应用数学和力学, 第 23 卷 第 1 期(2002 年 1 月)
Applied Mathematics and Mechanics
应用数学和力学编委会编
重庆出版社出版

文章编号: 1000_0887(2002) 01_0047_05

约束 Birkhoff 系统的形式不变性*

陈向炜[1], 罗绍凯[1,2], 梅凤翔[3]

(1. 商丘师范学院 物理系, 河南商丘 476000; 2 长沙大学数学力学与数学物理研究所, 长沙 410003;
3. 北京理工大学 应用力学系, 北京 100081)

(马兴瑞推荐)

摘要: 约束 Birkhoff 系统的形式不变性是约束 Birkhoff 方程在无限小变换下的一种不变性• 给出约束 Birkhoff 系统形式不变性的定义与判据, 并研究了这种形式不变性与 Noether 对称性之间的关系•

关 键 词: 约束 Birkhoff 系统; 形式不变性; Noether 对称性
中图分类号: O313 文献标识码: A

引 言

Birkhoff 系统动力学是近代经典力学的一个重要分支, 是数学物理学中的一个近代发展方向, 并可应用于量子力学、统计力学、生物物理、航天力学及一些现代工程领域[1,2]• 人们研究了 Birkhoff 方程和它的变换理论[1]、Noether 理论[3]、Poisson 理论[4] 以及 Lie 对称性[5] 等一系列重要内容并得到一系列重要结果•

力学系统的对称性与守恒量研究不仅具有数学重要性, 而且表现为深刻的物理规律[6,7]• 本文研究一种新的对称性: 约束 Birkhoff 方程在无限小变换下保持形式不变的性质• 给出其形式不变性的定义与判据, 并研究了这种形式不变性与 Noether 对称性之间的关系• 这种形式不变性不同于 Noether 对称性, 形式不变性也不一定导致守恒律•

1 约束 Birkhoff 系统的方程

Birkhoff 方程表为[1,2]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0 \qquad (\mu,\nu=1,\dots,2n), \tag{1}$$

其中 $B=B(t,\boldsymbol{a})$ 称为 Birkhoff 函数, $R_\mu=R_\mu(t,\boldsymbol{a})$ 称为 Birkhoff 函数组, a^μ 为变量• 用方程(1) 描述的力学系统称为 Birkhoff 系统•

如果 Birkhoff 系统的变量 a^μ 不是彼此独立的, 而受到一些限制, 这些限制表为约束方程

---

* 收稿日期: 2000_03_14; 修订日期: 2001_08_26
基金项目: 国家自然科学基金资助项目(19972010); 高校博士点专项基金和河南省自然科学基金资助课题(998040080)
作者简介: 陈向炜(1967—), 男, 博士, 副教授, 主要从事约束系统动力学的研究工作, 已发表论文 30 多篇.

$$f_\beta(\boldsymbol{a}, t) = 0 \quad (\beta = 1, \cdots, g)• \tag{2}$$

用通常的 Lagrange 乘子法, 得到

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = \omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = \lambda_\beta \frac{\partial f_\beta}{\partial a^\mu}$$

$$(\mu, \nu = 1, \cdots, 2n;\ \beta = 1, \cdots, g), \tag{3}$$

其中 $\omega_{\mu\nu}$ 称为 Birkhoff 张量• 在运动方程积分之前, 可由方程(2) 和(3) 解出 λ_β 如下:

$$\lambda_\beta = -\frac{\Delta_{\gamma\beta}}{\Delta}\left\{\frac{\partial f_\gamma}{\partial a^\rho}\omega^{\rho\mu}\left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right) + \frac{\partial f_\gamma}{\partial t}\right\} \quad (\mu, \rho = 1, \cdots, 2n;\ \beta, \gamma = 1, \cdots, g), \tag{4}$$

其中

$$\Delta = \left|\left(\frac{\partial f_\gamma}{\partial a^\rho}\omega^{\rho\mu}\frac{\partial f_\beta}{\partial a^\mu}\right)\right| \stackrel{\triangle}{=} |(b_{\gamma\beta})|•$$

令

$$P_\mu = \lambda_\beta \frac{\partial f_\beta}{\partial a^\mu} = P_\mu(\boldsymbol{a}, t),$$

其中 λ_β 已用式(4) 替代• 则约束 Birkhoff 方程(3) 变为

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = P_\mu \quad (\mu, \nu = 1, \cdots, 2n), \tag{5}$$

称方程(5) 为不带乘子的约束 Birkhoff 方程• 下面我们就研究该方程的形式不变性•

2 约束 Birkhoff 系统的形式不变性

引入无限小变换

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{a}), \quad a^{\mu^*} = a^\mu + \varepsilon\xi_\mu(t, \boldsymbol{a}), \tag{6}$$

其中 ε 为无限小参数, ξ_0, ξ_μ 称为无限小生成元• 在变换(6) 下, 函数 $B(t, \boldsymbol{a})$ 变为 $B(t^*, \boldsymbol{a}^*)$, $R_\mu(t, \boldsymbol{a})$ 变为 $R_\mu(t^*, \boldsymbol{a}^*)$, $P_\mu(t, \boldsymbol{a})$ 变为 $P_\mu(t^*, \boldsymbol{a}^*)$•

定义 在无限小变换(6) 下, 如果约束 Birkhoff 方程(5) 保持形式不变, 即成立

$$\left(\frac{\partial R^*_\nu}{\partial a^\mu} - \frac{\partial R^*_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B^*}{\partial a^\mu} - \frac{\partial R^*_\mu}{\partial t} = P^*_\mu \quad (\mu, \nu = 1, \cdots, 2n), \tag{7}$$

其中

$$B^* = B(t^*, \boldsymbol{a}^*), \quad R^*_\mu = R_\mu(t^*, \boldsymbol{a}^*), \quad P^*_\mu = P_\mu(t^*, \boldsymbol{a}^*), \tag{8}$$

则称这种不变性为约束 Birkhoff 系统的形式不变性•

展开式(8), 得

$$\left.\begin{aligned}
B^* &= B(t^*, \boldsymbol{a}^*) = B(t, \boldsymbol{a}) + \varepsilon\left(\frac{\partial B}{\partial t}\xi_0 + \frac{\partial B}{\partial a^\mu}\xi_\mu\right) + o(\varepsilon^2),\\
R^*_\mu &= R_\mu(t^*, \boldsymbol{a}^*) = R_\mu(t, \boldsymbol{a}) + \varepsilon\left(\frac{\partial R_\mu}{\partial t}\xi_0 + \frac{\partial R_\mu}{\partial a^\nu}\xi_\nu\right) + o(\varepsilon^2),\\
P^*_\mu &= P_\mu(t^*, \boldsymbol{a}^*) = P_\mu(t, \boldsymbol{a}) + \varepsilon\left(\frac{\partial P_\mu}{\partial t}\xi_0 + \frac{\partial P_\mu}{\partial a^\nu}\xi_\nu\right) + o(\varepsilon^2)•
\end{aligned}\right\} \tag{9}$$

判据 如果存在常数 k 和函数 $G_0 = G_0(t, \boldsymbol{a})$ 使得无限小生成元 ξ_0, ξ_μ 满足如下等式

$$\frac{\partial B}{\partial t}\xi_0 + \frac{\partial B}{\partial a^\mu}\xi_\mu = k\left(B - \frac{\partial G_0}{\partial t}\right), \tag{10a}$$

$$\frac{\partial R_\mu}{\partial t}\xi_0 + \frac{\partial R_\mu}{\partial a^\nu}\xi_\nu = k\left(R_\mu - \frac{\partial G_0}{\partial a^\mu}\right), \tag{10b}$$

$$\frac{\partial P_\mu}{\partial t}\xi_0 + \frac{\partial P_\mu}{\partial a^\nu}\xi_\nu = k\left(P_\mu + 2\frac{\partial^2 G_0}{\partial t\partial a^\mu}\right) \quad (\mu,\ \nu = 1,\ \cdots,\ 2n), \tag{10c}$$

那么约束 Birkhoff 系统(5) 在无限小变换(6) 下是形式不变的•

证明　将式(10) 代入式(9), 去掉 ε^2 及更高阶小项, 得

$$B^* = B + \varepsilon k\left(B - \frac{\partial G_0}{\partial t}\right),$$

$$R_\mu^* = R_\mu + \varepsilon k\left(R_\mu - \frac{\partial G_0}{\partial a^\mu}\right),$$

$$P_\mu^* = P_\mu + \varepsilon k\left(P_\mu + 2\frac{\partial^2 G_0}{\partial t\partial a^\mu}\right)•$$

将上式代入方程(7), 得到

$$\begin{aligned}
&\left(\frac{\partial R_\nu^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial a^\nu}\right)\dot a^\nu - \frac{\partial B^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial t} - P_\mu^* = \\
&\quad \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu + \frac{\partial}{\partial a^\mu}\left[\varepsilon k\left(R_\nu - \frac{\partial G_0}{\partial a^\nu}\right)\right]\dot a^\nu - \\
&\quad \frac{\partial}{\partial a^\nu}\left[\varepsilon k\left(R_\mu - \frac{\partial G_0}{\partial a^\mu}\right)\right]\dot a^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial}{\partial a^\mu}\left[\varepsilon k\left(B - \frac{\partial G_0}{\partial t}\right)\right] - \\
&\quad \frac{\partial R_\mu}{\partial t} - \frac{\partial}{\partial t}\left[\varepsilon k\left(R_\mu - \frac{\partial G_0}{\partial a^\mu}\right)\right] - P_\mu - \varepsilon k\left(P_\mu + 2\frac{\partial^2 G_0}{\partial t\partial a^\mu}\right) = \\
&\quad (1 + \varepsilon k)\left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu\right] = 0,
\end{aligned}$$

故约束 Birkhoff 方程(5) 在无限小变换(6) 下是形式不变的•

3　约束 Birkhoff 系统形式不变性与 Noether 对称性

约束 Birkhoff 系统的 Noether 理论指出, 如果无限小生成元 ξ_0, ξ_μ 满足 Noether 等式[8]

$$\begin{aligned}
&\left(\frac{\partial R_\mu}{\partial t}\dot a^\mu - \frac{\partial B}{\partial t}\right)\xi_0 + \left(\frac{\partial R_\nu}{\partial a^\mu}\dot a^\nu - \frac{\partial B}{\partial a^\mu}\right)\xi_\mu - B\dot\xi_0 + \\
&\quad R_\mu\dot\xi_\mu - P_\mu(\xi_\mu - \dot a^\mu\xi_0) + \dot G = 0,
\end{aligned} \tag{11}$$

其中 $G = G(t, \boldsymbol{a})$ 称为规范函数, 则相应的对称性为 Noether 对称性•　此时存在守恒量

$$I = R_\mu\xi_\mu - B\xi_0 + G = \text{const}• \tag{12}$$

如果对满足式(10) 的无限小生成元 ξ_0, ξ_μ, 使得 Noether 等式(11) 成立, 则形式不变性导致 Noether 对称性; 否则, 不是 Noether 对称性•

4　算　　例

已知 4 阶 Birkhoff 系统为

$$B = \frac{1}{2}\left\{(a^1)^2 + (a^3)^2 + (a^4)^2\right\}, \tag{13}$$

$$R_1 = a^3, \quad R_2 = a^4, \quad R_3 = R_4 = 0•$$

约束方程为

$$f_1 = a^1 a^3 - C_1^2 = 0, \quad f_2 = a^1 + a^4 - C_2 = 0, \tag{14}$$

试研究系统的形式不变性•

由 Birkhoff 函数组 R_μ 求得

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{15}$$

故约束 Birkhoff 方程为

$$\left.\begin{aligned} &-\dot{a}^3 - a^1 = \lambda_1 a^3 + \lambda_2, \\ &-\dot{a}^4 = 0, \\ &\dot{a}^1 - a^3 = \lambda_1 a^1, \\ &\dot{a}^2 - a^4 = \lambda_2. \end{aligned}\right\} \tag{16}$$

由约束方程(14)和方程(16)求得

$$\lambda_1 = -\frac{a^3}{a^1}, \quad \lambda_2 = -a^1 + \frac{(a^3)^2}{a^1}, \tag{17}$$

于是有

$$\left.\begin{aligned} &P_1 = \lambda_1 \frac{\partial f_1}{\partial a^1} + \lambda_2 \frac{\partial f_2}{\partial a^1} = \lambda_1 a^3 + \lambda_2 = -a^1, \quad P_2 = 0, \\ &P_3 = \lambda_1 a^1 = -a^3, \quad P_4 = \lambda_2 = -a^1 + \frac{(a^3)^2}{a^1}. \end{aligned}\right\} \tag{18}$$

判据(10)给出

$$\begin{aligned} &a^1\xi_1 + a^3\xi_3 + a^4\xi_4 = k\left(B - \frac{\partial G_0}{\partial t}\right), \\ &\xi_3 = k\left(a^3 - \frac{\partial G_0}{\partial a^3}\right), \quad \xi_4 = k\left(a^4 - \frac{\partial G_0}{\partial a^4}\right), \\ &0 = k\left(-\frac{\partial G_0}{\partial a^3}\right), \quad 0 = k\left(-\frac{\partial G_0}{\partial a^4}\right), \\ &-\xi_1 = k\left(-a^1 + 2\frac{\partial^2 G_0}{\partial t \partial a^1}\right), \quad 0 = k\left(2\frac{\partial^2 G_0}{\partial t \partial a^2}\right), \\ &-\xi_3 = k\left(-a^3 + 2\frac{\partial^2 G_0}{\partial t \partial a^3}\right), \\ &\left[-1 - \left(\frac{a^3}{a^1}\right)^2\right]\xi_1 + 2\frac{a^3}{a^1}\xi_3 = k\left[-a^1 + \frac{(a^3)^2}{a^1} + 2\frac{\partial^2 G_0}{\partial t \partial a^4}\right]. \end{aligned}$$

上方程组有解

$$k = 0, \quad \xi_0 = 1, \quad \xi_1 = 0, \quad \xi_2 = 1, \quad \xi_3 = 0, \quad \xi_4 = 0, \tag{19}$$

此时对应系统的形式不变性.

式(19)代入式(11)得

$$\begin{aligned} &P_1\dot{a}^1 + P_3\dot{a}^3 + P_4\dot{a}^4 + \dot{G} = \\ &\quad -a^1\dot{a}^1 - a^3\dot{a}^3 + \left(-a^1 + \frac{(a^3)^2}{a^1}\right)\dot{a}^4 + \dot{G} = 0. \end{aligned}$$

由此找不到规范函数 G. 因此, 对应的形式不变性不是 Noether 对称性.

[参 考 文 献]

[1] Santilli R M. Foundations of theoretical mechanics Ⅱ[M]. New York: Springer_Verlag, 1983.

[2] 梅凤翔, 史荣昌, 张永发, 等. Birkhoff 系统动力学[M]. 北京: 北京理工大学出版社, 1996.

[3] 梅凤翔. Birkhoff 系统的 Noether 理论: 中国科学(A 辑) [J]. 1993, **23**(7): 709—717.
[4] 梅凤翔. Birkhoff 系统的 Poisson 理论, 科学通报[J]. 1995, **40**(21): 1947—1950.
[5] 梅凤翔. Birkhoff 系统的 Lie 对称性和守恒律, 科学通报[J]. 1998, **43**(18): 1937—1939.
[6] 李子平. 经典和量子约束系统及其对称性质[M]. 北京: 北京工业大学出版社, 1993.
[7] 赵跃宇, 梅凤翔. 力学系统的对称性与不变量[M]. 北京: 科学出版社, 1999.
[8] 梅凤翔. 李群和李代数对约束力学系统的应用[M]. 北京: 科学出版社, 1999.

A Form Invariance of Constrained Birkhoffian System

CHEN Xiang_wei[1], LUO Shao_kai[1,2], MEI Feng_xiang[3]

(1. Department of Physics, Shangqiu Teachers College, Shangqiu, Henan 476000, P R China;
2. Institute of Mathematical Mechanics and Mathematical Physics, Changsha University, Changsha 410003, P R China;
3. Department of Applied Mechanics, Beijing Institute of Technology, Beijing 100081, P R China)

Abstract: The form invariance of constrained Birkhoffian system is a kind of invariance of the constrained Birkhoffian equations under infinitesimal transformations. The definition and criteria of the form invariance of constrained Birkhoffian system are given, and the relation of the form invariance and the Noether symmetry is studied.

Key words: constrained Birkhoffian system; form invariance; Noether symmetry