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具有周期边界的守恒型方程的守恒型差分格式*

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摘要:　研究守恒型奇摄动方程的周期边界问题,构造了一个守恒型差分格式,利用分解解的奇性项的方法,结合问题的渐近展开,证明所构造的差分格式为一阶一致收敛·

关　键　词: 守恒方程;　奇摄动;　周期边界;　守恒型格式;　一致收敛
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1　微分方程及其性质

本文讨论守恒型奇摄动常微分方程周期边值问题:

$$Lu(x) \equiv \varepsilon(p(x)u'(x))' + (q(x)u(x))' - r(x)u(x) = f(x) \qquad (0 < x < 1), \tag{1}$$

$$u(0) \equiv u(1),\ lu \equiv u'(1) - u'(0) = C/\varepsilon, \tag{2}$$

其中小参数 $0 < \varepsilon \ll 1$, 而 $p(x)$、$q(x)$、$r(x)$、$f(x)$ 是充分光滑的函数, 且满足:

$$\alpha > p(x) > \alpha > 0,\ \beta > p(x) > \beta > 0,\ r(x) > r > 0,\ p \geqslant p'(x) \geqslant 0,\ q'(x) \leqslant 0;$$

设

$$a(x, \varepsilon) = \frac{\varepsilon p'(x) + q(x)}{p(x)},\quad b(x) = \frac{r(x) - q'(x)}{p(x)},$$

$$a_1 = \frac{\beta}{\alpha},\quad a_2 = \frac{r}{\alpha},\quad a = \max_{x\in[0,1]} \frac{p'(x) + q(x)}{p(x)}$$

和 α 为小于 $\min(a_1, a_2)$ 的正数, 则: $a > a(x, \varepsilon) > a_1 > a > 0,\ b(x) > a_2 > a > 0$·

文[1]已经证得:

引理 1　若

$$|Lu(x)| \leqslant K\left\{1 + \varepsilon^{-1}\exp\left(-\frac{\alpha x}{2\varepsilon}\right)\right\},\ u(0) = u(1),\ |lu| \leqslant |C|/\varepsilon,$$

则 $|u(x)| \leqslant C$ 对 $x \in [0, 1]$ 均成立·

引理 2　微分方程(1)~(2)的解 $u(x)$ 满足 $u(x) = rv(x) + z(x)$,

其中

$$v(x) = \exp\left(-q(0)\cdot x/p(0)\cdot\varepsilon\right),\ |r| \leqslant C,$$

$$|z^{(i+1)}(x)| \leqslant C\left\{1 + \varepsilon^{-i}\cdot\exp\left(-\alpha x/2\varepsilon\right)\right\} \qquad (i = 0, 1, 2, \cdots)·$$

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2 差分方程

对区间[0, 1]进行等距划分: $x_i = ih$, $i = 0, 1, \cdots, N$, $Nh = 1$.

构造如下差分格式:

$$L^h u_i \equiv \varepsilon\delta(\sigma_i(\rho) p(x_i)\delta u_i) + D_0(q(x_i)u_i) - r(x_i)u_i = f(x_i) \quad (0 < i < N), \tag{3}$$

$$u_0 = u_N, \quad l^h u_i \equiv \frac{u_N - u_{N-1}}{h} - A\frac{u_1 - u_0}{h} = \frac{C}{\varepsilon}, \tag{4}$$

其中

$$\sigma_i = \frac{T(x_i)\rho}{2}\coth\frac{T(x_i)\rho}{2}, \quad \rho = \frac{h}{\varepsilon}, \quad T(0) = \frac{q(0)}{p(0)},$$

$$T(x_i) = \frac{q(x_i - 0.5h)}{p(x_i)} \quad (0 < i < N), \quad A = \frac{T(0)\rho}{1 - \exp(-T(0)\rho)}.$$

引理3 令

$$R(x) = \frac{q(x - 0.5h)}{2p(x)}, \quad S(x, \rho) = R(x)\rho\coth(R(x)\rho), \quad x \in [0.5h, 1 - 0.5h],$$

$$T_i = \frac{\operatorname{sh}\frac{T(0)\rho}{2}\operatorname{sh}\frac{T(0) - T(x_i)}{2}\rho}{\operatorname{sh}\frac{T(x_i)}{2}\rho}\exp\left(-\frac{T(0)x_i}{2\varepsilon}\right) \quad (0 < i < N),$$

则有:

1) $\varepsilon|S(x, \rho) - 1| \leqslant Ch$,

2) $\varepsilon S(x, \rho) \leqslant C$,

3) $\varepsilon|\partial S(x, \rho)/\partial x| \leqslant Ch$,

4) $\varepsilon h^{-1}|S(x_{i+0.5}, \rho)P(x_{i+0.5}) - S(x_{i-0.5}, \rho)P(x_{i-0.5})| \leqslant C(h + \varepsilon)$,

5) $\varepsilon|S(x_{i+0.5}, \rho)P(x_{i+0.5}) + S(x_{i-0.5}, \rho)P(x_{i-0.5})| \leqslant C(h + \varepsilon)$,

6) 存在一个充分小的正数 h_1, 当 $0 < h < h_1$ 时, $|T_i| \leqslant Ch$.

证明 容易证明 1) ~ 5) 成立, 现证 6):

记 $$Q_i = \left|\operatorname{sh}\frac{T(0) - T(x_i)}{2}\rho\right|\exp\left(-\frac{T(0)x_i}{4\varepsilon}\right), \quad t_i = x_i/\varepsilon,$$

当取 $T_0 = -\frac{8}{T(0)}\ln h$ 时, 对于满足 $t_i \geqslant \max(T_0, 1)$ 的 i, 利用不等式

$$C_1\exp(t) \leqslant \operatorname{sh}(t) \leqslant C_2\exp(t) \quad t \in (m, +\infty),$$

得

$$Q_i = \left|\operatorname{sh}\frac{T(0) - T(x_i)}{2x_i}ht_i\right|\exp\left(-\frac{T(0)x_i}{4\varepsilon}\right) \leqslant C\exp\left(\frac{T(0) - T(x_i)}{2x_i}ht_i - \frac{T(0)t_i}{4}\right).$$

设 $\max\limits_{0<i<N}\frac{T(0) - T(x_i)}{x_i} \leqslant C_1$, 取 $h_2 \leqslant \frac{T(0)}{C_1}$,

则当 $0 < h < h_2$ 时,

$$(T(0) - T(x_i))h/2x_i - T(0)/4 \leqslant -T(0)/8,$$

此时 $Q_i \leqslant C\exp\left(-T(0)T_0/8\right) \leqslant Ch$•

对于满足 $t_i \leqslant \max(T_0, 1)$ 的 i, 利用不等式 $C_1 t \leqslant \mathrm{sh}(t) \leqslant C_2 t,\ t \in (0, m)$ 得

$$Q_i \leqslant Cht_i\exp\left(-\frac{T(0)}{8}t_i\right) \leqslant Ch,$$

因此对于 $0 < i < N$, 当 $0 < h < h_2$ 时, $Q_i \leqslant Ch$ 成立•

另外

$$\frac{\mathrm{sh}(T(0)\rho/2)}{\mathrm{sh}(T(x_i)\rho/2)}\exp\left(-\frac{T(0)x_i}{4\varepsilon}\right) \leqslant C\exp\left(\frac{T(0)-T(x_i)}{2x_i}x_i\rho - \frac{T(0)x_i}{4\varepsilon}\right) \leqslant$$
$$C\exp\left(\frac{C_1 h}{2\varepsilon}x_i - \frac{T(0)x_i}{4\varepsilon}\right)•$$

取 $h_3 \leqslant T(0)/2C_1$, 则当 $0 < h < h_3$ 时,

$$\frac{\mathrm{sh}(T(0)\rho/2)}{\mathrm{sh}(T(xi)\rho/2)}\exp\left(-\frac{T(0)x_i}{4\varepsilon}\right) \leqslant C•$$

存在一个充分小的正数 $h_1 = \min(h_2, h_3)$, 当 $0 < h < h_1$ 时, $|T_i| \leqslant Ch$ 成立•

容易证明下列 3 个引理成立:

引理 4　微分方程(1)~(2)有渐近展开

$$u(x) = \phi_0(x) + v_0(x) + w(x),$$

其中 $$v_0(x) = \frac{C}{T(0)}\exp\left(-\frac{T(0)x}{\varepsilon}\right) = \frac{C}{T(0)}v(x),\ w(x) = O(\varepsilon),$$

而 $\phi_0(x)$ 满足:

$$(q(x)u(x))' - r(x)u(x) = f(x),$$
$$u(1) - u(0) = v_0(0) - v_0(1) = \frac{C}{T(0)}\left[1 - \exp\left(-\frac{T(0)}{\varepsilon}\right)\right]•$$

引理 5　若 $\mathrm{L}^h u_i \leqslant 0,\ u_0 = u_N,\ \mathrm{l}^h u_i \geqslant 0$, 则对于 $0 \leqslant i \leqslant N,\ u_i \geqslant 0$ 成立•

引理 6　若

$$|\mathrm{L}^h u_i| \leqslant C\left\{1 + \frac{1}{\max(h, \varepsilon)}\exp\left(-\frac{\alpha x_i}{2\varepsilon}\right)\right\},\ u_0 = u_N,\ |\mathrm{l}^h u_i| \leqslant |C|/\varepsilon,$$

则存在一个充分小的正数 h_0, 当 $0 < h < h_0$ 时, $|u_i| \leqslant C$ 对 $0 \leqslant i \leqslant N$ 均成立•

3　一致收敛性

设 v_i、z_i 分别为 $v(x_i)$、$z(x_i)$ 的数值解

定义

$$\mathrm{L}^h(rv_i + z_i) \equiv \mathrm{L}(rv(x_i) + z(x_i)) = f_i \qquad (0 < i < N), \tag{5}$$
$$(rv_N + z_N) - (rv_0 + z_0) = (rv(x_N) + z(x_N)) - (rv(x_0) + z(x_0)), \tag{6}$$
$$\mathrm{l}^h(rv_i + z_i) \equiv \mathrm{l}(rv(x_i) + z(x_i)) = c/\varepsilon, \tag{7}$$

则差分格式的解 u_i 可表示为: $u_i = rv_i + z_i \quad (0 \leqslant i \leqslant N)$•

设微分方程(1)~(2)的解为 $u(x)$, 则

$$u_0 - u(0) = u_N - u(1),$$
$$\mathrm{l}^h(u_i - u(x_i)) = \mathrm{l}^h(rv_i + z_i - rv(x_i) - z(x_i)) =$$
$$c/\varepsilon - \mathrm{l}^h(rv(x_i) - z(x_i)) = O\left(h/\varepsilon\right)•$$

类似于文[2]的证明易得: 当 $h \leqslant h_4 = 0.5\min\limits_{0<i<N}\frac{(T(0)-a)x_i}{T(0)-T(x_i)}$ 时,

$$|L^h(u_i - u(x_i))| \leqslant ch\left\{1 + \frac{1}{\varepsilon}\exp\left(-\frac{ax_i}{\varepsilon}\right)\right\}.$$

由引理6得如下定理:

定理1　设 $u(x)$ 为微分方程(1)～(2)的解, u_i 为差分格式(3)～(4)的解, 则当 $h \leqslant \varepsilon$ 且 $h \leqslant h_0 = \min(h_0, h_4)$ 时, $|u_i - u(x_i)| \leqslant Ch$ 对 $0 \leqslant i \leqslant N$ 均成立.

引理7　若 $h \geqslant \varepsilon$ 且 $h \leqslant h_1$, 则 $|w_i| \leqslant C(h + \varepsilon)$ 对 $0 \leqslant i \leqslant N$ 均成立, 其中 h_1 是给定的充分小的正数, $w_i = u_i - \phi_0(x_i) - v_0(x_i)$.

证明　因为 $w_N - w_0 = 0$, $|l^h w_i| \leqslant C\dfrac{h+\varepsilon}{\varepsilon}$ 而 $L^h w_i = L^h u_i - L^h\phi_0(x_i) - L^h v_0(x_i)$, 则:

1) $L^h\phi_0(x_i)$ 的估计如下:

$$L^h\phi_0(x_i) = \frac{\varepsilon}{h}\left\{\sigma_{i+0.5}p(x_{i+0.5})\frac{\phi_0(x_{i+1}) - \phi_0(x_i)}{h} - \sigma_{i-0.5}p(x_{i-0.5})\frac{\phi_0(x_i) - \phi_0(x_{i-1})}{h}\right\} + D_0(q(x_i)u_i) - r(x_i)u_i =$$

$$\frac{\varepsilon}{h}\left\{\sigma_{i+0.5}p(x_{i+0.5})[\phi_0'(x_i) + 0.5h\phi_0''(x_i) + O(h^2)] - \sigma_{i-0.5}p(x_{i-0.5})\times [\phi_0'(x_i) - 0.5h\phi_0''(x_i) + O(h^2)]\right\} + D_0(q(x_i)u_i) - r(x_i)u_i =$$

$$\frac{\varepsilon}{h}\left\{[\sigma_{i+0.5}p(x_{i+0.5}) - \sigma_{i-0.5}p(x_{i-0.5})]\phi_0'(x_i) + 0.5h\phi_0''(x_i)[\sigma_{i+0.5}p(x_{i+0.5}) + \sigma_{i-0.5}p(x_{i-0.5})] + C\varepsilon h\sigma_{i+0.5} + C\varepsilon h\sigma_{i-0.5}\right\} + D_0(q(x_i)u_i) - r(x_i)u_i,$$

所以　$L^h\phi_0(x_i) = O(h + \varepsilon) + f(x_i)$.

2) $L^h v_0(x_i)$ 的估计如下:

$$L^h v_0(x_i) = \frac{\varepsilon}{h}\left\{\sigma_{i+0.5}p(x_{i+0.5})\frac{v_0(x_{i+1}) - v_0(x_i)}{h} - \sigma_{i-0.5}p(x_{i-0.5})\frac{v_0(x_i) - v_0(x_{i-1})}{h}\right\} + \frac{q(x_{i+1})v_0(x_{i+1}) - q(x_{i-1})v_0(x_{i-1})}{2h} - r(x_i)v_0(x_i) = F_1 + F_2 + F_3 - r(x_i)v_0(x_i),$$

其中

$$F_1 = \frac{\varepsilon}{h^2}\sigma_{i-0.5}p(x_{i-0.5})[v_0(x_{i+1}) - 2v_0(x_i) + v_0(x_{i-1})] + \frac{1}{2h}q(x_{i-1})[v_0(x_{i+1}) - v_0(x_{i-1})],$$

$$F_2 = \frac{\varepsilon}{h}\left\{\sigma_{i+0.5}p(x_{i+0.5}) - \sigma_{i-0.5}p(x_{i-0.5})\right\}\frac{v_0(x_{i+1}) - v_0(x_i)}{h},$$

$$F_3 = \frac{q(x_{i+1}) - q(x_{i-1})}{2h}v_0(x_{i+1}).$$

容易证得:

$$|F_i| \leqslant Cv(x_i) \qquad (i = 2, 3),$$

进一步化简 F_1 得:

$$F_1 = -\frac{2q(x_i - h)}{h} v(x_i) \frac{\mathrm{sh}\frac{T(0)}{2}\rho \,\mathrm{sh}\frac{T(x_i) - T(0)}{2}\rho}{\mathrm{sh}\frac{T(x_i)}{2}\rho}.$$

根据引理 3 得: 当 $0 < h < h_1$ 时,

$$|L^h w_i| \leqslant c(h + \varepsilon)\left\{1 + \frac{1}{h}\exp\left(-\frac{ax_i}{\varepsilon}\right)\right\}.$$

根据引理 7 得: 当 $0 < h < h_1 = \min(h_0, h_1)$ 时,

$$|u_i - u(x_i)| = |w_i - w(x_i)| \leqslant C(h + \varepsilon).$$

取 $h = \min(h_0, h_1)$ 可得以下关于 ε 的一致收敛定理:

定理 2　设 $u(x)$ 为微分方程(1) ~ (2) 的解, u_i 为差分格式(3) ~ (4) 的解, 则当 $h \leqslant h$ 时, $|u_i - u(x_i)| \leqslant Ch$ 对 $0 \leqslant i \leqslant N$ 均成立.

当 $h \geqslant h$ 时, 容易证得: $|u_i| \leqslant C$, $|u(x_i)| \leqslant C$, 因此 $|u_i - u(x_i)| \leqslant Ch$ 仍成立.

本文的主要结论如下:

定理 3　设 $u(x)$ 为微分方程(1) ~ (2) 的解, u_i 为差分格式(3) ~ (4) 的解, 则对 $0 \leqslant i \leqslant N$ 均有 $|u_i - u(x_i)| \leqslant Ch$.

限于篇幅, 本文的数值例子从略.

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A Conservative Difference Scheme for Conservative Differential Equation With Periodic Boundary

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Abstract: The conservative form and singular perturbed ordinary differential equation with periodic boundary value problem were studied, and a conservative difference scheme was constructed. Using the method of decomposing the singular term from its solution and combining an asymptotic expansion of the equation, it is proved that the scheme converges uniformly to the solution of differential equation with order one.

Key words: conservative equation; singular perturbation; periodic boundary; conservative difference scheme; uniform convergence