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U 型波纹管整体弯曲问题的一般解*

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摘要: 利用朱卫平等提出的柔性细环壳整体弯曲的一般解和相应的环板方程的解给出了 U 型波纹管在子午面内受纯弯矩作用整体弯曲的应力分布和转角公式. 把计算的结果和按 EJMA(Standards of the Expansion Joint Manufacturers Association)计算的结果以及黎廷新等的实验结果作了分析比较.

关 键 词: 柔性壳; 旋转壳; 圆环壳; 波纹管; U 型波纹管
中图分类号: O343.2 **文献标识码:** A

引 言

U 型波纹管,如图 1 所示,是一种应用广泛的弹性元件,一般受法向压力、轴向力和弯曲载荷作用. 在研究和设计这类波纹管时,人们通常把它简化成环壳和环板的组合件.

U 型波纹管的轴对称问题已得到广泛深入的讨论[1~5]. 关于这类波纹管的非轴对称问题,M.Hamada 等[6](1971)用有限差分法分析了非对称弯曲载荷作用下的 U 型波纹管. Э.Л. Аксельрад(E.L.Axelrad)(1976,1987)[7,8]从一般的旋转壳方程出发,针对薄壁弯管、波纹管这类壳体的变形特点,引入了柔性壳假设(或半薄膜假设,即子午向按弯曲理论处理,环向按薄膜理论处理),得到较为简化的线性的、非线性的旋转壳方程. 基于这样的线性方程,Axelrad 介绍了波纹管整体弯曲问题的三角级数解. 文[9](1992)根据 Axelrad 提供的线性方程用打靶法计算了 S 型波纹管的整体弯曲刚度. EJMA[10](1993)主要以曲梁为模型计算波纹管,用“等效法”把波纹管的非轴对称问题近似地按轴对称问题处理. 文[11](1994)讨论了 EJMA 中 U 型波纹管的位移和应力公式,做了电测实验. 文[12]根据 Axelrad 的柔性壳理论,导出了柔性圆环壳在子午面内整体弯曲问题的复变量方程和相应的细环壳方程,并借助于钱伟长轴对称细环壳的一般解[13]给出了细环壳整体弯曲问题的一般解. 文[14]用这样的解计算了 C 型波纹管整体纯弯曲的应力分布和角位移.

本文作为文[12]的具体应用,处理了 U 型波纹管在子午面内受纯弯矩作用的整体弯曲问题. 对波纹管的环壳部分,根据柔性细环壳在子午面内整体弯曲的一般解计算;对环板部分按相应的环板方程的解计算;利用环壳和环板之间的连接条件确定积分常数. 给出了 U 型波纹管在子午面内受纯弯矩作用时整体弯曲的应力分布和端面转角的解析公式. 算例显示:文[11]的实验结果除位于波纹管波谷处的个别点外,其余的与本文的理论计算结果基本一致;按

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EJMA 计算,得到的整体弯曲刚度比本文的计算结果小 10%左右.

1 基本方程

1.1 基本符号

U 型波纹管的受力和几何情况如图 1 和图 2 所示.

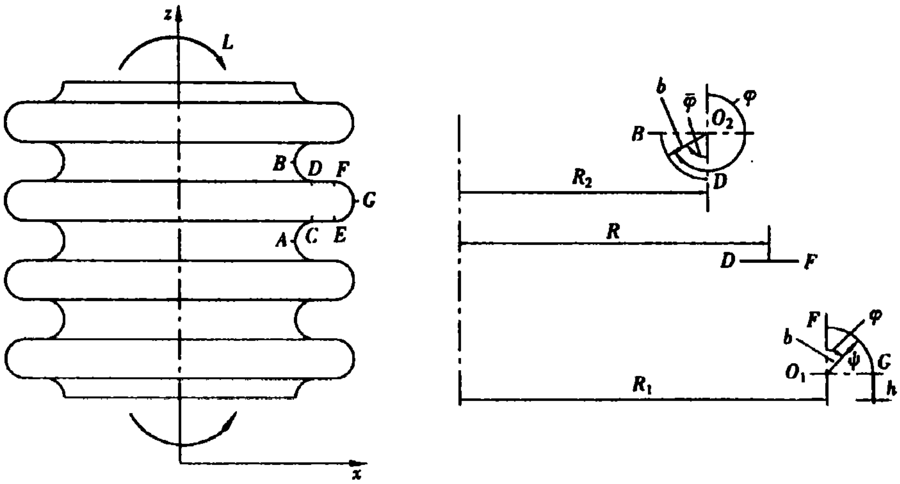


图 1 U 型波纹管的整体纯弯曲

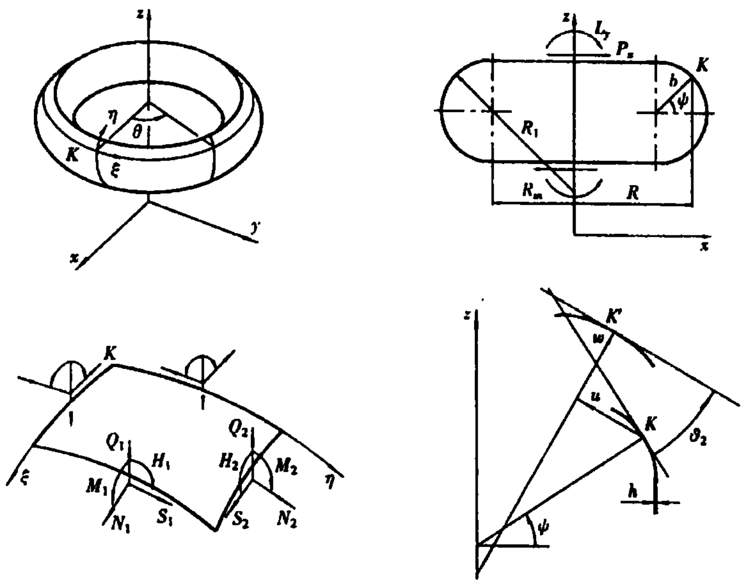


图 2 环壳在子午面内整体弯曲的受力和变形

图中,h 为环壳、环板的厚度;b 为圆环壳在子午面内的曲率半径;R_m,R_1,R_2 为圆环壳在子午面内圆心到旋转轴的距离;R 为环壳、环板中面上的点到旋转轴的距离;E、ν、D 为弹性模量、泊桑比和板壳抗弯刚度,$D = Eh^3/[12(1-\nu^2)]$;L 为平面集中力偶;L^0、L_1^0、L_2^0 为环壳任一平行圆截面上(ψ = const)总等效力矩的量纲为一的形式,$L^0 = Lb^2/(\pi D R_m^2)$,$L_1^0 = Lb^2/(\pi D R_1^2)$,$L_2^0 = Lb^2/(\pi D R_2^2)$;$L_3^0$ 为作用在环板上径向平面内集中力偶的量纲为一的形式,

$L_3^0 = L/(\pi D)$；α、α_1、α_2 为环壳几何参数，$\alpha = b/R_m$，$\alpha_1 = b/R_1$，$\alpha_2 = b/R_2$；μ、μ_1、μ_2 为环壳特征参数，

$$\mu = \sqrt{3(1-\nu^2)}\,\frac{b^2}{R_m h},\ \mu_1 = \sqrt{3(1-\nu^2)}\,\frac{b^2}{R_1 h},\ \mu_2 = \sqrt{3(1-\nu^2)}\,\frac{b^2}{R_2 h}.$$

1.2 细环壳微分方程

环壳单元的内力和变形如图 2 所示，在柔性细环壳条件下其基本微分方程为[12]

$$\frac{d^2 V}{d\psi^2} + 2i\mu\cos\psi V = -L^0\sin\psi, \tag{1a}$$

其中

$$V(\psi) = \chi(\psi) + iT(\psi), \quad (i = \sqrt{-1}), \tag{1b}$$

$$\chi = \vartheta_2^1 - \frac{u_z^1}{R}, \tag{1c}$$

$$T = V_1^1\sqrt{12(1-\nu^2)}\,\frac{1}{Eh^2},\ \frac{d}{b\,d\psi}(RV_1^1) = RN_1^1, \tag{1d,e}$$

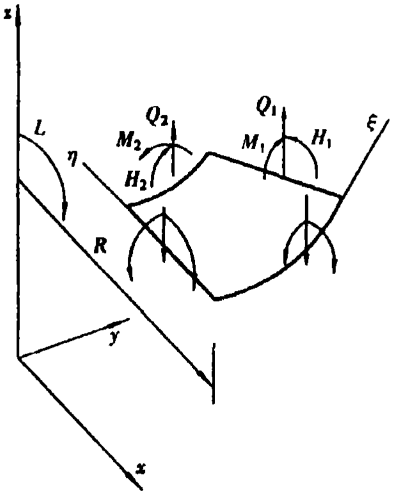


图 3 环板单元体上的内力

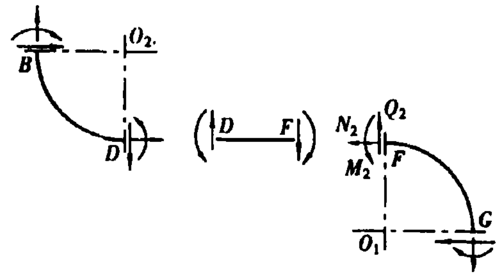


图 4 子午面内半波单元连接点上的内力

ϑ_2^1、u_z^1、N_1^1 和 M_1^1，如图 2 所示，分别为子午线绕纬线的转角 ϑ_2、中面线位移在回转轴 z 上的投影 u_z、纬线方向的薄膜力 N_1 和弯矩 M_1 在弯曲对称面($\theta = 0$ 的子午面)上所对应的量．

1.3 环板微分方程

环板在径向平面内(xz 平面内)受集中力偶作用，单元体受力如图 3 所示，不计薄膜力．其基本微分方程可由文[12]式(7)退化($\alpha = 1, \psi = \pi/2, \chi \to V, d\eta \to dr, L_3^0 \to L^0$)而成[8]

$$r d^2\chi/dr^2 + d\chi/dr - 4\chi/r = -L_3^0/r, \tag{2}$$

其中，$r = R/R_1$；χ 为环壳方程复变量 $V = \chi + iT$ 的实部．从文[12]的式(12)和(16)，我们得到内力矩：

$$\left.\begin{aligned} R_1M_1^1/D &= \nu(\chi/r + d\chi/dr) - \chi/r,\\ R_1M_2^1/D &= \chi/r + d\chi/dr - \nu\chi/r,\\ R_1H^1/D &= -(1-\nu)\chi/r \end{aligned}\right\} \tag{3a,b,c}$$

及外力偶所在平面内环板内边缘相对于外边缘的转角：

$$\vartheta_2^1 = \int_{r_1}^{r_2} \frac{\mathrm{d}(r\chi)}{r}, \qquad (r_1 = 1, r_2 = R_2/R_1). \tag{4}$$

2　细环壳的一般解

首先考虑 GF 段环壳，其解为记 V_{I}．由图1和1.1中基本符号可知，方程(1)中的参数应作相应的变换：　$\to R_m, \alpha_1 \to \alpha, \mu_1 \to \mu, L_1^0 \to L^0$；并作坐标变换：$\varphi \to \psi(\psi = \pi/2 - \varphi)$，因而，$-L_1^0(\varphi) \to L_1^0(\psi$．于是，方程(1)在 GF 区域内可以写成

$$\frac{\mathrm{d}^2 V_{\mathrm{I}}}{\mathrm{d}\varphi^2} + 2\mu_1 \mathrm{i}\sin\varphi V_{\mathrm{I}} = L_1^0 \cos\varphi. \tag{5}$$

根据文[12]，方程(5)的解 V_{I} 由非齐次解 V_{I}^* 和齐次解 V_{I}^0 两部分组成，即

$$V_{\mathrm{I}} = V_{\mathrm{I}}^* + V_{\mathrm{I}}^0, \tag{6a}$$

其中

$$\begin{aligned} V_{\mathrm{I}}^* = -\frac{L_1^0}{\mu_1}\{ & A_1\cos\varphi + A_2\sin2\varphi - A_3\cos3\varphi - A_4\sin4\varphi + A_5\cos5\varphi + \\ & A_6\sin6\varphi - A_7\cos7\varphi - A_8\sin8\varphi + \cdots\}, \end{aligned} \tag{6b}$$

$$\begin{aligned} V_{\mathrm{I}}^0 = \frac{L_1^0}{\mu_1}(C_{\mathrm{I}} + \mathrm{i}C'_{\mathrm{I}})\mathrm{e}^{-\lambda_1\pi/2}\Big\{ & [F_1(\varphi) + \mathrm{i}F_2(\varphi)]\cosh\lambda_1\left(\frac{\pi}{2} - \varphi\right) - \\ & [G_1(\varphi) + \mathrm{i}G_2(\varphi)]\sinh\lambda_1\left(\frac{\pi}{2} - \varphi\right)\Big\}, \end{aligned} \tag{6c}$$

$$\left.\begin{aligned} F_1(\varphi) &= -b_2\sin2\varphi - b_4\sin4\varphi - \cdots - b_{2n}\sin2n\varphi - \cdots, \\ F_2(\varphi) &= b_1\cos\varphi + b_3\cos3\varphi + \cdots + b_{2n+1}\cos(2n+1) + \cdots, \\ G_1(\varphi) &= 1 + a_2\cos2\varphi + a_4\cos4\varphi + \cdots + a_{2n}\cos2n\varphi + \cdots, \\ G_2(\varphi) &= a_1\sin\varphi + a_3\sin3\varphi + \cdots + a_{2n}\sin(2n+1) + \cdots, \\ & \qquad (n = 1,2,3,\cdots). \end{aligned}\right\} \tag{6d}$$

式(6c)已经用了波纹管整体纯弯曲时每个波的变形关于波峰（G 点）对称的条件，使积分常数由原来的4个减少到2个(C_{I} 和 C'_{I})．实指数 $\lambda_1 \equiv \lambda(\mu_1)$，级数项系数 A_1、A_{2n+1}、$A_{2n} = \mathrm{i}B_{2n}$、$a_n$ 和 b_n 是 μ 的函数(此处 $\mu = \mu_1$)，其算式详见文献[12,13]．把式(6)代入文[12]的式(21)，得到内力

$$\begin{aligned} -N_1^1\Big/\frac{2\alpha_1^2 L}{b^2\pi} = {} & 2B_2\cos2\varphi - 4B_4\cos4\varphi + 6B_6\cos6\varphi - 8B_8\cos8\varphi + \cdots - \\ & C_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\Big\{[F_2(\varphi) + \lambda_1 G_2(\varphi)]\cosh\lambda_1\left(\frac{\pi}{2} - \varphi\right) - \\ & [G_2(\varphi) + \lambda_1 F_2(\varphi)]\sinh\lambda_1\left(\frac{\pi}{2} - \varphi\right)\Big\} - \\ & C'_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\Big\{[F_1(\varphi) + \lambda_1 G_1(\varphi)]\cosh\lambda_1\left(\frac{\pi}{2} - \varphi\right) - \\ & [G_1(\varphi) + \lambda_1 F_1(\varphi)]\sinh\lambda_1\left(\frac{\pi}{2} - \varphi\right)\Big\}, \end{aligned} \tag{7a}$$

$$\begin{aligned} N_2^1\Big/\frac{\alpha_1^2 L}{b^2\pi} = {} & (B_2 - 1)\sin\varphi + (B_2 - B_4)\sin3\varphi - (B_4 - B_6)\sin5\varphi + \\ & (B_6 - B_8)\sin7\varphi - (B_8 - B_{10})\sin9\varphi + \cdots - \end{aligned}$$

$$2C_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\cos\varphi\Big[F_2(\varphi)\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-G_2(\varphi)\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big]-$$
$$2C'_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\cos\varphi\Big[F_1(\varphi)\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-G_1(\varphi)\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big],\tag{7b}$$

$$-S^1\Big/\frac{2\alpha_1^2L}{b^2\pi}=B_2\sin2\varphi-B_4\sin4\varphi+B_6\sin6\varphi-B_8\sin8\varphi+\cdots-$$
$$C_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\Big[F_2(\varphi)\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-G_2(\varphi)\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big]-$$
$$C'_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\Big[F_1(\varphi)\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-G_1(\varphi)\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big],\tag{7c}$$

$$-Q_2^1\Big/\frac{\alpha_1^2L}{b^2\pi}=(1+B_2)\cos\varphi-(B_2+B_4)\cos3\varphi+(B_4+B_6)\cos5\varphi-$$
$$(B_6+B_8)\sin7\varphi+(B_8+B_{10})\cos9\varphi+\cdots-$$
$$2C_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\sin\varphi\Big[F_2(\varphi)\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-G_2(\varphi)\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big]-$$
$$2C'_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\sin\varphi\Big[F_1(\varphi)\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-G_1(\varphi)\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big],\tag{7d}$$

$$M_2^1\Big/\frac{\alpha_1^2L}{\mu_1\pi b}=A_1\sin\varphi-3A_3\sin3\varphi+5A_5\sin5\varphi-7A_7\sin7\varphi+\cdots+$$
$$C_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\Big\{[F_1(\varphi)+\lambda_1G_1(\varphi)]\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-$$
$$[G_1(\varphi)+\lambda_1F_1(\varphi)]\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big\}-$$
$$C'_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\Big\{[F_2(\varphi)+\lambda_1G_2(\varphi)]\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-$$
$$[G_2(\varphi)+\lambda_1F_2(\varphi)]\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big\},\tag{7e}$$

$$M_1^1=\nu M_2^1,\tag{7f}$$

$$H^1\Big/(1-\nu)\frac{\alpha_1^3L}{\mu_1\pi b}=A_1\cos\varphi-A_3\cos3\varphi+A_5\cos5\varphi-A_7\cos7\varphi+\cdots-$$
$$C_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\Big[F_1(\varphi)\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-G_1(\varphi)\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big]+$$
$$C'_{\mathrm{I}}\mathrm{e}^{-\lambda_1\pi/2}\Big[F_2(\varphi)\cosh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)-G_2(\varphi)\sinh\lambda_1\Big(\frac{\pi}{2}-\varphi\Big)\Big].\tag{7g}$$

由文[12]的式(22)得到平行圆截面 F 相对 G 的转角：

$$\Omega_1=-\vartheta_2^1=\mathrm{Re}V_{\mathrm{I}}(0)-\alpha_1\int_0^{\pi/2}\mathrm{Re}V_{\mathrm{I}}\cos\varphi\mathrm{d}\varphi.\tag{8a}$$

把式(6)代入(8a)，积分后得：

$$\Omega_1=-\frac{L_1^0}{\mu_1}\Big\{\Lambda+\frac{C_{\mathrm{I}}}{2}(1-\mathrm{e}^{-\lambda_1\pi})G_1(0)+\frac{C'_{\mathrm{I}}}{2}(1+\mathrm{e}^{-\lambda_1\pi})F_2(0)-$$
$$\alpha_1\Big[\frac{\pi}{4}A_1+\frac{C_{\mathrm{I}}}{2}(1+\mathrm{e}^{\lambda_1\pi})\Gamma_{\mathrm{I}}+\frac{C'_{\mathrm{I}}}{2}(1-\mathrm{e}^{\lambda_1\pi})\Gamma'_{\mathrm{I}}\Big]\Big\},\tag{8b}$$

$$\Lambda=A_1-A_3+A_5-A_7+\cdots,\tag{8c}$$

$$G_1(0)=1+a_2+a_4+a_6+\cdots,\tag{8d}$$

$$F_2(0) = b_1 + b_3 + b_5 + \cdots, \tag{8e}$$

$$\Gamma_{\mathrm{I}} = \frac{1}{2}\left\{\frac{1}{\lambda_1^2+1}[\lambda_1(2+a_2)+b_2] + \frac{1}{\lambda_1^2+3^2}[\lambda_1(a_2+a_4)+3(b_2+b_4)] + \frac{1}{\lambda_1^2+5^2}[\lambda_1(a_4+a_6)+5(b_4+b_6)] + \cdots\right\}, \tag{8f}$$

$$\Gamma'_{\mathrm{I}} = \frac{1}{2}\left\{\frac{b_1}{\lambda_1} + \frac{1}{\lambda_1^2+2}[\lambda_1(b_1+b_3)-2(a_1+a_3)] + \frac{1}{\lambda_1^2+4^2}[\lambda_1(b_3+b_5) - 4(a_3+a_5)] + \frac{1}{\lambda_1^2+6^2}[\lambda_1(b_5+b_7)-6(a_5+a_7)] + \cdots\right\}, \tag{8g}$$

其中,级数项系数 A_{2n-1}, a_n 和 $b_n(n = 1,2,3,\cdots)$ 是 μ 的函数;此处 $\mu = \mu_1$.

类似地,考虑 DB 段环壳,其解记为 V_{II}. 对方程(1)中的参数应作相应的变换:$R_2 \to R_m$, $a_2 \to a, \mu_2 \to \mu, L_2^0 \to L^0$;并作坐标变换:$\bar{\varphi} \to \psi(\psi = 3\pi/2 - \bar{\varphi})$,并且,$L_2^0(\bar{\varphi}) \to L_2^0(\psi)$. 于是,方程(1)在 DB 区域内可以写成

$$\frac{\mathrm{d}^2 V_{\mathrm{II}}}{\mathrm{d}\bar{\varphi}^2} - 2\mu_2 \mathrm{i}\sin\bar{\varphi} V_{\mathrm{II}} = L_2^0 \cos\bar{\varphi}. \tag{9}$$

方程(9)的解可以写成

$$V_{\mathrm{II}} = V_{\mathrm{II}}^* + V_{\mathrm{II}}^0, \tag{10a}$$

$$V_{\mathrm{II}}^* = -\frac{L_2^0}{\mu_2}\{A_1\cos\bar{\varphi} - A_2\sin2\bar{\varphi} - A_3\cos3\bar{\varphi} + A_4\sin4\bar{\varphi} + A_5\cos5\bar{\varphi} - A_6\sin6\bar{\varphi} - A_7\cos7\bar{\varphi} + A_8\sin8\bar{\varphi} + \cdots\}, \tag{10b}$$

$$V_{\mathrm{II}}^0 = \frac{2L_2^0}{\mu_2}(C_{\mathrm{II}} + \mathrm{i}C'_{\mathrm{II}})e^{-\lambda_2\pi/2}\left\{-[F_1(\bar{\varphi}) - \mathrm{i}F_2(\bar{\varphi})]\cosh\lambda_2\left(\frac{\pi}{2} - \bar{\varphi}\right) + [G_1(\bar{\varphi}) - \mathrm{i}G_2(\bar{\varphi})]\sinh\lambda_2\left(\frac{\pi}{2} - \bar{\varphi}\right)\right\}. \tag{10c}$$

式(10c)已经用了波纹管整体纯弯曲时每个波的变形关于波谷(B 点)对称的条件,使积分常数由原来的4个减少到2个(C_{II} 和 C'_{II}). $F_1(\bar{\varphi})$、$F_2(\bar{\varphi})$、$G_1(\bar{\varphi})$ 和 $G_2(\bar{\varphi})$ 由 $\varphi = \pi + \bar{\varphi}$ 代入式(6d)而得,即 $F_1(\bar{\varphi})$ 表示 $F_1(\pi + \bar{\varphi})$,余类推,实指数 $\lambda_2 = \lambda(\mu_2)$,级数项系数 A_1、A_{2n+1}、$A_{2n} = \mathrm{i}B_{2n}$、$a_n$ 和 b_n 是 μ 的函数(此处 $\mu = \mu_2$),其算式详见文献[12,13]. 把式(10)代入文[12]的式(21),得到内力

$$N_1^1\Big/\frac{2a_2^2L}{b^2\pi} = 2B_2\cos2\bar{\varphi} - 4B_4\cos4\bar{\varphi} + 6B_6\cos6\bar{\varphi} - 8B_8\cos8\bar{\varphi} + \cdots + 2C_{\mathrm{II}}e^{-\lambda_2\pi/2}\left\{[F_2(\bar{\varphi}) + \lambda_2G_2(\bar{\varphi})]\cosh\lambda_2\left(\frac{\pi}{2} - \bar{\varphi}\right) - [G_2(\bar{\varphi}) + \lambda_2F_2(\bar{\varphi})]\sinh\lambda_2\left(\frac{\pi}{2} - \bar{\varphi}\right)\right\} - 2C'_{\mathrm{II}}e^{-\lambda_2\pi/2}\left\{[F_1(\bar{\varphi}) + \lambda_2G_1(\bar{\varphi})]\cosh\lambda_2\left(\frac{\pi}{2} - \bar{\varphi}\right) - [G_1(\varphi) + \lambda_2F_1(\varphi)]\sinh\lambda_2\left(\frac{\pi}{2} - \bar{\varphi}\right)\right\}, \tag{11a}$$

$$N_2^1\Big/\frac{a_2^2L}{b^2\pi} = (B_2 + 1)\sin\bar{\varphi} + (B_2 - B_4)\sin3\bar{\varphi} - (B_4 - B_6)\sin5\bar{\varphi} + (B_6 - B_8)\sin7\bar{\varphi} - (B_8 - B_{10})\sin9\bar{\varphi} + \cdots +$$

$$4C_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\cos\bar{\varphi}\left[F_2(\bar{\varphi})\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-G_2(\bar{\varphi})\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right]-$$
$$4C'_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\cos\bar{\varphi}\left[F_1(\bar{\varphi})\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-G_1(\bar{\varphi})\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right], \tag{11b}$$

$$S^1\bigg/\frac{2\alpha_2^2L}{b^2\pi}=B_2\sin2\bar{\varphi}-B_4\sin4\bar{\varphi}+B_6\sin6\bar{\varphi}-B_8\sin8\bar{\varphi}+\cdots+$$
$$2C_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\left[F_2(\bar{\varphi})\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-G_2(\bar{\varphi})\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right]-$$
$$2C'_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\left[F_1(\bar{\varphi})\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-G_1(\bar{\varphi})\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right], \tag{11c}$$

$$Q_2^1\bigg/\frac{\alpha_2^2L}{b^2\pi}=(B_2-1)\cos\bar{\varphi}-(B_2+B_4)\cos3\bar{\varphi}+(B_4+B_6)\cos5\bar{\varphi}-$$
$$(B_6+B_8)\sin7\bar{\varphi}+(B_8+B_{10})\cos9\bar{\varphi}+\cdots+$$
$$4C_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\sin\bar{\varphi}\left[F_2(\bar{\varphi})\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-G_2(\bar{\varphi})\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right]-$$
$$4C'_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\sin\bar{\varphi}\left[F_1(\bar{\varphi})\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-G_1(\bar{\varphi})\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right], \tag{11d}$$

$$M_2^1\bigg/\frac{\alpha_2^2L}{\mu_2\pi b}=A_1\sin\bar{\varphi}-3A_3\sin3\bar{\varphi}+5A_5\sin5\bar{\varphi}-7A_7\sin7\bar{\varphi}+\cdots-$$
$$2C_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\left\{[F_1(\bar{\varphi})+\lambda_2G_1(\bar{\varphi})]\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-\right.$$
$$\left.[G_1(\bar{\varphi})+\lambda_2F_1(\bar{\varphi})]\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right\}-$$
$$2C'_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\left\{[F_1(\bar{\varphi})+\lambda_2G_2(\bar{\varphi})]\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-\right.$$
$$\left.[G_2(\varphi)+\lambda F_2(\varphi)]\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right\}, \tag{11e}$$

$$\bar{M}_1^1=\nu\bar{M}_2^1, \tag{11f}$$

$$H^1\bigg/(1-\nu)\frac{\alpha_2^3L}{\mu_2\pi b}=A_1\cos\bar{\varphi}-A_3\cos3\bar{\varphi}+A_5\cos5\bar{\varphi}-A_7\cos7\bar{\varphi}+\cdots+$$
$$2C_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\left[F_1(\bar{\varphi})\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-G_1(\bar{\varphi})\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right]+$$
$$2C'_{\mathrm{II}}\mathrm{e}^{-\lambda_2\pi/2}\left[F_2(\bar{\varphi})\cosh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)-G_2(\bar{\varphi})\sinh\lambda_2\left(\frac{\pi}{2}-\bar{\varphi}\right)\right], \tag{11g}$$

由文[12]的式(22)得到平行圆截面 B 相对 D 的转角：

$$\Omega_2=-\vartheta_2^1=\mathrm{Re}V_{\mathrm{II}}(0)-\alpha_2\int_0^{\pi/2}\mathrm{Re}V_{\mathrm{II}}\cos\bar{\varphi}\mathrm{d}\bar{\varphi}. \tag{12a}$$

把式(10)代入(12a)，积分后得：

$$\Omega_2=\frac{L_2^0}{\mu_2}\left\{\Lambda-C_{\mathrm{II}}(1-\mathrm{e}^{-\lambda_2\pi})G_1(0)+C'_{\mathrm{II}}(1+\mathrm{e}^{\lambda_2\pi})F_2(0)+\right.$$
$$\left.\alpha_2\left[\frac{\pi}{4}A_1-C_{\mathrm{II}}(1+\mathrm{e}^{\lambda_2\pi})\Gamma_{\mathrm{I}}+C'_{\mathrm{II}}(1-\mathrm{e}^{\lambda_2\pi})\Gamma'_{\mathrm{I}}\right]\right\}, \tag{12b}$$

$$\Lambda=A_1-A_3+A_5-A_7+\cdots, \tag{12c}$$

$$G_1(0)=1+a_2+a_4+a_6+\cdots, \tag{12d}$$

$$F_2(0) = b_1 + b_3 + b_5 + \cdots, \tag{12e}$$

$$\Gamma_{\mathrm{II}} = \frac{1}{2}\left\{\frac{1}{\lambda_2^2+1}[\lambda_2(2+a_2)+b_2] + \frac{1}{\lambda_2^2+3^2}[\lambda_2(a_2+a_4)+3(b_2+b_4)] + \frac{1}{\lambda_2^2+5^2}[\lambda_2(a_4+a_6)+5(b_4+b_6)] + \cdots\right\}, \tag{12f}$$

$$\Gamma'_{\mathrm{II}} = \frac{1}{2}\left\{\frac{b_1}{\lambda_2} + \frac{1}{\lambda_2^2+2}[\lambda_2(b_1+b_3)-2(a_1+a_3)] + \frac{1}{\lambda_2^2+4^2}[\lambda_2(b_3+b_5) - 4(a_3+a_5)] + \frac{1}{\lambda_2^2+6^2}[\lambda_2(b_5+b_7)-6(a_5+a_7)] + \cdots\right\}, \tag{12g}$$

其中,级数项系数 A_{2n-1}, a_n 和 $b_n(n = 1,2,3,\cdots)$ 是 μ 的函数,此处 $\mu = \mu_2$.

3 环板的通解

环板基本方程(2)通解可以写成

$$\chi = \frac{L_3^0}{4}(1 + Br^{-2} + B'r^2), \tag{13}$$

其中,B 和 B' 为积分常数,$r = R/R_1$. 把式(13)代入式(3)和(4)得内力矩

$$\left.\begin{aligned}
\frac{R_1M_1^1}{D} &= -\frac{L_3^0}{4}[(1-\nu)r^{-1} + B(1+\nu)r^{-3} + B'(1-3\nu)r],\\
\frac{R_1M_2^1}{D} &= \frac{L_3^0}{4}[(1-\nu)r^{-1} - B(1+\nu)r^{-3} + B'(3-\nu)r],\\
\frac{R_1H^1}{D} &= -\frac{L_3^0}{4}(1-\nu)(r^{-1} + Br^{-3} + B'r)
\end{aligned}\right\} \tag{14a,b,c}$$

和角位移(外力偶所在平面内环板内边缘相对于外边缘的转角):

$$\Omega_3 = -\vartheta_2^1 = -\frac{L_3^0}{4}\left[\ln r_2 + \frac{B}{2}(r_2^2 - 1) + \frac{3B'}{2}(r_2^2-1)\right\}. \tag{15}$$

波纹管一个波($ACEGFDB$ 段)的相对转角,由于变形关于 G 点对称(该点的转角为零),可以写成:

$$\Omega = 2(\Omega_1 + \Omega_2 + \Omega_3), \tag{16}$$

其中 Ω_1, Ω_2 和 Ω_3 由式(8),(12)和(15)确定.

4 连续条件

上述各解中共含有6个待定的积分常数:$C_{\mathrm{I}}, C'_{\mathrm{I}}, C_{\mathrm{II}}, C'_{\mathrm{II}}, B$ 和 B'. 这些积分常数由环壳和环板连接处的连续性条件确定,可表述为:基本应变函数 χ 彼此相等,板壳外表面子午向合应力、环向合应力彼此相等. 由图4和式(6),(7),(10),(11),(13),(14),得

F 点,$\varphi = 0, r = 1$:

$$(\mathrm{Re}V_{\mathrm{I}})_{\mathrm{shell}} = (\chi)_{\mathrm{plate}}, \tag{17a}$$

$$\left(\frac{N_2^1}{h} + \frac{6M_2^1}{h^2}\right)_{\mathrm{shell}} = \left(\frac{6M_2^1}{h^2}\right)_{\mathrm{plate}}, \tag{17b}$$

$$\left(\frac{N_1^1}{h} + \frac{6M_1^1}{h^2}\right)_{\mathrm{shell}} = \left(\frac{6M_1^1}{h^2}\right)_{\mathrm{plate}}. \tag{17c}$$

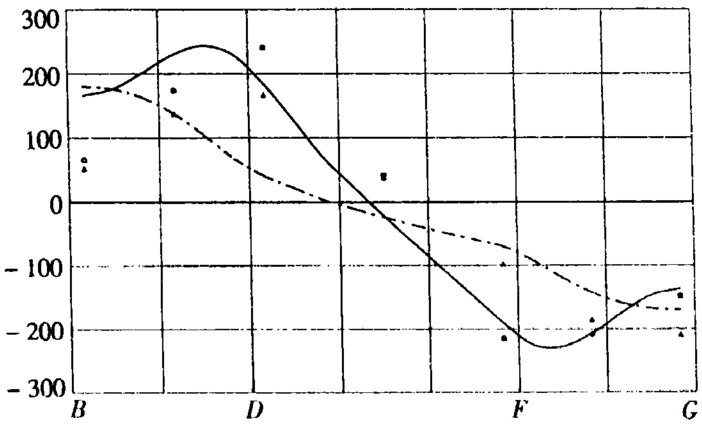
D 点,$\bar{\varphi} = 0, r = R_2/R_1 = r_2$:

$$(\mathrm{Re}V_{\mathrm{II}})_{\mathrm{shell}} = (\chi)_{\mathrm{plate}}, \tag{18a}$$

$$\left(\frac{N_2^1}{h}-\frac{6M_2^1}{h^2}\right)_{\text{shell}}=\left(\frac{6M_2^1}{h^2}\right)_{\text{plate}},\tag{18b}$$

$$\left(\frac{N_1^1}{h}-\frac{6M_1^1}{h^2}\right)_{\text{shell}}=\left(\frac{6M_1^1}{h^2}\right)_{\text{plate}}.\tag{18c}$$

这 6 个代数方程彼此独立,联立求解,得到 C_{I}、C'_{I}、C_{II}、C'_{II},B 和 B'. 至此,问题的解完全确定.



-·- 环向应力理论曲线; — 子午向应力理论曲线;

▲—环向实验应力点; ■—子午向实验应力点

图 5 U 型波纹管弯曲时受拉一侧外表面合应力(MPa)

5 算 例

以文献[11]实验模型为例. U 型波纹管圆筒内径 D_b = 277.5 mm,波高 w = 27.0 mm,波距 q = 29.0 mm,壁厚 h = 0.5 mm,b = 7.25 mm,R_1 = 159 mm,R_2 = 146.25 mm,波数 N = 4,转角 Ω = 3°(逆时针转动,与图 1 相反),材料为 1Cr18Ni9Ti,取 $E = 200\times10^3$ MPa,ν = 0.3.

用本文提出的公式,经过计算得到:

特征参数:

$$h/b\approx 1/15,\ h/(R_1-R_2)\approx 1/26,\ r_1=1, r_2=0.920,\ \alpha_1=0.046,$$
$$\mu_1=1.09,\ \lambda_1=0.781\,06_8,\ \alpha_2=0.050,\ \mu_2=1.19,\ \lambda_2=0.815\,98_3;$$

GF 段环壳级数项系数:

$$A_1=0.847\,59_4,\ A_3=-0.026\,89_0,\ A_5=0.000\,08_0;\ B_2=0.224\,11_9,$$
$$B_4=-0.001\,83_6,\ B_6=0.000\,00_0;\ a_1=0.558\,51_5,\ a_3=-0.028\,28_9,$$
$$a_5=0.000\,00_0;\ b_2=0.322\,26_1,\ b_4=0.000\,88_9, b_6=0.000\,00_0;$$

DB 段环壳级数项系数:

$$A_1=0.886\,66_8,\ A_3=-0.033\,11_9,\ A_5=0.000\,11_6;\ B_2=0.253\,43_1,$$
$$B_4=-0.002\,44_9,\ B_6=0.000\,00_0;\ a_1=0.560\,69_7,\ a_3=-0.037\,41_5,$$
$$a_5=0.000\,00_0;\ b_2=0.363\,90_9,\ b_4=0.009\,11_1, b_6=0.000\,00_0;$$

积分常数:

$$C_{\text{I}}=-0.795,\ C'_{\text{I}}=1.479,\ C_{\text{II}}=0.245,\ C'_{\text{II}}=0.620,$$

$$B = -0.488,\ B' = -0.540;$$

外力偶矩:

$$L = 90697\ \text{N}\cdot\text{mm}.$$

波纹管 *BDFG* 段(本例中为波纹管受拉一侧)外表面子午向合应力和环向合应力(弯曲应力和膜应力的代数和)按文[12]式(14)计算,应力曲线如图5所示. 子午向最大拉应力243.26 MPa,作用点离开 *B* 点60°;子午向最大压应力228.31 MPa,作用点离开 *G* 点60°;此外,按文[12]式(13),正应力沿环向按余弦规律变化. 点▲,■是文[11]的实验结果.

按EJMA计算,得到外力偶矩81 819 N·mm;子午向最大合应力234.45 MPa,作用在波纹管的波峰或波谷处.

可见,实验结果除位于波纹管波谷处的个别点外,其余的与本文的理论计算结果基本一致;按EJMA计算,得到的整体弯曲刚度比本文的计算结果小10%左右. 这在意料之中,因为EJMA的计算公式是以曲梁模型为基础的,它放松了环向约束,降低了刚度,因而也不能准确地反映应力分布规律.

6 结 论

本文根据文[12]提出的方程和解,给出了U型波纹管在子午面内受纯弯矩作用整体弯曲的解析解,并通过具体算例与文[11]的实验结果及按EJMA计算的结果作了比较. 算例中选用的U型波纹管,即文[11]的实验模型是目前最常见的U型波纹管. 其波高和波距都比较小:环壳的几何参数 α 一般小于0.1(多在0.05左右);环板的外径和内径之差约等于环壳的截面半径. 把这类波纹管简化成细环壳和小挠度环板是切合实际的;采用本文提出的解进行计算是有效的.

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General Solution for U-Shaped Bellows Overall-Bending Problems

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Abstract: The formulae for stresses and angular displacements of U-shaped bellows overall bending in a meridian plane under pure bending moments are presented based on the general solution for slender ring shells proposed by Zhu Weiping, et al. and the solution for ring plates. The results evaluated in this paper are compared with those on EJMA (standards of the expansion joint manufacturers association) and of the experiment given by Li Tingxin, et al.

Key words: flexible shells; shells of revolution; circular ring shells; bellows; U-shaped bellows