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抛物方程的时空有限元方法*

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摘要: 讨论了一类半线性抛物方程的自适应有限元方法,即空间连续、时间间断的时空有限元方法• 利用有限元方法和有限差分方法相结合的技巧,不对时空网格施加限制条件,证明弱解的存在唯一,并且给出了时间最大模、空间 L^2 模,即 $L^\infty(L^2)$ 模的误差估计,同时给出了数值分析结果,并对理论结果作了验证•

关 键 词: 半线性抛物方程; 时空有限元方法; 存在唯一; 误差估计

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引 言

本文考虑如下形式的方程

$$\begin{cases} u_t - \Delta u = f(u), & \Omega \times [0, T], \\ u|_{\partial\Omega} = 0, & \partial\Omega \times [0, T], \\ u(\cdot, 0) = u^0, & \Omega, \end{cases} \tag{1}$$

其中, $\Omega \in \mathbf{R}^2$, 函数 $f(u)$ 满足:

$$|f(u)| \leqslant c|u|, \qquad \forall u \in \mathbf{C}(\Omega)\bullet \tag{2}$$

而且, $f(u)$ 是 Lipschitz 连续函数, 即满足

$$|f(u) - f(v)| \leqslant L|u - v|, \qquad \forall u, v \in \mathbf{C}(\Omega), \tag{3}$$

L 为 Lipschitz 常数, c 为正常数•

我们知道, 对于上述形式的抛物方程, 其解结构特征复杂, 在有限时间内会出现奇点, 因此, 为求得准确的数值解, 有必要选择适应间断问题的有效数值方法• 空间连续, 时间间断的时空有限元方法, 在时间和空间两个方向同时使用有限元离散, 不仅实现了时空变量的统一处理, 而且能够灵活处理间断问题• 更为重要的是, 它可以在不同的时空片采用不同的网格剖分, 形成高度自适应方法• 自适应时空有限元方法, 有两个基本特点, 一是可靠性, 在计算过程中, 通过无结构时空网格的自动选择来控制离散误差, 尽可能减少误差积累; 一是有效性, 通过自适应算法实现几乎最优网格的生成, 并实现尽可能少的自由度•

利用自适应时空有限元方法求解抛物方程, 文[1]对线性模型进行了讨论, 并给出空间 L_2

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模误差估计• 在[2]中,首次给出了抛物型问题自适应方法的有效性和可靠性分析,并给出最优$L_\infty(L_2)$和$L_\infty(L_\infty)$模误差估计• 进一步,在[3]中推广到一般非线性问题• 在这些文章中,其共同特点是,在进行先验和后验误差估计时,以连续或离散的对偶问题的强稳定性和误差估计为基础,同时利用了局部误差估计结果、Galerkin正交性等性质,而且对时空网格施加了限制条件$k_n \geqslant ch^2$• 虽然在文[4]中重新定义依赖于网格步长的模,但网格还必须满足上述限制条件• 本文对方程(1)所讨论的间断时空有限元方法,类似于文[5]中对Schrödinger方程的讨论,不考虑对偶问题,把有限元方法和有限差分方法相结合,在时间离散区间I_n内,利用Radau点处的Lagrange插值多项式的特性,在对时空网格没有附加限制条件的情况下,证明了弱解的存在唯一,并给出时间最大模,空间L^2模,即$L^\infty(L^2)$模误差估计•

文中首先引入一些定义和概念,第2节给出弱解的存在唯一性证明,然后证明了有限元解的$L^\infty(L^2)$模误差估计,最后给出数值分析结果•

1 定义和概念

为了引进方程(1)的时空有限元方法,首先对时间区间$[0,T]$离散• 设$0=t^0<t^1<\cdots<t^N=T$, $I_n=(t^n,t^{n+1}]$,时间步长$k_n=t^{n+1}-t^n$, $n=0,1,2,\cdots,N-1$• 定义时空区域$Q:=\Omega\times[0,T]$,时空片$S^n:=\Omega\times I_n$• 并记$\{\Gamma_{hn}\}$是S^n的一种剖分,τ是剖分单元,e单元τ的边界• h_τ是单元τ的直径,$h_n=\max\limits_{\tau\in\Gamma_{hn}}h_\tau$, $h=\max\limits_n h_n$•

定义1 对每一时间区间I_n,定义有限元空间

$$S_h^n=\left\{\chi\in H_0^1(\Omega):\ \chi|_\tau\in P_{r-1}(\tau),\ \tau\in\Gamma_{hn}\right\},$$

其中,$P_{r-1}(\tau)$表示τ上的$r-1$次多项式,$n=0,1,\cdots,N-1$•

定义2 空间$V_h^n=\left\{\varphi|_{S^n}:=\varphi=\sum\limits_{j=0}^{q-1}t^j\chi_j(x),\ \chi\in S_h^n\right\}$• 即对$\forall t\in I_n$, $\varphi\in S_h^n$,而对$\forall x\in\Omega$, φ是t的$q-1$次多项式,并且在时间剖分点$t^n(n=0,1,\cdots,N-1)$处允许间断•

文中仍用$H^m(\Omega)$、$L^2(\Omega)$以及$L^\infty(\Omega)$表示通常的Sobolev空间,且它们的模分别记为$\|\cdot\|_m$, $\|\cdot\|$,和$\|\cdot\|_\infty$•

定义3 空间$L^2(I_n,L^2(\Omega))$上的范数:$|\!|\!|v|\!|\!|_n:=\left(\int_{I_n}\|v(t)\|^2\mathrm{d}t\right)^{1/2}$, $\forall v\in L^2(I_n,L^2(\Omega))$•

定义4 空间$L^\infty(I_n,L^2(\Omega))$上的范数为$\max\limits_{I_n}\|\cdot\|$,其中$\|\cdot\|$表示Sobolev空间$L^2(\Omega)$上的范数•

定义5 对给定的$s,m=0,1,\cdots$, $\forall v\in H^m(\Omega)$,定义模

$$\|h_n^s v\|_{m,h}=\left(\sum_{\tau\in\Gamma_{hn}}h_\tau^{2s}\|v\|_{m\tau}^2\right)^{1/2}•$$

定义6 椭圆投影π_n: $H_0^1(\Omega)\to S_h^n$,满足

$$(\nabla(\pi_n u-u),\nabla\chi)=0,\qquad\forall\chi\in S_h^n•$$

引理1 椭圆投影的误差估计

$$\|\nabla(u-\pi_n u)\|\leqslant c\|h_n^{s-1}u\|_{s,h},\qquad u\in H^s\cap H_0^1,\ 2\leqslant s\leqslant r•$$

$$\|u-\pi_n u\|\leqslant c\|h_n^s u\|_{s,h},\qquad u\in H^s\cap H_0^1,\ 2\leqslant s\leqslant r•$$

此引理的证明可参看[1, 2, 3]•

另外, 文中出现的 c 表示正常数, 并且每个 c 不一定相同•

2 弱解的存在唯一性

首先给出方程(1) 的弱形式• 为此, 用 $v \in C(Q)$ 乘方程的两端, 得

$$\int_{I_n}(u_t, v)\,\mathrm{d}t + \int_{I_n}(\nabla u, \nabla v)\,\mathrm{d}t + ([u^n], v_+^n) = \int_{I_n}(f(u), v)\,\mathrm{d}t•$$

其中时间跳跃项

$$[u^n] = u_+^n - u_-^n, \quad u_\pm^n(t) = \lim_{\varepsilon \to 0^\pm}(t^n + \varepsilon)•$$

$([u^n], v_+^n)$ 是 L^2 投影算子, 表示不同时空片之间的数据输运过程, 也体现了该方法在时间节点 $t^0, t^1, t^2, \cdots, t^{N-1}$ 处允许间断的特点• 利用对时间 t 的积分: $\int_{I_n} u_t\,\mathrm{d}t = u^{n+1} - u_+^n$, 并代入相应的有限元空间近似函数, 经整理, 方程的弱形式为: 求 $U \in V_h^n$, 使得对 $\forall v_h \in V_h^n$, 有

$$-\int_{I_n}(U, v_{ht})\,\mathrm{d}t + \int_{I_n}(\nabla U, \nabla v_h)\,\mathrm{d}t + (U^{n+1}, v_h^{n+1}) - (U^n, v_{h+}^n) = \int_{I_n}(f(U), v_h)\,\mathrm{d}t• \tag{4}$$

下面证明弱解的存在唯一性• 为此, 对于每一 $q \geqslant 1$, 考虑 Radau 方法

$$\int_0^1 g(s)\,\mathrm{d}s \approx \sum_{j=1}^q w_j g(s_j), \qquad 0 < s_1 < s_2 < \cdots < s_q = 1• \tag{5}$$

此积分法则具有 $2q-2$ 阶精确度•

对于固定的 $q \geqslant 1$, 在插值节点 $s_1, s_2, \cdots, s_q$ 处, 定义 Lagrange 插值基函数 $\{l_i\}_{i=1}^q$, $l_i(s) = \prod_{j \neq i, j=1}^q (s - s_j)/(s_i - s_j)$• 进一步, 作线性变换 $t = t^n + sk_n$, 把 $[0, 1]$ 区间映射到区间 I_n, 则有

$$t^{n,j} = t^n + s_j k_n, \quad t^{n,q} = t^{n+1}, \quad l_{n,j}(t) = l_j(s),$$

$$w_{n,j}(t) = \int_{t^n}^{t^{n+1}} l_{n,j}(t)\,\mathrm{d}t = k_n \int_0^1 l_j(s)\,\mathrm{d}s = k_n w_j \qquad (j = 1, 2, \cdots, q)•$$

这样, 记 $U^{n,j}(x) = U(x, t^{n,j}) \in S_h^n$, 利用 Lagrange 插值, $U(x, t) \in \Omega \times I_n$ 可表示为

$$U(x, t) = \sum_{j=1}^q l_{n,j}(t)\, U^{n,j}(x)• \tag{6}$$

进一步, 在(4) 中, 取 $v_h = l_{n,i}(t)\,\phi(x) \in V_h^n$, $\phi(x) \in S_h^n$, 并令 $w_{n,j} = \int_{I_n} l_{n,j}^2\,\mathrm{d}t > 0$, 代入式(4), 整理得

$$\begin{aligned}
&-\int_{I_n}\left(\sum_{j=1}^q l_{n,j}(t)\,U^{n,j}, \sum_{k=1}^q l_{n,k}(t)\,l'_{n,i}(t^{n,k})\,\phi\right)\mathrm{d}t + \int_{I_n}\left(\sum_{j=1}^q l_{n,j}(t)\,\nabla U^{n,j},\right.\\
&\left.\sum_{k=1}^q l_{n,k} l_{n,i}(t^{n,k})\,\nabla \phi\right)\mathrm{d}t + (U^{n,q} l_{n,q}, l_{n,i}(t^{n+1})\,\phi) - l_{n,i}(t^n)(U^n, \phi) = \\
&-\sum_{j=1}^q w_{n,j} l'_{n,i}(t^{n,j})(U^{n,j}, \phi) + k_n w_i(\nabla U^{n,i}, \nabla \phi) + \\
&\delta_{q,i}(U^{n,q}, \phi) - l_{n,i}(t^n)(U^n, \phi) = \\
&\int_{I_n}(f(U), l_{n,i}(t)\,\phi)\,\mathrm{d}t•
\end{aligned}$$

定义不依赖于 k_n 的$q\times q$ 矩阵$\boldsymbol{M}$、$\boldsymbol{N}$, 且 $\boldsymbol{N}_{ij}=w_{n,j}l'_{n,i}(t^{n,j})=w_j l'_i(s_j)$, $\boldsymbol{M}_{ij}=\boldsymbol{e}_q\boldsymbol{e}_q^{\mathrm{T}}-\boldsymbol{N}_{ij}$, 其中 $\boldsymbol{e}_q^{\mathrm{T}}=(0,0,\cdots,0,1)\in\mathbf{R}^q$. 则对 $\forall\,\boldsymbol{Y}=(y^{n,1},\cdots,y^{n,q})^{\mathrm{T}}\in\mathbf{R}^q$, 有

$$\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{Y}=\sum_{i=1}^{q}\delta_{qi}y^{n,q}y^{n,i}-\sum_{i,j=1}^{q}w_{n,j}l'_{n,i}(t^{n,j})y^{n,i}y^{n,j}.$$

引理 2　设 $\boldsymbol{D}=\mathrm{diag}\left\{s_1,s_2,s_3,\cdots,s_q\right\}$, 令

$$\widetilde{\boldsymbol{M}}=\boldsymbol{D}^{-1/2}\boldsymbol{M}\boldsymbol{D}^{1/2},\quad \alpha=\frac{1}{2}\min\left\{\frac{w_1}{s_1},\cdots,\frac{w_{q-1}}{s_{q-1}},1+w_q\right\}>0,$$

则 $\widetilde{\boldsymbol{M}}$ 是正定的, 即对 $\forall\boldsymbol{X}\in\mathbf{R}^q$ 有

$$\boldsymbol{X}^{\mathrm{T}}\widetilde{\boldsymbol{M}}\boldsymbol{X}\geqslant\alpha\,|\boldsymbol{X}|^2=\alpha\left(\sum_{i=1}^{q}x_i^2\right).$$

此引理证明参看[5].

令 $\widetilde{U}^{n,j}=s_j^{-1/2}U^{n,j}\in S_h^n$, 则对 $\forall(x,t)\in\Omega\times I_n$, $U(x,t)$ 可表示为

$$U(x,t)=\sum_{j=1}^{q}s_j^{1/2}\widetilde{U}^{n,j}(x)l_{n,j}(t).$$

进一步, 在(4)中, 令 $v_h=s_i^{-1/2}l_{n,i}\phi$, $\phi\in S_h^n$, 则有

$$\begin{aligned}&\delta_{qi}(\widetilde{U}^q,\phi)-(U^n,s_i^{-1/2}l_{n,i}(t^n)\phi)-\sum_{j=1}^{q}w_{n,j}l'_{n,i}(t^{n,j})s_i^{-1/2}s_j^{1/2}(\widetilde{U}^{n,j},\phi)+\\&\qquad k_nw_i(\nabla\widetilde{U}^{n,i},\nabla\phi)=\int_{I_n}(f(U),s_i^{-1/2}l_{n,i}\phi)\,\mathrm{d}t\qquad(i=1,2,\cdots,q).\end{aligned}\tag{7}$$

定理 1　设 U^n 是S_h^{n-1} 中给定的解, 则对充分小的 k_n, 存在$\left\{\widetilde{U}^{n,j}\right\}_{j=1}^{q}\in\left\{S_h^n\right\}^q$ 满足(7), 且(4) 存在唯一解 $U\in V_h^n$.

证明　设$\left\{S_h^n\right\}^q$ 是有穷维空间, $\forall\boldsymbol{X}=(\chi_1,\chi_2,\cdots,\chi_q)$, $\Psi=(\phi_1,\phi_2,\cdots,\phi_q)\in\left\{S_h^n\right\}^q$, 其内积定义为: $(\boldsymbol{X},\Psi)=\sum_{i=1}^{q}(\chi_i,\phi_i)$. 相应的模定义为 $|||\boldsymbol{X}|||^2=\sum_{i=1}^{q}\|\chi_i\|^2$.

定义映射 F: $\left\{S_h^n\right\}^q\to\left\{S_h^n\right\}^q$,

$$\begin{aligned}(F(V)_i,\phi)=\;&\delta_{qi}(v_q,\phi)-\sum_{j=1}^{q}w_jl'_i(s_j)s_j^{1/2}s_i^{-1/2}(v_j,\phi)+k_nw_i(\nabla v_i,\nabla\phi)-\\&\int_{I_n}\left(f\left(\sum_{j=1}^{q}s_j^{1/2}l_{n,j}v_j\right),s_i^{-1/2}l_{n,i}\phi\right)\mathrm{d}t+(v_i,\phi)-(U^n,s_i^{-1/2}l_{n,i}(t^n)\phi)\\&\qquad\forall\phi\in S_h^n,\quad i=1,2,\cdots,q.\end{aligned}\tag{8}$$

由于f 是连续的, 所以 F 是连续映射. 根据Brower 不动点定理, F 存在不动点, $(F(V)_i,\phi)=(v_i,\phi)$. 因此(7) 有解存在, 不妨设为 V. 下面证明解的唯一性.

设 V 和V^* 是方程的两个解, 在(8) 中, 令 $\phi=v_i-v_i^*$, 并对 i 从 1 到 q 求和, 得

$$\begin{aligned}&\sum_{i=1}^{q}(v_i-v_i^*,v_i-v_i^*)=|||V-V^*|||^2=\sum_{i=1}^{q}(F(V-V^*)_i,v_i-v_i^*)=\\&\quad\sum_{i=1}^{q}\delta_{qi}(v_q-v_q^*,v_i-v_i^*)-\sum_{i,j=1}^{q}w_jl'_i(s_j)s_j^{1/2}s_i^{-1/2}(v_j-v_j^*,v_i-v_i^*)+\\&\quad\sum_{i=1}^{q}k_nw_i(\nabla(v_i-v_i^*),\nabla(v_i-v_i^*))+\sum_{i=1}^{q}(v_i-v_i^*,v_i-v_i^*)-\\&\quad\sum_{i=1}^{q}\int_{I_n}\left(f\left(\sum_{j=1}^{q}s_j^{1/2}l_{n,j}(v_j-v_j^*)\right),s_i^{-1/2}l_{n,i}(v_i-v_i^*)\right)\mathrm{d}t.\end{aligned}\tag{9}$$

对上式右端前两项有

$$(\boldsymbol{M}(V-V^*), V-V^*) \geqslant \alpha \| V-V^* \|^2.$$

对右端第 5 项, 由 f 满足的条件(2), 有

$$\left| \sum_{i=1}^{q} \int_{I_n} \left(f\left(\sum_{j=1}^{q} s_j^{1/2} l_{n,j}(v_j - v_j^*) \right), s_i^{-1/2} l_{n,i}(v_i - v_i^*) \right) \mathrm{d}t \right| \leqslant$$

$$\sum_{i=1}^{q} \int_{I_n} \left| \left(\sum_{j=1}^{q} s_j^{1/2} l_{n,j}(v_j - v_j^*), s_i^{-1/2} l_{n,i}(v_i - v_i^*) \right) \right| \mathrm{d}t \leqslant c k_n \| V-V^* \|^2.$$

所以由式(9) 有 $(\alpha - ck_n) \| V-V^* \|^2 \leqslant 0$. 取 $k_n \leqslant \alpha/2c$, 则得 $\| V-V^* \|^2 \leqslant 0$, 即 $V = V^*$.

故问题(4) 的解存在唯一.

3 误差估计

为给出有限元解的空间 L^2 模, 时间最大模, 即 $L^\infty(L^2)$ 模误差估计结果. 首先, 定义时间区间 $I_n = (t_n, t_{n+1}]$ 上的通常 Lagrange 插值算子, $\tilde{I}_n: C(I_n) \rightarrow P_{q-1}(I_n)$, 使得: $\tilde{I}_h y(t^{n,j}) = y(t^{n,j})$. $t^{n,j}$ 定义如前, $j = 1, 2, \cdots, q$, 插值节点为 Radau 点 $s_1, s_2, \cdots, s_q$, 可以看出对 $\forall x \in \Omega$, $\tilde{I}_h u(x, \cdot) \in P_{q-1}(I_n)$, 且 $\tilde{I}_h u(x, t^{n+1}) = u(t^{n+1})$. 设 $W = \tilde{I}_n \pi_h u(x, t)$, 则有

引理 3 对上面定义的 W, 有如下的误差估计

$$\| u - W \|_n \leqslant c k_n^q \| u^{(q)} \|_n + c k_n^{1/2} \max_{I_n} \| h_n^s u \|_{s,h}, \qquad 2 \leqslant s \leqslant r, \tag{10}$$

$$\max_{I_n} \| u - W \| \leqslant c k_n^q \max_{I_n} \| u^{(q)} \| + c \max_{I_n} \| h_n^s u \|_{s,h}, \qquad 2 \leqslant s \leqslant r. \tag{11}$$

其中, $u^{(q)}$ 表示函数 u 的 q 阶导数. 此引理的证明参看[6, 7].

定理 2 设 u 和 U 分别是(1) 和(4) 的解, 则有如下的 $L^\infty(L^2(\Omega))$ 模误差估计

$$\max_{t \in [0,T]} \| u(t) - U(t) \| \leqslant c \max_m k_m^q \max_{I_m} (\| u^{(q)} \| + \| u^{(q+1)} \| + \| \Delta u^{(q)} \|) +$$

$$c \max_m \max_{I_m} (\| h_m^s u_t \|_{s,h} + \| h_m^s u \|_{s,h}) + c \max_m \| [\eta^m] \|, \qquad 2 \leqslant s \leqslant r.$$

其中, $\eta^m = u^m - \pi_h u^m$.

证明 利用前面定义的 W, 误差 $e = U - u$ 可写为

$$e = U - u = (U - W) + (W - u) = \theta + \rho.$$

对 ρ 有引理 3 的估计结果, 因此只需估计 θ. 为此, 考虑 θ 满足的方程:

$$- \int_{I_n} (\theta, v_{h,t}) \mathrm{d}t + \int_{I_n} (\nabla\theta, \nabla v_h) \mathrm{d}t + (\theta^{n+1}, v_h^{n+1}) - (\theta^n, v_{h+}^n) =$$

$$\int_{I_n} (f(U), v_h) \mathrm{d}t + \int_{I_n} (W, v_{h,t}) \mathrm{d}t - \int_{I_n} (\nabla W, \nabla v_h) \mathrm{d}t - (W^{n+1}, v_h^{n+1}) + (W^n, v_{h,+}^n) =$$

$$\int_{I_n} ((f(U) - f(W)), v_h) \mathrm{d}t + \int_{I_n} (f(W), v_h) \mathrm{d}t + \int_{I_n} (W, v_{h,t}) \mathrm{d}t -$$

$$\int_{I_n} (\nabla W, \nabla v_h) \mathrm{d}t - (W^{n+1}, v_h^{n+1}) + (W^n, v_{h+}^n).$$

即 θ 满足的方程为

$$- \int_{I_n} (\theta, v_{h,t}) \mathrm{d}t + \int_{I_n} (\nabla\theta, \nabla v_h) \mathrm{d}t + (\theta^{n+1}, v_h^{n+1}) - \int_{I_n} (f(U) - f(W), v_h) \mathrm{d}t =$$

$$(\theta^n, v_{h+}^n) + \int_{I_n}(f(W), v_h)\mathrm{d}t + \int_{I_n}(W, v_{h,t})\mathrm{d}t -$$

$$\int_{I_n}(\nabla W, \nabla v_h)\mathrm{d}t - (W^{n+1}, v_h^{n+1}) + (W^n, v_{h+}^n)\cdot \tag{12}$$

其中 $W^0 = \pi_0 u^0$, $\theta^0 = U^0 - W^0 = u^0 - \pi_0 u^0$•

上式中, 令 $v_h = l_{n,i}\phi$, $\phi \in S_h^n$, 并利用插值 $W = I_h\pi_n u(x,t) = \sum_{j=1}^{q} l_{n,j}\pi_n u(x, t^{n,j})$, $\theta = U - W = \sum_{j=1}^{q} l_{n,j}\theta(x, t^{n,j}) = \sum_{j=1}^{q} l_{n,j}(U^{n,j} - \pi_n u(x, t^{n,j}))$, 则(12) 式左端可以写为

$$-\int_{I_n}\left(\sum_{j=1}^{q} l_{n,j}\theta^{n,j}, l'_{n,i}(t)\phi\right)\mathrm{d}t + \int_{I_n}\left(\sum_{j=1}^{q} l_{n,j}\nabla\theta^{n,j}, l_{n,i}\nabla\phi\right)\mathrm{d}t +$$

$$(l_{n,q}\theta^{n,q}, l_{n,i}(t^{n,q})\phi) - \int_{I_n}(f(U) - f(W), l_{n,i}\phi)\mathrm{d}t =$$

$$-\int_{I_n}\left(\sum_{j=1}^{q} l_{n,j}\theta^{n,j}, \sum_{j=1}^{q} l'_{n,i}(t^{n,j})l_{n,j}\phi\right)\mathrm{d}t +$$

$$\int_{I_n}\left(\sum_{j=1}^{q} l_{n,j}\nabla\theta^{n,j}, \sum_{j=1}^{q} l_{n,j}l_{n,i}(t^{n,j})\nabla\phi\right)\mathrm{d}t +$$

$$(l_{n,q}\theta^{n,q}, l_{n,i}(t^{n,q})\phi) - \int_{I_n}(f(U) - f(W), l_{n,i}\phi)\mathrm{d}t =$$

$$-\sum_{j=1}^{q} w_{n,j}l'_{n,i}(t^{n,j})(\theta^{n,j}, \phi) + k_n w_i(\nabla\theta^{n,i}, \nabla\phi) + \delta_{qi}(\theta^{n,q}, \phi) -$$

$$\int_{I_n}(f(U) - f(W), l_{n,i}\phi)\mathrm{d}t\cdot$$

而利用椭圆投影的定义, (12) 式右端为

$$(\theta^n, l_{n,i}(t^n)\phi) + \int_{I_n}(f(W), l_{n,i}\phi)\mathrm{d}t + \int_{I_n}\Big(\sum_{j=1}^{q} l_{n,j}\pi_n u(x, t^{n,j}), \sum_{j=1}^{q} l_{n,j}l'_{n,i}(t^{n,j})\phi\Big)\mathrm{d}t -$$

$$\int_{I_n}\left(\sum_{j=1}^{q} l_{n,j}\nabla\pi_n u(x, t^{n,j}), \sum_{j=1}^{q} l_{n,j}l_{n,i}(t^{n,j})\nabla\phi\right)\mathrm{d}t -$$

$$\delta_{qi}(\pi_n u(x, t^{n,q}), l_{n,i}(t^{n,q})\phi) + (W^n, l_{n,i}(t^n)\phi) =$$

$$(\theta^n, l_{n,i}(t^n)\phi) + \int_{I_n}(f(W), l_{n,i}\phi)\mathrm{d}t + \sum_{j=1}^{q} w_{n,j}l'_{n,i}(t^{n,j})(\pi_n u(x, t^{n,j}), \phi) -$$

$$w_i k_n(\nabla\pi_n u^{n,i}, \nabla\phi) - \delta_{qi}(\pi_n u(x, t^{n,q}), l_{n,i}(t^{n,q})\phi) + (W^n, l_{n,i}(t^n)\phi) =$$

$$(\theta^n, l_{n,i}(t^n)\phi) + \int_{I_n}(f(W), l_{n,i}\phi)\mathrm{d}t + \sum_{j=1}^{q} w_{n,j}l'_{n,i}(t^{n,j})(\pi_n u(x, t^{n,j}), \phi) -$$

$$w_i k_n(\nabla u^{n,i}, \nabla\phi) - \delta_{qi}(\pi_n u(x, t^{n,q}), l_{n,i}(t^{n,q})\phi) + (W^n, l_{n,i}(t^n)\phi)\cdot$$

令 $\eta = u - \pi_n u$, 并注意

$$-\int_{I_n}(u, v_t)\mathrm{d}t + \int_{I_n}(\nabla u, \nabla v)\mathrm{d}t + (u^{n+1}, v^{n+1}) - (u^n, v_+^n) = \int_{I_n}(f(u), v)\mathrm{d}t\cdot \tag{13}$$

(12) 式可以写为

$$-\sum_{j=1}^{q} w_{n,j}l'_{n,i}(t^{n,j})(\theta^{n,j}, \phi) + k_n w_i(\nabla\theta^{n,i}, \nabla\phi) + \delta_{qi}(\theta^{n,q}, \phi) -$$

$$
\begin{aligned}
&\int_{I_n}(f(U)-f(W),\ l_{n,i}\varphi)\,\mathrm{d}t= \\
&(\theta^n,\ l_{n,i}(t^n)\varphi)+\int_{I_n}(f(W),\ l_{n,i}\varphi)\,\mathrm{d}t+\sum_{j=1}^{q}w_{n,j}l'_{n,i}(t^{n,j})(\pi_n u(x,t^{n,j}),\ \varphi)- \\
&w_ik_n(\nabla u^{n,i},\ \nabla\varphi)-\delta_{qi}(\pi_n u(x,t^{n,q}),\ \varphi)+(W^n,\ l_{n,i}(t^n)\varphi)= \\
&(\theta^n,\ l_{n,i}(t^n)\varphi)+\int_{I_n}(f(W)-f(u),\ l_{n,i}\varphi)\,\mathrm{d}t+\sum_{j=1}^{q}w_{n,j}l'_{ni}(t^{n,j})(\pi_n u(x,t^{n,j}),\ \varphi)- \\
&w_ik_n(\nabla u^{n,i},\ \nabla\varphi)-\delta_{qi}(\pi_n u(x,t^{n,q}),\ \varphi)+(W^n,\ l_{n,i}(t^n)\varphi)-\int_{I_n}(u,\ l'_{n,i}\varphi)\,\mathrm{d}t+ \\
&\int_{I_n}(\nabla u,\ l_{n,i}\nabla\varphi)\,\mathrm{d}t+(u^{n+1},\ l_{n,i}(t^{n+1})\varphi)-(u^n,\ l_{n,i}(t^n)\varphi)= \\
&(\theta^n,\ l_{n,i}(t^n)\varphi)+\int_{I_n}(f(W)-f(u),\ l_{n,i}\varphi)\,\mathrm{d}t+\sum_{j=1}^{q}w_{n,j}l'_{ni}(t^{n,j})(u(x,t^{n,j}),\ \varphi)- \\
&\int_{I_n}(l'_{n,i}u,\ \varphi)\,\mathrm{d}t+w_ik_n(\Delta u^{n,i},\ \varphi)-\int_{I_n}(\Delta u,\ l_{n,i}\varphi)\,\mathrm{d}t+ \\
&\sum_{j=1}^{q}w_{n,j}l'_{n,i}(t^{n,j})(\pi_n u(x,t^{n,j})-u(x,t^{n,j}),\ \varphi)- \\
&(u^n-\pi_n u^n,\ l_{n,i}(t^n)\varphi)-(\pi_n u^n,\ l_{n,i}(t^n)\varphi)-\delta_{qi}(\pi_n u^{n,q}-u^{n,q},\ \varphi)- \\
&\delta_{qi}(u^{n,q},\ \varphi)+(W^n,\ l_{n,i}(t^n)\varphi)+(u^{n+1},\ l_{n,i}(t^{n+1})\varphi)= \\
&(\theta^n,\ l_{n,i}(t^n)\varphi)+\int_{I_n}(f(W)-f(u),\ l_{n,i}\varphi)\,\mathrm{d}t+ \\
&(\Sigma_1+\Sigma_2+\Sigma_3,\ \varphi)+([\eta^n],\ l_{n,i}(t^n)\varphi).
\end{aligned}
$$

其中

$$
\begin{aligned}
\Sigma_1={}&\sum_{j=1}^{q}w_{n,j}l'_{ni}(t^{n,j})(\pi_n u(x,t^{n,j})-u(x,t^{n,j}))- \\
&(u^n-\pi_n u^n)l_{n,i}(t^n)-\delta_{qi}(\pi_n u^{n,q}-u^{n,q}), \\
\Sigma_2={}&\sum_{j=1}^{q}w_{n,j}l'_{n,i}(t^{n,j})u(x,t^{n,j})-\int_{I_n}l'_{n,i}u\,\mathrm{d}t, \\
\Sigma_3={}&w_ik_n\Delta u^{n,i}-\int_{I_n}(\Delta u l_{n,i})\,\mathrm{d}t, \\
[\eta^n]={}&[u-\pi_n u]=u_+^n-\pi_n u_+^n-u_-^n+\pi_n u_-^n= \\
&-\pi_n u_+^n+\pi_n u_-^n=-(\pi_n u+\pi_{n-1}u)_{(t^n)}.
\end{aligned}
$$

即误差方程为

$$
\begin{aligned}
-&\sum_{j=1}^{q}w_{n,j}l'_{n,i}(t^{n,j})(\theta^{n,j},\ \varphi)+k_nw_i(\nabla\theta^{n,i},\ \nabla\varphi)+\delta_{qi}(\theta^{n,q},\ \varphi)- \\
&\int_{I_n}(f(U)-f(W),\ l_{n,i}\varphi)\,\mathrm{d}t= \\
&(\theta^n,\ l_{n,i}(t^n)\varphi)+\int_{I_n}(f(W)-f(u),\ l_{n,i}\varphi)\,\mathrm{d}t+ \\
&(\Sigma_1+\Sigma_2+\Sigma_3,\ \varphi)+([\eta^n]l_{n,i}(t^n),\ \varphi).
\end{aligned}
\tag{14}
$$

对(12)式左端，$v_h=\theta$ 时有

$$-\frac{1}{2}\int_{I_n}\frac{d}{dt}\|\theta\|^2dt+\int_{I_n}\|\nabla\theta\|^2dt+(\theta^{n+1},\theta^{n+1})-\int_{I_n}(f(U)-f(W),\theta)dt=$$
$$\frac{1}{2}\|\theta^{n+1}\|^2+\frac{1}{2}\|\theta_+^n\|^2+\int_{I_n}\|\nabla\theta\|^2-\int_{I_n}(f(U)-f(W),\theta)dt.$$

在(14)式中,令 $\phi=\theta^{n,i}$,并对 i 从1到 q 求和,则它和(12)式的左端取 $v_h=\theta$ 时相同,所以有

$$\frac{1}{2}\|\theta^{n+1}\|^2+\frac{1}{2}\|\theta_+^n\|^2\leqslant\int_{I_n}(f(U)-f(W),\theta)dt+(\theta^n,\theta_+^n)+$$
$$\int_{I_n}(f(W)-f(u),\theta)dt+\sum_{i=1}^{q}(\Sigma_1+\Sigma_2+\Sigma_3,\theta^{n,i})+([\eta^n],\theta_+^n).\tag{15}$$

上式中,对右端第1项,由于 f 是 Lipschitz 连续的,所以

$$\left|\int_{I_n}(f(U)-f(W),\theta)dt\right|\leqslant c|||\theta|||_n^2.\tag{16}$$

对右端第3项利用引理3和Holder 不等式有

$$\left|\int_{I_n}(f(W)-f(u),\theta)dt\right|\leqslant c\left(\int_{I_n}\|W-u\|^2\right)^{1/2}\left(\int_{I_n}\|\theta\|^2dt\right)^{1/2}\leqslant$$
$$(ck_n^q|||u^{(q)}|||_n+ck_n^{1/2}\max_{I_n}\|h_n^su\|_{s,h})|||\theta|||_n,\qquad 2\leqslant s\leqslant r.\tag{17}$$

在右端第4项中,对 Σ_1 有估计

$$\|\Sigma_1\|^2\leqslant ck_n\left(\int_{I_n}\|h_n^su_t\|_{s,h}^2\right),\qquad 2\leqslant s\leqslant r.\tag{18}$$

这是因为,对 $i=1,2,\cdots,q$ 有

$$\delta_{qi}-\sum_{j=1}^{q}w_{n,j}l'_{n,j}(t^{n,j})-l_{n,i}(t^n)=\delta_{qi}-\int_{t^n}^{t^{n+1}}l'_{n,i}dt-l_{n,i}(t^n)=0.$$

这意味着存在常数 c_{ij}(不依赖于 n)使得

$$\|\Sigma_1\|=\left\|\delta_{qi}\eta^{n,q}-\sum_{j=1}^{q}w_jl'_j(s_j)\eta^{n,j}-l_i(0)\eta^{n+}\right\|=$$
$$\left\|\sum_{j=1}^{q}c_{ij}(\eta^{n,j}-\eta^{n,j-1})\right\|=\left\|\sum_{j=1}^{q}c_{ij}\int_{t^{n,j-1}}^{t^{n,i}}\eta_t(s)ds\right\|=$$
$$\left\|\sum_{j=1}^{q}c_{ij}\int_{t^{n,j-1}}^{t^{n,j}}(u_t-\pi_nu_t)_{(s)}ds\right\|\leqslant c\int_{I_n}\|u_t-\pi_nu_t\|ds\leqslant$$
$$c\int_{I_n}\|h_n^su_t\|_{s,h}ds\leqslant ck_n^{1/2}\left(\int_{I_n}\|h_n^su_t\|_{s,h}^2ds\right)^{1/2},\qquad 2\leqslant s\leqslant r.$$

为了估计 Σ_2,设 $I_h^q: C(I^n)\to P_q(I_n)$ 为区间 I_n 上的 q 次插值算子,插值节点除Radau 节点外还有点 t_n,而且满足 $I_h^qy(t^{n,j})=y(t^{n,j})$, $I_h^qy(t^n)=y(t_+^n)$, $j=1,2,\cdots,q$. 显然,对每一 $x\in\Omega$, $l'_{ni}I_h^qu$ 为 $2q-2$ 次多项式.

$$\|\Sigma_2\|=\left\|\sum_{j=1}^{q}w_{n,j}l'_{ni}(t^{n,j})u^{n,j}-\int_{I_n}l'_{n,i}udt\right\|=\left\|\sum_{j=1}^{q}w_{n,j}l'_{n,i}(t^{n,j})I_h^q(u^{n,j})-\int_{I_n}l'_{n,i}udt\right\|=$$
$$\left\|\int_{I_n}l'_{n,i}(I_h^qu-u)dt\right\|\leqslant\left(\int_{I_n}|l'_{n,i}(t)|^2dt\right)^{1/2}|||I_h^qu-u|||_n\leqslant$$
$$c\left(k_n^{-1}\int_0^1|l'_i(s)|^2)ds\right)^{1/2}k_n^{q+1}|||u^{(q+1)}|||_n\leqslant ck_n^{q+1/2}|||u^{(q+1)}|||_n.\tag{19}$$

对于 Σ_3, 由于 $l_{n,i}\hat{I}_h u$ 是关于 t 的 $2q-2$ 次多项式, 所以

$$\|\Sigma_3\| = \left\|w_{n,i}\Delta u^{n,i} - \int_{I_n}\Delta u l_{n,i}\,\mathrm{d}t\right\| = \left\|\sum_{j=1}^{q} w_{n,j} l_{n,j}(t^{n,j})\Delta u^{n,j} - \int_{I_n}\Delta u l_{n,i}\,\mathrm{d}t\right\| =$$

$$\left\|\int_{I_n} l_{n,i}(\hat{I}_n - I)\Delta u\,\mathrm{d}t\right\| \leqslant c\left(\int_{I_n}|l_{n,i}|^2\mathrm{d}t\right)^{1/2}\left(\int_{I_n}\|\hat{I}_n\Delta u - \Delta u\|^2\mathrm{d}t\right)^{1/2} \leqslant$$

$$ck_n^{q+1/2}|||\Delta u^{(q)}|||_n. \tag{20}$$

利用(参看[5]) $|||\theta|||_n = \left(\sum_{j=1}^{q} w_{n,j}\|\theta^{n,j}\|^2\right)^{1/2} = k_n^{1/2}\left(\sum_{j=1}^{q} w_j\|\theta^{n,j}\|^2\right)^{1/2}$. (15) 式右端第 4 项有如下估计

$$\sum_{i=1}^{q}(\Sigma_1+\Sigma_2+\Sigma_3,\theta^{n,i}) \leqslant (\|\Sigma_1\|+\|\Sigma_2\|+\|\Sigma_3\|)\,|||\theta||| \leqslant$$

$$c\left(\int_{I_n}\|h_n^s u_t\|_{s,h}^2\mathrm{d}s\right) + ck_n^{2q}|||u^{(q+1)}|||_n^2 + k_n^{2q}|||\Delta u^{(q)}|||_n^2 + |||\theta|||_n^2. \tag{21}$$

这样对(15)式, 利用式(16)~(20)以及式(21), 有

$$\|\theta^{n+1}\|^2 \leqslant c|||\theta|||_n^2 + c\|\theta^n\|^2 + ck_n^{2q}|||u^{(q)}|||_n^2 + ck_n\max_{I_n}\|h_n^s u\|_{s,h}^2 +$$

$$c\int_{I_n}\|h_n^s u_t\|_{s,h}^2\mathrm{d}s + ck_n^{2q}|||u^{(q+1)}|||_n^2 + ck_n^{2q}|||\Delta u^{(q)}|||_n^2 + c\|[\eta^n]\|^2. \tag{22}$$

对于 $|||\theta|||_n$, 我们有如下估计结果

$$|||\theta|||_n^2 \leqslant ck_n\left\{\|\theta^n\|^2 + k_n^{2q+1}(|||u^{(q)}|||_n^2 + |||u^{(q+1)}|||_n^2 + |||\Delta u|||_n^2) +\right.$$

$$\left. k_n\left(\int_{I_n}\|h_n^s u_t\|_{s,h}^2\mathrm{d}t + k_n\max_{I_n}\|h_n^s u\|_{s,h}^2\right) + \|[\eta^n]\|^2\right\}. \tag{23}$$

为证明(23)式, 把 θ 写为 I_n 上的插值形式, $\theta = \sum_{j=1}^{q} s_j^{1/2} l_{nj}\theta^{n,j}$, 代入(14)式, 并且两边同乘 $s_i^{-1/2}$, 得

$$-\sum_{j=1}^{q} w_{n,j} l'_{n,i}(t^{n,j}) s_j^{1/2} s_i^{-1/2}(\theta^{n,j},\phi) + k_n w_i(\nabla\theta^{n,i},\nabla\phi) + \delta_{qi}(\theta^{n,q},\phi) -$$

$$\int_{I_n} s_i^{-1/2}(f(U)-f(W), l_{n,i}\phi)\,\mathrm{d}t =$$

$$s_i^{-1/2}(\theta^n, l_{n,i}(t^n)\phi) + \int_{I_n}(f(W)-f(u), l_{n,i}\phi) +$$

$$(\Sigma_1+\Sigma_2+\Sigma_3,\phi) + ([\eta^n]l_{n,i}(t^n),\phi). \tag{24}$$

在上式中取 $\phi = \theta^{n,i}$, 并对 i 从 1 到 q 求和, 并注意

$$\sum_{i=1}^{q}\delta_{qi}(\theta^{n,q},\theta^{n,i}) - \sum_{i,j=1}^{q} w_{n,j} l'_{ni}(t^{n,j}) s_j^{1/2} s_i^{-1/2}(\theta^{n,j},\theta^{n,i}) = (\boldsymbol{M}\Psi^n,\Psi^n).$$

其中 $\Psi^n = (\theta^{n,q},\theta^{n,2},\dots,\theta^{n,q})$. 而 $(\boldsymbol{M}\Psi^n,\Psi^n) \geqslant \alpha\sum_{j=1}^{q}\|\theta^{n,j}\|^2$ 且

$$\left|\sum_{i=1}^{q} s_i^{-1/2}\int_{I_n}(f(U)-f(W), l_{n,i}\theta^{n,i})\,\mathrm{d}t\right| \leqslant$$

$$\left|\sum_{i=1}^{q} s_i^{-1/2}\int_{I_n}\left(\sum_{j=1}^{q} l_{n,j}\theta^{n,j}, l_{n,i}\theta^{n,i}\right)\mathrm{d}t\right| = \left|\sum_{i=1}^{q} s_i^{-1/2} w_i k_n(\theta^{n,i},\theta^{n,i})\right| \leqslant$$

$$ck_n^{1/2}\left(\sum_{i=1}^{q} s_i^{-1/2} w_i \|\theta^{n,i}\| \cdot \|\theta^{n,i}\| k_n^{1/2}\right) \leqslant$$

$$k_n^{1/2}\left(\sum_{i=1}^{q} w_i^2 \|\theta^{n,i}\|^2 s_i^{-1}\right)^{1/2} k_n^{1/2}\left(\sum_{i=1}^{q} \|\theta^{n,i}\|^2\right)^{1/2} \leqslant$$

$$ck_n^{1/2}\left(\sum_{i=1}^{q} \|\theta^{n,i}\|^2\right)^{1/2} ||| \theta |||_n \cdot$$

所以有

$$\alpha \sum_{j=1}^{q} \|\theta^{n,j}\|^2 \leqslant c\left\{\sum_{j=1}^{q} \|\theta^{n,j}\|^2\right\}^{1/2} \left\{ \|\theta^n\| + ck_n^{1/2} ||| \theta |||_n + \right.$$

$$\left. ck_n^{1/2} ||| u - W |||_n + \left(\sum_{i=1}^{q} (\|\Sigma_1\|^2 + \|\Sigma_2\|^2 + \|\Sigma_3\|^2\right)^{1/2} + \|[\eta^n]\| \right\} \cdot$$

当 k_n 充分小时

$$||| \theta_n |||_n \leqslant ck_n^{1/2} \|\theta^n\| + ck_n ||| u - W |||_n +$$

$$ck_n^{1/2}\left(\sum_{i=1}^{q} (\|\Sigma_1\|^2 + \|\Sigma_2\|^2 + \|\Sigma_3\|^2)\right)^{1/2} + ck_n^{1/2} \|[\eta^n]\| \cdot$$

将(10)和(21)代入上式, 可得(23)•

将(23)代入(22)式, 得

$$\|\theta^{n+1}\|^2 \leqslant c(1+k_n) \|\theta^n\|^2 + ck_n^{2q}(||| u^{(q)} |||_n^2 + ||| u^{(q+1)} |||_n^2 + ||| \Delta u^{(q)} |||_n^2) +$$

$$c\int_{I_n} \|h_n^s u_t\|_{s,h}^2 \mathrm{d}t + ck_n \max_{I_n} \|h_n^s u\|_{s,h}^2 + c(k_n+1) \|[\eta^n]\|^2 \cdot$$

利用叠代得

$$\|\theta^{n+1}\|^2 \leqslant \prod_{j=0}^{n} c(1+k_j) \|\theta^0\|^2 + c\sum_{m=0}^{n}\left(\prod_{j=m+1}^{n} c(1+k_j)\right)\left[k_m^{2q}\left\{||| u^{(q)} |||_m^2 + \right.\right.$$

$$\left. ||| u^{(q+1)} |||_m^2 + ||| \Delta u^{(q)} |||_m^2 \right\} + c\int_{I_m} \|h_m^s u_t\|_{s,h}^2 \mathrm{d}t +$$

$$\left. ck_m \max_{I_m} \|h_m^s u\|_{s,h}^2 + (ck_m + c) \|[\eta^m]\|^2 \right] \cdot$$

即

$$\|\theta^{n+1}\| \leqslant c\|u^0 - \pi_0 u^0\| + c\left\{\sum_{m=0}^{n} (k_m^{2q}(||| u^{(q)} |||_m^2 + ||| u^{(q+1)} |||_m^2 + ||| \Delta u^{(q)} |||)_m^2) + \right.$$

$$\left. \int_{I_m} \|h_m^s u_t\|_{s,h}^2 \mathrm{d}t + k_m \max_{I_m} \|h_m^s u\|_{s,h}^2 \right\}^{1/2} + c\left(\sum_{m=1}^{n} \|[\eta^m]\|^2\right)^{1/2} \cdot$$

由式(23)得

$$||| \theta |||_n \leqslant ck_n^{1/2} \|h_0^s u^0\|_{s,h} + ck_n^{1/2}\left\{\sum_{m=0}^{n} (k_m^{2q}(||| u^{(q)} |||_m^2 + ||| u^{(q+1)} |||_m^2 + \right.$$

$$\left. ||| \Delta u^{(q)} |||_m^2)) + \int_{I_m} \|h_m^s u_t\|_{s,h}^2 \mathrm{d}t + k_m \max_{I_m} \|h_m^s u\|_{s,h}^2 \right\}^{1/2} +$$

$$ck_n^{1/2}\left(\sum_{m=1}^{n} \|[\eta^m]\|^2\right)^{1/2} \cdot$$

因为 $\theta|_{I_n} \in V_{hk}^n$, 逆不等式 $\max_{I_n} |y(t)| \leqslant ck_n^{-1/2}\left(\int_{I_n} |y(t)|^2\right)^{1/2}$, $\forall y \in \boldsymbol{P}_{q-1}(I_n)$• 成立, 因此

$$\max_{I_n}\|\theta\| \leqslant c\|h_0^s u^0\|_{s,h} + c\left\{\sum_{m=0}^{n}\left(k_m^{2q}\left(|\!|\!| u^{(q)} |\!|\!|_m^2 + |\!|\!| u^{(q+1)} |\!|\!|_m^2 + |\!|\!| \Delta u^{(q)} |\!|\!|_m^2\right) + \int_{I_m}\|h_m^s u_t\|_{s,h}^2 + k_m \max_{I_m}\|h_m^s u\|_{s,h}\right)\right\}^{1/2} + c\left(\sum_{m=1}^{n}\|[\eta^m]\|^2\right)^{1/2}.$$

至此,联合引理 3 可得定理的证明•

4 数值结果

利用最简单的线性基,这里取 $q = 1$,给出一维热传导方程和含有源项 $2u$ 的抛物方程的数值计算结果• 首先讨论具有间断初值条件的齐次热传导方程一维 Cauchy 问题

$$\begin{cases} u_t - u_{xx} = 0 & x \in [-3, 3],\ t \in [0, T], \\ u(-3, t) = u(3, t) = 0 & t \in [0, T], \\ u|_{t=0} = \begin{cases} 0 & -3 \leqslant x \leqslant -1, \\ 1.0 & -1 \leqslant x \leqslant 1, \\ 0 & 1 \leqslant x \leqslant 3. \end{cases} \end{cases} \tag{25}$$

图 1 中,给出了初值和 $t = 0.1$ 以及 $t = 0.5$ 时的解• 可以看出利用间断时空有限元方法可以很好地处理间断初值问题,得到稳定的数值解•

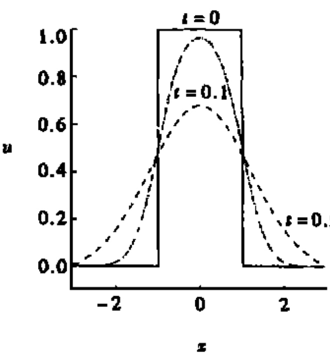

图 1 Cauchy 问题的解, $\Delta x = \Delta t = 0.01$

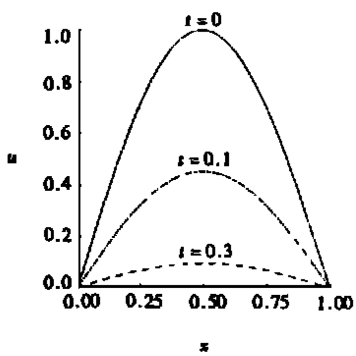

图 2 含有源项方程的解, $\Delta x = \Delta t = 0.01$

表 1 有源项抛物方程对时间变量的误差和相应的收敛阶

时间步长	0.02	0.01	0.005	0.002 5	0.001 25	0.000 625
误差	3.920 4e-002	2.060 7e-002	1.057 6e-002	5.359 4e-003	2.698 2e-003	1.354 0e-003
收敛阶		0.927 8	0.962 3	0.980 7	0.990 1	0.994 8

另外一个数值例子是含有源项的如下方程

$$\begin{cases} u_t - u_{xx} = 2u & 0 \leqslant x \leqslant 1,\ t > 0, \\ u(0, t) = u(1, t) = 0 & t > 0, \\ u(x, 0) = \sin \pi x & 0 < x < 1 \cdot \end{cases} \tag{26}$$

其精确解为 $u(x, t) = \mathrm{e}^{(2-\pi^2)t} \sin \pi x$• 图2中给出在 $t = 0.1$ 和 $t = 0.3$ 时解的结构, 表1中给出了空间剖分网格数为800时, 对时间变量的误差估计, 计算时间 $t = 0.1$• 可以看出, 间断时空有限元方法, 利用线性基函数, 对时间变量能够达到我们所希望的精度, 并与前面定理的结果相一致•

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The Space_Time Finite Element Method for Parabolic Problems

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Abstract: Adaptive space_time finite element method, continuous in space but discontinuous in time for semi_linear parabolic problems is discussed. The approach is based on a combination of finite element and finite difference techniques. The existence and uniqueness of the weak solution are proved without any assumptions on choice of the space_time meshes. Basic error estimates in $L^\infty(L^2)$ norm, that is maximum_norm in time, L^2_norm in space are obtained. The numerical results are given in the last part and the analysis between theoretic and experimental results are obtained.

Key words: semi_linear parabolic equations; space time finite element method; existence and uniqess; error estimate