4} \sum_{i=1,3,\dots} \frac{1}{i^4} \left[ \operatorname{th} \frac{\beta_i}{2} - \frac{\beta_i}{2} \frac{1}{\operatorname{ch}^2 \frac{\beta_i}{2}} \right] \sin \frac{i\pi y}{b}, \quad \beta_i = \frac{i\pi a}{b}. \quad (4)$$

$$(V_y)_{y=b} = -\frac{2qa}{\pi^2} \sum_{m=1,3,\dots} \frac{1}{m^2} \left[ (3 - \mu) \operatorname{th} \frac{\alpha_m}{2} - (1 - \mu) \frac{\alpha_m}{2} \frac{1}{\operatorname{ch}^2 \frac{\alpha_m}{2}} \right] \sin \frac{m\pi x}{a}. \quad (5)$$

式中,  $D$  为板的弯曲刚度,  $\mu$  为泊松比.

2) 一简支边矩形板, 分布在  $y = 0$  这边的弯矩为:  $M(x) = \sum_{m=1} E_m \sin \frac{m\pi x}{a}$

由此得:

$$W = \frac{a^2}{2D\pi^2} \sum_{m=1} \frac{E_m}{m^2} \left[ -\frac{\alpha_m}{\operatorname{sh}^2 \alpha_m} \sin \frac{m\pi y}{a} - \frac{m\pi y}{a} \operatorname{sh} \frac{m\pi y}{a} + \operatorname{cth} \alpha_m \frac{m\pi y}{a} \operatorname{ch} \frac{m\pi y}{a} \right] \sin \frac{m\pi x}{a}. \quad (6)$$

$$\left( \frac{\partial W}{\partial y} \right)_{y=0} = \frac{a}{2\pi D} \sum_{m=1} \frac{E_m}{m} \left[ \operatorname{cth} \alpha_m - \frac{\alpha_m}{\operatorname{sh}^2 \alpha_m} \right] \sin \frac{m\pi x}{a}. \quad (7)$$

$$\left( \frac{\partial W}{\partial x} \right)_{x=0} = \frac{2b^2}{\pi^2 a D} \sum_{i=1} \sum_{m=1} \frac{iE_m}{m^3 \left( \frac{b^2}{a^2} + \frac{i^2}{m^2} \right)^2} \sin \frac{i\pi y}{b}. \quad (8)$$

$$(V_y)_{y=b} = -\frac{\pi}{2a} \sum_{m=1} \frac{mE_m}{\operatorname{sh} \alpha_m} \left[ 1 + \frac{1 - \mu}{1 + \mu} \alpha_m \operatorname{cth} \alpha_m \right] \sin \frac{m\pi x}{a}. \quad (9)$$

3) 一简支边矩形板, 沿  $x = 0$  这边弯矩为

$$M(y) = \sum_{i=1} F_i \sin \frac{i\pi y}{b}$$

由此得:

$$W = \frac{b^2}{2D\pi^2} \sum_{i=1} F_i \left[ -\frac{\beta_i}{\text{sh}^2 \beta_i} \text{sh} \frac{i\pi x}{b} - \frac{i\pi x}{b} \text{sh} \frac{i\pi x}{b} + \text{cth} \beta_i \frac{i\pi x}{b} \text{ch} \frac{i\pi x}{b} \right] \sin \frac{i\pi y}{b}, \beta_i = \frac{i\pi a}{b}. \quad (10)$$

$$\left( \frac{\partial W}{\partial y} \right)_{y=0} = \frac{2a^2}{\pi^2 b D} \sum_{m=1} \sum_{i=1} \frac{F_i m}{i^3 \left( \frac{a^2}{b^2} + \frac{m^2}{i^2} \right)^2} \sin \frac{m\pi x}{a}, \quad (11)$$

$$\left( \frac{\partial W}{\partial x} \right)_{x=0} = \frac{b}{2\pi D} \sum_{i=1} \frac{F_i}{i} \left[ \text{cth} \beta_i - \frac{\beta_i}{\text{sh}^2 \beta_i} \right] \sin \frac{i\pi y}{b}, \quad (12)$$

$$(V_y)_{y=b} = \frac{2}{b} \sum_{m=1} \sum_{i=1} \frac{F_i m \left[ \frac{a^2}{b^2} + (2-\mu) \frac{m^2}{i^2} \right]}{i \left( \frac{a^2}{b^2} + \frac{m^2}{i^2} \right)^2} \cos i\pi \sin \frac{m\pi x}{a}. \quad (13)$$

4) 矩形板,  $x=0, x=a$  与  $y=0$  三边简支,  $y=b$  为广义简支边, 其挠度为:

$$(W)_{y=b} = \sum_{m=1} a_m \sin \frac{m\pi x}{a}$$

得:

$$W = \frac{1-\mu}{2} \sum_{m=1} \frac{a_m}{\text{sh} \alpha_m} \left[ \left( \frac{2}{1-\mu} + \alpha_m \text{cth} \alpha_m \right) \text{sh} \frac{m\pi y}{a} - \frac{m\pi y}{a} \text{ch} \frac{m\pi y}{a} \right] \sin \frac{m\pi x}{a}. \quad (14)$$

$$\left( \frac{\partial W}{\partial y} \right)_{y=0} = \frac{1-\mu}{2a} \pi \sum_{m=1} \frac{m a_m}{\text{sh} \alpha_m} \left[ \frac{1+\mu}{1-\mu} + \alpha_m \text{cth} \alpha_m \right] \sin \frac{m\pi x}{a}. \quad (15)$$

$$\left( \frac{\partial W}{\partial x} \right)_{x=0} = -\frac{2}{a} \sum_{m=1} \sum_{i=1} a_m i \cos i\pi \frac{\frac{i^2}{2} + (2-\mu) \frac{b^2}{a^2}}{m \left( \frac{i^2}{m^2} + \frac{b^2}{a^2} \right)^2} \sin \frac{i\pi y}{b}. \quad (16)$$

$$(V_y)_{y=b} = \frac{D}{2} (1-\mu)^2 \sum_{m=1} \frac{m^3 \pi^3}{a^3} a_m \left[ \frac{3+\mu}{1-\mu} \text{cth} \alpha_m + \frac{\alpha_m}{\text{sh}^2 \alpha_m} \right] \sin \frac{m\pi x}{a}. \quad (17)$$

### 3 叠加法

要满足沿固定边  $y=0$  各点的斜度为零, 即  $\left( \frac{\partial W}{\partial y} \right)_{y=0} = 0$ , 将式(3)、(7)、(11)和(15)所给斜度相加, 并使其和为零, 得到第一个方程。

其次, 要满足固定边  $x=0$  的斜度为零, 即  $(\partial W / \partial x)_{x=0} = 0$ , 只需将式(4)、(8)、(12)和(16)所给斜度相加, 并使其和为零, 可得到第二个方程。

最后, 要满足沿自由边  $y=b$  的剪力为零, 即  $(V_y)_{y=b} = 0$ , 需将式(5)、(9)、(13)和(17)所给的剪力相加, 并使其和为零, 可得到第三个方程。由以上三个无穷方程联立, 解出未知数

$a_m$ 、 $\frac{a^2}{D} E_m$  和  $\frac{b^2}{D} F_i$ 。

### 4 算例

取  $\mu = 0.3$ , 对均布载荷作用下两邻边  $x=0, y=0$  固定,  $x=a$  简支,  $y=a$  自由的正方

形板的计算结果如下：

表 1 沿自由边部分点的挠度值

$x(\times a)$	0	0.25	0.5	0.55	0.6	0.65	0.75	1
$(W)_{y=a} \left( \times 10^{-3} \frac{qa^4}{D} \right)$	0	2.5913	5.5196	5.7025	7.0927	5.5316	4.6363	0

1) 沿自由边的挠度

沿自由边  $y = a$  的一些点的挠度见表 1。沿自由边  $y = a$  的挠度曲线见图 2。

2) 沿固定边的弯矩

沿固定边  $y = 0$  和  $x = 0$  的部分点的弯矩值分别列于表 2 和表 3。其弯矩分布曲线分别见图 3 和图 4。可见固定边  $x = 0$  在自由边附近弯矩较大。

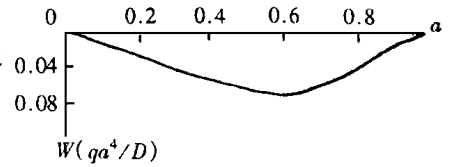


图 2 沿自由边  $y = a$  的挠度曲线

表 2 沿固定边  $y = 0$  部分点的弯矩值

$x(\times a)$	0	0.1	0.25	0.4	0.5	0.55
$M(x)(\times 10^{-2} qa^2)$	0	-0.76557	-3.9253	-6.6596	-7.7082	-7.9472
$x(\times a)$	0.6	0.65	0.75	0.85	0.95	1
$M(x)(\times 10^{-2} qa^2)$	-7.9709	-7.7867	-6.7403	-4.7675	-3.4191	0

表 3 沿固定边  $x = 0$  部分点的弯矩值

$y(\times a)$	0	0.25	0.5	0.75	0.85
$M(y)(\times 10^{-1} qa^2)$	0	-0.40352	-0.80585	-1.1796	-1.3445
$y(\times a)$	0.95	0.975	0.99	1	
$M(y)(\times 10^{-1} qa^2)$	-1.7204	-1.3763	-0.62955	0	

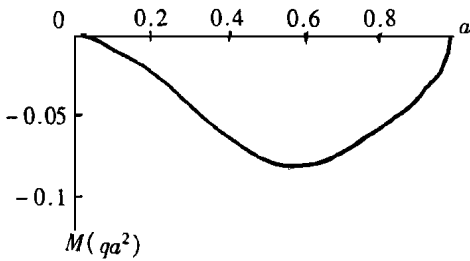


图 3 沿固定边  $y = 0$  的弯矩分布

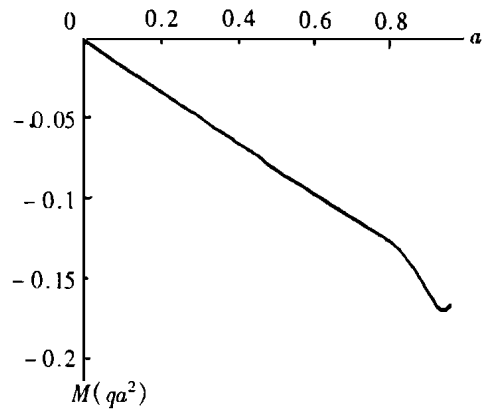


图 4 沿固定边  $x = 0$  的弯矩分布

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## Bending of Uniformly Loaded Rectangular Plates With Two Adjacent Edges Clamped, One Edge Simply Supported and the Other Edge Free

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**Abstract:** In this paper, an exact solution for an uniformly loaded rectangular plate with two adjacent edges clamped, one edge simply supported and the other edge free, was given by using the concept of generalized simply supported edges and superposition method. The numerical results were given for the deflections along the free edge and bending moments along the clamped edges of a square plate.

**Key words:** bending of rectangular plates; generalized simply supported edges; superposition