 应用数学和力学编委会编
重 庆 出 版 社 出 版

文章编号: 1000_0887(2002) 04_0338_09

弹性线夹杂的相互干扰*

陶 敏, 张明焕, 汤任基

(上海交通大学 工程力学系, 上海 200030)

(我刊编委汤任基来稿)

摘要: 利用线夹杂的工程计算模型以及无限平面中单夹杂的基本解, 分析了无限平面中两根径向弹性线夹杂的相互干扰问题• 并将线夹杂和线夹杂相互作用的问题归结为解一组柯西型奇异积分的积分方程组, 计算了夹杂端点的应力强度因子和夹杂界面应力• 给出了一些数值例子• 这里的结果对于研究短纤维复合材料有一定的参考价值•

关 键 词: 弹性线夹杂; 相互干扰; 应力强度因子; 界面应力
中图分类号: O346.1 文献标识码: A

引 言

在纤维增强复合材料中, 短纤维增强复合材料既能增强基体, 又能避免长纤维复合材料的不足, 很有发展前途• 有关它的断裂、疲劳和损伤等微观力学问题是非常复杂的• 文献上大都将短纤维视为直线夹杂, 使用弹性力学方法进行研究• 有些文献如[1]、[2]研究了多个刚性线夹杂的相互作用, 而对于多根弹性线夹杂问题及夹杂的界面应力分析则很少• 本文使用了线夹杂的工程计算模型, 讨论了无限平面中两根径向弹性线夹杂的相互干扰问题• 这里的结果对于研究短纤维复合材料有一定的参考价值•

1 问题的提出及线夹杂基本解

考虑图 1 所示问题: 沿坐标轴 Ox_1 和 Ox_2 轴各有一弹性线夹杂 $L_1 = L_1(a_1, b_1)$ 和 $L_2 = L_2(a_2, b_2)$, 其间夹角为 θ• 基体 Ω 的弹性常数为(μ, ν), 夹杂 L_1 和 L_2 的弹性常数为(μ_1, ν_1)、(μ_2, ν_2), 其中 μ、μ_1、μ_2 和 ν、ν_1、ν_2 为材料的剪切弹性模量和泊松比• 以下均假定 $\nu_1 = \nu_2 = \nu$• 夹杂厚度为 h_1、h_2• 若在无穷远处受外应力$(\sigma_{yy}^\infty, \sigma_{xx}^\infty, \sigma_{xy}^\infty)$ 作用, 则夹杂相互干扰•

本文将夹杂看作是一根材料力学细杆, 它的抗拉压刚度为 EA (A 为夹杂的横截面面积), 抗弯刚度 EI (I 为夹杂横截面的惯性矩), 在外力作用下, 夹杂产生轴向变形和横向弯曲变形, 因而它的上下侧($L\pm$) 的界面应力发生间断, 夹杂对原基体连续场的干扰可用图2所示等效夹杂模型计算• 记夹杂 L_1 对基体的约束力为(p_1, q_1), 夹杂 L_2 对基体的约束力为(p_2, q_2), 它们由下式决定:

$$\begin{cases} p_i(x_i) = \sigma_{yy}(x_i, -0) - \sigma_{yy}(x_i, +0), \\ q_i(x_i) = \sigma_{xy}(x_i, -0) - \sigma_{xy}(x_i, +0) \qquad x_i \in L_i,\ i = 1, 2, \end{cases} \tag{1a, b}$$

* 收稿日期: 2000_12_12; 修订日期: 2001_11_28
作者简介: 陶 敏(1969—), 女, 浙江湖州人, 讲师, 硕士.

其中 σ_{yy} 为法向正应力, σ_{xy} 为切向应力•

由约束力 (p_1, q_1) 和 (p_2, q_2) 在域 Ω 上产生的位移记为 $[u_1(x_1, y_1), v_1(x_1, y_1)]$ 和 $[u_2(x_2, y_2), v_2(x_2, y_2)]$, 为[3]:

$$u_i(x_i, y_i) = \frac{1}{2\pi\mu(1+\kappa)}\int_{a_i}^{b_i}\left[\frac{\kappa}{2}\ln\frac{1}{(x_i-t)^2+y_i^2}+\frac{(x_i-t)^2}{(x_i-t)^2+y_i^2}\right]q_i(t)\,\mathrm{d}t+ \frac{1}{2\pi\mu(1+\kappa)}\int_{a_i}^{b_i}\frac{(x_i-t)y_i}{(x_i-t)^2+y_i^2}p_i(t)\,\mathrm{d}t+B_i \qquad (i=1,2), \tag{2a}$$

$$v_i(x_i, y_i) = \frac{1}{2\pi\mu(1+\kappa)}\int_{a_i}^{b_i}\left[\frac{\kappa}{2}\ln\frac{1}{(x_i-t)^2+y_i^2}+\frac{y_i^2}{(x_i-t)^2+y_i^2}\right]p_i(t)\,\mathrm{d}t+ \frac{1}{2\pi\mu(1+\kappa)}\int_{a_i}^{b_i}\frac{(x_i-t)y_i}{(x_i-t)^2+y_i^2}q_i(t)\,\mathrm{d}t+C_i \qquad (i=1,2), \tag{2b}$$

式中 B_i、C_i 为任意常数•

由外应力 $(\sigma_{yy}^\infty, \sigma_{xx}^\infty, \sigma_{xy}^\infty)$ 在域 Ω 上产生的位移记为 $[u_0(x_1, y_1), v_0(x_1, y_1)]$:

$$u_0(x_1, y_1) = \frac{1+\kappa}{8\mu}\left(\sigma_{xx}^\infty-\frac{3-\kappa}{1+\kappa}\sigma_{yy}^\infty\right)x_1+\frac{\sigma_{xy}^\infty}{2\mu}y_1+B_0, \tag{3a}$$

$$v_0(x_1, y_1) = \frac{1+\kappa}{8\mu}\left(\sigma_{yy}^\infty-\frac{3+\kappa}{1+\kappa}\sigma_{yy}^\infty\right)y_1+\frac{\sigma_{xy}^\infty}{2\mu}x_1+C_0, \tag{3b}$$

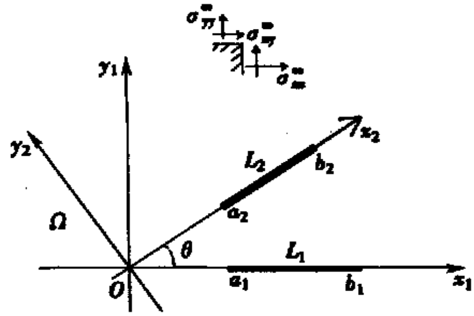

图 1

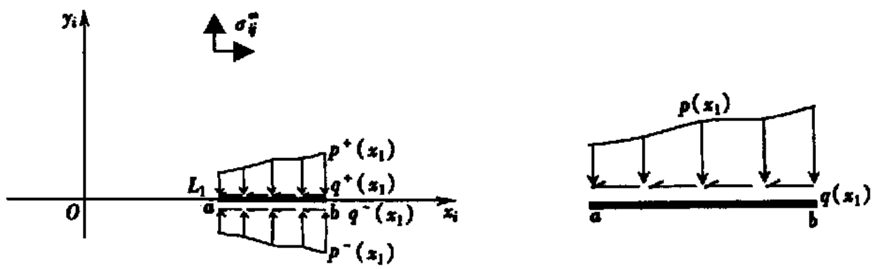

图 2

其中 B_0 和 C_0 为刚体平动, 可任意置值•

由虚拟夹杂 L_1 和 L_2 产生的位移分别记为 $[u_3(x_1), v_3(x_1)]$ 和 $[u_4(x_2), v_4(x_2)]$, 则:

$$\begin{cases} u_3(x_1) = \dfrac{\kappa_1+1}{8\mu_1 h_1}\displaystyle\int_{a_1}^{x_1}(x_1-u)q(u)\,\mathrm{d}u+B_1^*, & \\ & x_1\in L_1, \\ v_3(x_1) = \theta_1(x_1-a)-\dfrac{\kappa_1+1}{4\mu_1 h_1^3}\displaystyle\int_{a_1}^{x_1}(x_1-u)^3p(u)\,\mathrm{d}u+C_1^*, & \\ u_4(x_2) = \dfrac{\kappa_2+1}{8\mu_2 h_2}\displaystyle\int_{a_2}^{x_2}(x_2-u)q(u)\,\mathrm{d}u+B_2^*, & \\ & x_2\in L_2, \\ v_4(x_2) = \theta_2(x_2-a)-\dfrac{\kappa_2+1}{4\mu_2 h_2^3}\displaystyle\int_{a_2}^{x_2}(x_2-u)^3p(u)\,\mathrm{d}u+C_2^*, & \end{cases} \tag{4a ~ d}$$

式中 B_1^*、C_1^* 为夹杂 L_1 端点 a_1 的轴向位移和横向位移, θ_1 为端点 a_1 的转角, B_2^*、C_2^* 为夹杂 L_2 端点 a_2 的轴向位移和横向位移, θ_2 为端点 a_2 的转角•

2 问题的积分方程的建立

根据图 2 所示的工程近似模型, 由约束力产生的位移及由外应力和虚拟夹杂产生的位移应使基体的总位移与夹杂的变形协调, 即满足以下联结条件:

$$u_1(x_1, +0) + u_{12}(x_1, +0) + u_0(x_1, +0) = u_3(x_1) \qquad a_1 < x_1 < b_1, \tag{5a}$$

$$v_1(x_1, +0) + v_{12}(x_1, +0) + v_0(x_1, +0) = v_3(x_1) \qquad a_1 < x_1 < b_1, \tag{5b}$$

$$u_2(x_2, +0) + u_{21}(x_2, +0) + u_0(x_2, +0) = u_4(x_2) \qquad a_2 < x_2 < b_2, \tag{5c}$$

$$v_2(x_2, +0) + v_{21}(x_2, +0) + v_0(x_2, +0) = v_4(x_2) \qquad a_2 < x_2 < b_2, \tag{5d}$$

式中

$$u_{12}(x_1, +0) = u_2\cos\theta - v_2\sin\theta, \quad v_{12}(x_1, +0) = u_2\sin\theta + v_2\cos\theta, \qquad (x_2 = x_1\cos\theta,\ y_2 = -x_1\sin\theta),$$

$$u_{21}(x_2, +0) = u_1\cos\theta + v_1\sin\theta, \quad v_{21}(x_2, +0) = -u_1\sin\theta + v_1\cos\theta, \qquad (x_1 = x_2\cos\theta,\ y_1 = x_2\sin\theta)•$$

以上联结条件(5)式亦可由以下微分关系代替:

$$\frac{\partial u_1(x_1, +0)}{\partial x_1} + \frac{\partial u_{12}(x_1, +0)}{\partial x_1} - \frac{\partial u_3(x_1)}{\partial x_1} = -\frac{\partial u_0(x_1, +0)}{\partial x_1}, \tag{6a}$$

$$\frac{\partial v_1(x_1, +0)}{\partial x_1} + \frac{\partial v_{12}(x_1, +0)}{\partial x_1} - \frac{\partial v_3(x_1)}{\partial x_1} = -\frac{\partial v_0(x_1, +0)}{\partial x_1}, \tag{6b}$$

$$\frac{\partial u_2(x_2, +0)}{\partial x_2} + \frac{\partial u_{21}(x_2, +0)}{\partial x_2} - \frac{\partial u_4(x_2)}{\partial x_2} = -\frac{\partial u_0(x_2, +0)}{\partial x_2}, \tag{6c}$$

$$\frac{\partial v_2(x_2, +0)}{\partial x_2} + \frac{\partial v_{21}(x_2, +0)}{\partial x_2} - \frac{\partial v_4(x_2)}{\partial x_2} = -\frac{\partial v_0(x_2, +0)}{\partial x_2}• \tag{6d}$$

由以上条件, 得到问题的积分方程为:

$$\frac{1}{\pi}\int_{a_1}^{b_1}\frac{p_1(t)}{t-x_1}\mathrm{d}t + \frac{3\pi\mu(1+\kappa)(1+\kappa_1)}{2\kappa\mu_1^* h_1^3}\frac{1}{\pi}\int_{a_1}^{b_1}\mathrm{H}(x_1-t)(x_1-t)^2 p_1(t)\mathrm{d}t + \frac{1}{\kappa\pi}\int_{a_2}^{b_2}Q_{yy1}^{12}(x_1, 0, t)q_2(t)\mathrm{d}t + \frac{1}{\kappa\pi}\int_{a_2}^{b_2}P_{yy1}^{12}(x_1, 0, t)p_2(t)\mathrm{d}t - \frac{2\mu(1+\kappa)}{\kappa}\Theta_{01} = -\frac{1+\kappa}{\kappa}\sigma_{yy}^{\infty} \qquad a_1 < x_1 < b_1, \tag{7a}$$

$$\frac{1}{\pi}\int_{a_1}^{b_1}\frac{q_1(t)}{t-x_1}\mathrm{d}t - \frac{\pi\mu(1+\kappa)(1+\kappa_1)}{4\kappa\mu_1^* h_1}\frac{1}{\pi}\int_{a_1}^{b_1}\mathrm{H}(x_1-t)q_1(t)\mathrm{d}t + \frac{1}{\kappa\pi}\int_{a_2}^{b_2}Q_{xx1}^{12}(x_1, 0, t)q_2(t)\mathrm{d}t + \frac{1}{\kappa\pi}\int_{a_2}^{b_2}P_{xx1}^{12}(x_1, 0, t)p_2(t)\mathrm{d}t = -\frac{(1+\kappa)^2}{4\kappa}\left(\sigma_{xx}^{\infty} - \frac{3-\kappa}{1+\kappa}\sigma_{yy}^{\infty}\right) \qquad a_1 < x_1 < b_1, \tag{7b}$$

$$\frac{1}{\pi}\int_{a_2}^{b_2}\frac{p_2(t)}{t-x_2}\mathrm{d}t + \frac{3\pi\mu(1+\kappa)(1+\kappa_2)}{2\kappa\mu_2^* h_2^3}\frac{1}{\pi}\int_{a_2}^{b_2}\mathrm{H}(x_2-t)(x_2-t)^2 p_2(t)\mathrm{d}t + \frac{1}{\kappa\pi}\int_{a_1}^{b_1}P_{yy2}^{21}(x_2, 0, t)p_1(t)\mathrm{d}t + \frac{1}{\kappa\pi}\int_{a_1}^{b_1}Q_{yy2}^{21}(x_2, 0, t)q_1(t)\mathrm{d}t - \frac{2\mu(1+\kappa)}{\kappa}\Theta_{02} =$$

$$-\frac{1+\kappa}{\kappa}[(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})\sin\theta\cos\theta+\sigma_{xy}^{\infty}\cos 2\theta] \qquad a_2<x_2<b_2, \tag{7c}$$

$$\frac{1}{\pi}\int_{a_2}^{b_2}\frac{q_2(t)}{t-x_2}\mathrm{d}t-\frac{\pi\mu(1+\kappa)(1+\kappa_2)}{4\kappa\mu_2^* h_2^3}\frac{1}{\pi}\int_{a_2}^{b_2}\mathrm{H}(x_2-t)q_2(t)\mathrm{d}t+$$
$$\frac{1}{\kappa\pi}\int_{a_1}^{b_1}P_{xx2}^{21}(x_2,0,t)p_1(t)\mathrm{d}t+\frac{1}{\kappa\pi}\int_{a_1}^{b_1}Q_{xx2}^{21}(x_2,0,t)q_1(t)\mathrm{d}t=$$
$$-\frac{(1+\kappa)^2}{4\kappa}\Big\{\sigma_{yy}^{\infty}\sin^2\theta+\sigma_{xx}^{\infty}\cos^2\theta+\sigma_{xy}^{\infty}\sin 2\theta-$$
$$\frac{3-\kappa}{1+\kappa}[(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})\sin\theta\cos\theta+\sigma_{xy}^{\infty}\cos 2\theta]\Big\} \qquad a_2<x_2<b_2, \tag{7d}$$

式中 $\mu_1^*=\mu_1-\mu$, $\mu_2^*=\mu_2-\mu$ 分别是夹杂L_1和夹杂L_2与基体的模量差, κ、κ_1、κ_2 分别是基体、夹杂L_1和夹杂L_2的材料常数• $\kappa=3-4\nu$(平面应变), $\kappa=(3-\nu)/(1+\nu)$(平面应力), Θ_{01}、Θ_{02} 是夹杂L_1和夹杂L_2的初参数倾角, 其 $\mathrm{H}(x)$ 是Heaviside函数• 积分核由以下函数决定:

$$P_{xx1}^{12}(x_1,0,t)=\frac{(1-\kappa)(t-x_2)\sin\theta\cos\theta+y_2(\cos^2\theta-\kappa\sin^2\theta)}{(t-x_2)^2+y_2^2}-$$
$$\frac{2y_2(t-x_2)[(t-x_2)\cos 2\theta+y_2\sin 2\theta]}{[(t-x_2)^2+y_2^2]^2},$$

$$Q_{xx1}^{12}(x_1,0,t)=\frac{(\kappa-1)y_2\sin\theta\cos\theta+(x_2-t)(\sin^2\theta-\kappa\cos^2\theta)}{(t-x_2)^2+y_2^2}-$$
$$\frac{2y_2(t-x_2)[y_2\cos 2\theta+(x_2-t_2)\sin 2\theta]}{[(t-x_2)^2+y_2^2]^2},$$

$$P_{yy1}^{12}(x_1,0,t)=\frac{(1+\kappa)y_2\sin\theta\cos\theta-(x_2-t)(\sin^2\theta+\kappa\cos^2\theta)}{(t-x_2)^2+y_2^2}+$$
$$\frac{2y_2(t-x_2)[(x_2-t)\sin 2\theta+y_2\cos 2\theta]}{[(t-x_2)^2+y_2^2]^2},$$

$$Q_{yy1}^{12}(x_1,0,t)=\frac{(1+\kappa)(t-x_2)\sin\theta\cos\theta+y_2[1+(\kappa-1)\sin^2\theta]}{(t-x_2)^2+y_2^2}-$$
$$\frac{2y_2(t-x_2)[(t-x_2)\cos 2\theta+y_2\sin 2\theta]}{[(t-x_2)^2+y_2^2]^2},$$
$$(x_2=x_1\cos\theta,\ y_2=-x_1\sin\theta);$$

$$P_{yy2}^{21}(x_2,0,t)=\frac{-(1+\kappa)y_1\sin\theta\cos\theta-(x_1-t)(\sin^2\theta+\kappa\cos^2\theta)}{(t-x_1)^2+y_1^2}-$$
$$\frac{2y_1(t-x_1)[(x_1-t)\sin 2\theta-y_1\cos 2\theta]}{[(t-x_1)^2+y_1^2]^2},$$

$$Q_{yy2}^{21}(x_2,0,t)=\frac{(1+\kappa)(x_1-t)\sin\theta\cos\theta+y_1(\cos^2\theta+\kappa\sin^2\theta)}{(t-x_1)^2+y_1^2}+$$
$$\frac{2y_1(t-x_1)[(x_1-t)\cos 2\theta+y_1\sin 2\theta]}{[(t-x_1)^2+y_1^2]^2},$$

$$P_{xx2}^{21}(x_2,0,t)=\frac{(1-\kappa)(x_1-t)\sin\theta\cos\theta+y_1(\cos^2\theta-\kappa\sin^2\theta)}{(t-x_1)^2+y_1^2}+$$

$$\frac{2y_1(t-x_1)[(x_1-t)\cos 2\theta+y_1\sin 2\theta]}{[(t-x_1)^2+y_1^2]^2},$$

$$Q_{xx2}^{21}(x_2,0,t)=\frac{(1-\kappa)y_1\sin\theta\cos\theta-(x_1-t)(\kappa\cos^2\theta-\sin^2\theta)}{(t-x_1)^2+y_1^2}+$$
$$\frac{2y_1(t-x_1)[(x_1-t)\sin 2\theta-y_1\cos 2\theta]}{[(t-x_1)^2+y_1^2]^2},$$
$$(x_1=x_2\cos\theta,\ y_1=x_2\sin\theta)\cdot$$

此外,为使夹杂平衡,函数 (p_1, q_1) 和 (p_2, q_2) 应满足以下 6 个补充方程

$$\int_{a_1}^{b_1}p_1(t)\mathrm{d}t=\int_{a_1}^{b_1}q_1(t)\mathrm{d}t=\int_{a_1}^{b_1}(t-a_1)p_1(t)\mathrm{d}t=0, \tag{8a}$$

$$\int_{a_2}^{b_2}p_2(t)\mathrm{d}t=\int_{a_2}^{b_2}q_2(t)\mathrm{d}t=\int_{a_2}^{b_2}(t-a_2)p_2(t)\mathrm{d}t=0\cdot \tag{8b}$$

方程组(7)为标准的柯西型奇异积分方程,在定解条件下可解得所有的未知数• 而后将未知量代入位移表达式,可得到域 Ω 中任一点位移• 于是图示两径向弹性夹杂的相互作用问题可以从理论上获得解决•

3 夹杂的界面应力和应力强度因子

由夹杂对基体的约束力 (p_1, q_1) 和 (p_2, q_2) 产生的应力分量[4]为:

$$\begin{cases}\sigma_{xx}^{ii}(x_i,y_i)=\int_{a_i}^{b_i}[I_{xx}(x_i,y_i,t)q_i(t)+J_{xx}(x_i,y_i,t)p_i(t)]\mathrm{d}t,\\ \sigma_{yy}^{ii}(x_i,y_i)=\int_{a_i}^{b_i}[I_{yy}(x_i,y_i,t)q_i(t)+J_{yy}(x_i,y_i,t)p_i(t)]\mathrm{d}t,\\ \sigma_{xy}^{ii}(x_i,y_i)=\int_{a_i}^{b_i}[I_{xy}(x_i,y_i,t)q_i(t)+J_{xy}(x_i,y_i,t)p_i(t)]\mathrm{d}t,\end{cases}\quad (i=1,2), \tag{9a ~ c}$$

式中

$$I_{xx}(x,y,t)=B_1\left\{-(\kappa-1)\frac{x-t}{(x-t)^2+y^2}-\frac{4(x-t)^3}{[(x-t)^2+y^2]^2}\right\},$$

$$J_{xx}(x,y,t)=B_1\left\{(\kappa-1)\frac{y}{(x-t)^2+y^2}-\frac{4y(x-t)^2}{[(x-t)^2+y^2]^2}\right\},$$

$$I_{yy}(x,y,t)=B_1\left\{(\kappa-5)\frac{x-t}{(x-t)^2+y^2}+\frac{4(x-t)^3}{[(x-t)^2+y^2]^2}\right\},$$

$$J_{yy}(x,y,t)=B_1\left\{-(\kappa+3)\frac{y}{(x-t)^2+y^2}+\frac{4y(x-t)^2}{[(x-t)^2+y^2]^2}\right\},$$

$$I_{xy}(x,y,t)=B_1\left\{-(\kappa-1)\frac{y}{(x-t)^2+y^2}-\frac{4y(x-t)^2}{[(x-t)^2+y^2]^2}\right\},$$

$$J_{xy}(x,y,t)=B_1\left\{-(\kappa+3)\frac{x-t}{(x-t)^2+y^2}+\frac{4(x-t)^3}{[(x-t)^2+y^2]^2}\right\}\cdot$$

记夹杂 L_1 上的法向和切向界面应力为 $\sigma_{yy}(x_1,\pm 0)$, $\sigma_{xy}(x_1,\pm 0)$,它们由三部分组成:

$$\begin{cases}\sigma_{yy}(x_1,\pm 0)=\sigma_{yy}^{10}(x_1,\pm 0)+\sigma_{yy}^{11}(x_1,\pm 0)+\sigma_{yy}^{12}(x_1,\pm 0),\\ \sigma_{xy}(x_1,\pm 0)=\sigma_{xy}^{10}(x_1,\pm 0)+\sigma_{xy}^{11}(x_1,\pm 0)+\sigma_{xy}^{12}(x_1,\pm 0),\end{cases} \tag{10a, b}$$

式中

$$\sigma_{yy}^{10}(x_1,\ \pm 0)=\sigma_{yy}^{\infty},\quad \sigma_{xy}^{10}(x_1,\ \pm 0)=\sigma_{xy}^{\infty},$$

$$\sigma_{yy}^{11}(x_1,\ \pm 0)=\frac{1-\kappa}{2\pi(\kappa+1)}\int_{a_1}^{b_1}\frac{q_1(t)}{t-x_1}\mathrm{d}t \mp \frac{1}{2}p_1(x_1),$$

$$\sigma_{xy}^{11}(x_1,\ \pm 0)=-\frac{1-\kappa}{2\pi(\kappa+1)}\int_{a_1}^{b_1}\frac{p_1(t)}{t-x_1}\mathrm{d}t \mp \frac{1}{2}q_1(x_1),$$

$$\sigma_{yy}^{12}(x_1,\ \pm 0)=\sigma_{yy}^{22}\cos^2\theta+\sigma_{xx}^{22}\sin^2\theta+\sigma_{xy}^{22}\sin 2\theta,$$

$$\sigma_{xy}^{12}(x_1,\ \pm 0)=-(\sigma_{yy}^{22}-\sigma_{xx}^{22})\sin\theta\cos\theta+\sigma_{xy}^{22}\cos 2\theta.$$

以上公式中作变量代换 $x_2=x_1\cos\theta,\ y_2=-x_1\sin\theta$.

记夹杂 L_2 上的法向和切向界面应力为 $\sigma_{yy}(x_2,\ \pm 0)$、$\sigma_{xy}(x_2,\ \pm 0)$, 同理有:

$$\begin{cases}\sigma_{yy}(x_2,\ \pm 0)=\sigma_{yy}^{20}(x_2,\ \pm 0)+\sigma_{yy}^{21}(x_2,\ \pm 0)+\sigma_{yy}^{22}(x_2,\ \pm 0),\\ \sigma_{xy}(x_2,\ \pm 0)=\sigma_{xy}^{20}(x_2,\ \pm 0)+\sigma_{xy}^{21}(x_2,\ \pm 0)+\sigma_{xy}^{22}(x_2,\ \pm 0),\end{cases}\tag{11a, b}$$

其中

$$\sigma_{yy}^{20}(x_2,\ \pm 0)=\sigma_{yy}^{\infty}\cos^2\theta+\sigma_{xx}^{\infty}\sin^2\theta-\sigma_{xy}^{\infty}\sin 2\theta,$$

$$\sigma_{xy}^{20}(x_2,\ \pm 0)=(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})\sin\theta\cos\theta+\sigma_{xy}^{\infty}\cos 2\theta,$$

$$\sigma_{yy}^{22}(x_2,\ \pm 0)=\frac{1-\kappa}{2\pi(\kappa+1)}\int_{a_2}^{b_2}\frac{q_2(t)}{t-x_2}\mathrm{d}t \mp \frac{1}{2}p_2(x_2),$$

$$\sigma_{xy}^{22}(x_2,\ \pm 0)=-\frac{1-\kappa}{2\pi(\kappa+1)}\int_{a_2}^{b_2}\frac{p_2(t)}{t-x_2}\mathrm{d}t \mp \frac{1}{2}q_2(x_2),$$

$$\sigma_{yy}^{21}(x_2,\ \pm 0)=\sigma_{yy}^{11}\cos^2\theta+\sigma_{xx}^{11}\sin^2\theta-\sigma_{xy}^{11}\sin 2\theta,$$

$$\sigma_{xy}^{21}(x_2,\ \pm 0)=(\sigma_{yy}^{11}-\sigma_{xx}^{11})\sin\theta\cos\theta+\sigma_{xy}^{11}\cos 2\theta.$$

以上公式中作变量代换 $x_1=x_2\cos\theta,\ y_1=x_2\sin\theta$.

夹杂 L_1 和 L_2 端点的应力强度因子由以下式子给出:

$$K_{\mathrm{I}}(a_1)=\lim_{x_1\to a_1}\sqrt{2(a_1-x_1)}\,\sigma_{yy}(x_1,0)=-\frac{\kappa-1}{2(\kappa+1)}\lim_{x_1\to a_1}\sqrt{2(x_1-a_1)}\,q_1(x_1),\tag{12a}$$

$$K_{\mathrm{I}}(b_1)=\lim_{x_1\to b_1}\sqrt{2(x_1-b_1)}\,\sigma_{yy}(x_1,0)=\frac{\kappa-1}{2(\kappa+1)}\lim_{x_1\to b_1}\sqrt{2(b_1-x_1)}\,q_1(x_1),\tag{12b}$$

$$K_{\mathrm{I}}(a_1)=\lim_{x_1\to a_1}\sqrt{2(a_1-x_1)}\,\sigma_{xx}(x_1,0)=\frac{\kappa+3}{2(\kappa+1)}\lim_{x_1\to a_1}\sqrt{2(x_1-a_1)}\,q_1(x_1),\tag{12c}$$

$$K_{\mathrm{I}}(b_1)=\lim_{x_1\to b_1}\sqrt{2(x_1-b_1)}\,\sigma_{xx}(x_1,0)=-\frac{\kappa+3}{2(\kappa+1)}\lim_{x_1\to b_1}\sqrt{2(b_1-x_1)}\,q_1(x_1),\tag{12d}$$

$$K_{\mathrm{II}}(a_1)=\lim_{x_1\to a_1}\sqrt{2(a_1-x_1)}\,\sigma_{xy}(x_1,0)=\frac{\kappa-1}{2(\kappa+1)}\lim_{x_1\to a_1}\sqrt{2(x_1-a_1)}\,p_1(x_1),\tag{12e}$$

$$K_{\mathrm{II}}(b_1)=\lim_{x_1\to b_1}\sqrt{2(x_1-b_1)}\,\sigma_{xy}(x_1,0)=-\frac{\kappa-1}{2(\kappa+1)}\lim_{x_1\to b_1}\sqrt{2(b_1-x_1)}\,p_1(x_1),\tag{12f}$$

$$K_{\mathrm{I}}(a_2)=\lim_{x_2\to a_2}\sqrt{2(a_2-x_2)}\,\sigma_{yy}(x_2,0)=-\frac{\kappa-1}{2(\kappa+1)}\lim_{x_2\to a_2}\sqrt{2(x_2-a_2)}\,q_2(x_2),\tag{12g}$$

$$K_{\mathrm{I}}(b_2)=\lim_{x_2\to b_2}\sqrt{2(x_2-b_2)}\,\sigma_{yy}(x_2,0)=\frac{\kappa-1}{2(\kappa+1)}\lim_{x_2\to b_2}\sqrt{2(b_2-x_2)}\,q_2(x_2),\tag{12h}$$

$$K_{\mathrm{I}}(a_2)=\lim_{x_2\to a_2}\sqrt{2(a_2-x_2)}\,\sigma_{xx}(x_2,0)=\frac{\kappa+3}{2(\kappa+1)}\lim_{x_2\to a_2}\sqrt{2(x_2-a_2)}\,q_2(x_2),\tag{12i}$$

$$K_{\mathrm{I}}(b_2)=\lim_{x_2\to b_2}\sqrt{2(x_2-b_2)}\,\sigma_{xx}(x_2,0)=-\frac{\kappa+3}{2(\kappa+1)}\lim_{x_2\to b_2}\sqrt{2(b_2-x_2)}\,q_2(x_2),\tag{12j}$$

$$K_{\mathrm{II}}(a_2)=\lim_{x_2\to a_2}\sqrt{2(a_2-x_2)}\,\sigma_{xy}(x_2,0)=\frac{\kappa-1}{2(\kappa+1)}\lim_{x_2\to a_2}\sqrt{2(x_2-a_2)}\,p_2(x_2),\qquad(12\mathrm{k})$$

$$K_{\mathrm{II}}(b_2)=\lim_{x_2\to b_2}\sqrt{2(x_2-b_2)}\,\sigma_{xy}(x_2,0)=-\frac{\kappa-1}{2(\kappa+1)}\lim_{x_2\to b_2}\sqrt{2(b_2-x_2)}\,p_2(x_2),\qquad(12\mathrm{l})$$

式中应力强度因子 $K_{\mathrm{I}}(a_1)$ 和 $K_{\mathrm{I}}(a_1)$ 分别由夹杂端点 a_1 的奇性应力 $\sigma_{yy}(x_1,0)$ 和 $\sigma_{xx}(x_1,0)$ 的定义获得• 当 $q(x_1)>0$ 时, $K_{\mathrm{I}}(a_1)>K_{\mathrm{I}}(a_1)$, 此时夹杂端点 a_1 沿垂直方向开裂要比沿水平方向开裂更容易, 故此处与通常的裂纹问题不同, 还引进了如上的 $K_{\mathrm{I}}(a_1)$ 等新的应力强度因子•

4 数 值 结 果

在积分方程组(7)中, 未知函数在夹杂上的积分都存在柯西奇异核, 因此可按奇异积分方程组的数值法进行离散• 本文利用 Gauss_Jacobi 求积公式[5] 求解• 以下给出本文获得的数值结果•

例 1 两共线刚性夹杂的相互干扰

两共线刚性夹杂的几何位置及受力如图 3、图 4 所示, 基体的剪切弹性模量和泊松比取为 $(\mu, \nu=0.3)$, 无量纲韧带取 $\lambda/(b_1-a_1)=2/1\,000$, 无穷远处作用的外载为 $\sigma_{yy}^{\infty}=\sigma^{\infty}$• 这里使用不同的无量纲长度比 $(b_1-a_1)/(b_2-a_2)$, 对夹杂的相互影响作了计算• 图 3 为夹杂 L_1 和 L_2 端点的无量纲应力强度因子

$$K_{\mathrm{I}}^{*}(a_1)=\frac{K_{\mathrm{I}}(a_1)}{\sigma^{\infty}\sqrt{(b_1-a_1)/2}},\quad K_{\mathrm{I}}^{*}(b_1)=\frac{K_{\mathrm{I}}(b_1)}{\sigma^{\infty}\sqrt{(b_1-a_1)/2}},$$

$$K_{\mathrm{I}}^{*}(a_2)=\frac{K_{\mathrm{I}}(a_2)}{\sigma^{\infty}\sqrt{(b_2-a_2)/2}},\quad K_{\mathrm{I}}^{*}(b_2)=\frac{K_{\mathrm{I}}(b_2)}{\sigma^{\infty}\sqrt{(b_2-a_2)/2}}$$

随参数 $(b_1-a_1)/(b_2-a_2)$ 的变化• 图 4 为夹杂 L_1 上的无量纲界面应力 $\sigma_{xy}^{*}(\tau_1,+0)$ $(\sigma_{xy}^{*}(\tau_1,+0)=\sigma_{xy}(\tau_1,+0)/\sigma^{\infty})$ 随参数 $(b_1-a_1)/(b_2-a_2)$ 的变化• 无量纲法向界面应力经计算为一常数: $\sigma_{yy}^{*}(\tau_1,+0)=0.941\,1$, 这与理论分析十分接近的•

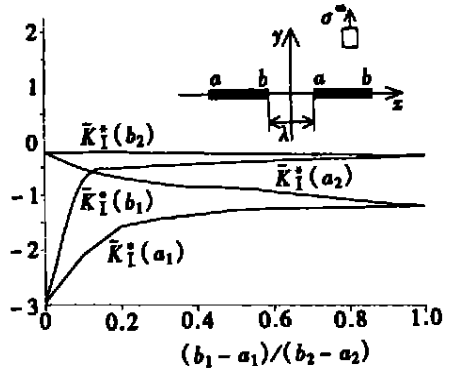

图 3

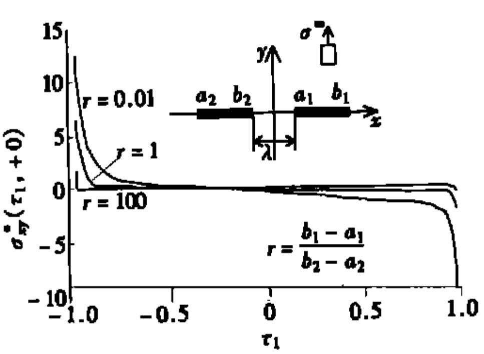

图 4

例 2 两共线弹性夹杂的相互干扰

两共线弹性夹杂的几何位置及受力与图 3、图 4 同, 基体的性质同上, 夹杂的剪切弹性模量和泊松比取为 $(\mu^{*}=10\mu, \nu^{*}=\nu=0.3)$, 无量纲韧带取 $\lambda/(b_1-a_1)=2/1\,000$, 无穷远处作用的外载为 $\sigma_{yy}^{\infty}=\sigma^{\infty}$• 表 1 为夹杂端点无量纲应力强度因子随参数 $(b_1-a_1)/(b_2-a_2)$ 变

化的结果•

从表中可以看到,随着两根夹杂相对位置的靠近,应力强度因子是单调增加的• 表中 $K_{\mathrm{I}}^{*}(a_1)$、$K_{\mathrm{I}}^{*}(b_1)$ 是负值,代表夹杂端点产生压应力,说明端点不易在垂直方向开裂•

例 3　两垂直刚性夹杂的相互干扰

两垂直刚性夹杂的几何位置及受力如图 5、图 6 所示,其中 $a_2=-b_2$, $b_2-a_2=2(b_1-a_1)$, $a_1=0.1(b_1-a_1)$• 基体的剪切弹性模量和泊松比与例 1 相同• 图 5 图 6 表示夹杂 L_1 在对称轴上时,夹杂 L_1 的界面应力 $\sigma_{yy}^{*}(\tau_1,+0)=\sigma_{yy}(\tau_1,+0)/\sigma^{\infty}$, $\sigma_{xy}^{*}(\tau_1,+0)=\sigma_{xy}(\tau_1,+0)/\sigma^{\infty}$ 随 τ_1 变化的分布,由于对称,只画了上岸的界面应力•

表 1

$(b_1-a_1)/(b_2-a_2)$	0.01	0.1	0.2	0.3
$K_{\mathrm{I}}^{*}(a_1)$	0.0287	0.0287	0.0288	0.0288
$K_{\mathrm{I}}^{*}(b_1)$	0.0287	0.0287	0.0287	0.0288
$K_{\mathrm{I}}^{*}(a_2)$	−0.1353	−0.1354	−0.1356	−0.1360
$K_{\mathrm{I}}^{*}(b_2)$	−0.1353	−0.1354	−0.1355	−0.1357
$(b_1-a_1)/(b_2-a_2)$	0.4	0.5	0.6	0.7
$K_{\mathrm{I}}^{*}(a_1)$	0.0290	0.0292	0.0296	0.0302
$K_{\mathrm{I}}^{*}(b_1)$	0.0289	0.0289	0.0290	0.0291
$K_{\mathrm{I}}^{*}(a_2)$	−0.1367	−0.1378	−0.1396	−0.1426
$K_{\mathrm{I}}^{*}(b_2)$	−0.1360	−0.1364	−0.1369	−0.1374
$(b_1-a_1)/(b_2-a_2)$	0.8	0.9	0.95	0.99
$K_{\mathrm{I}}^{*}(a_1)$	0.0314	0.0343	0.0386	0.0549
$K_{\mathrm{I}}^{*}(b_1)$	0.0293	0.0295	0.0296	0.0298
$K_{\mathrm{I}}^{*}(a_2)$	−0.1481	−0.1618	−0.1817	−0.2589
$K_{\mathrm{I}}^{*}(b_2)$	−0.1381	−0.1389	−0.1395	−0.1404

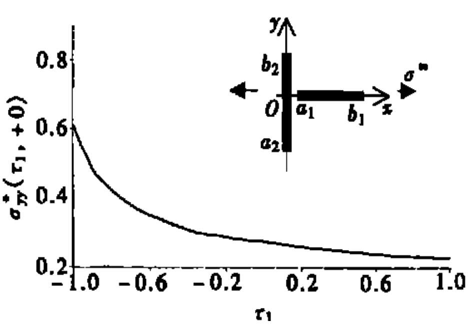

图　5

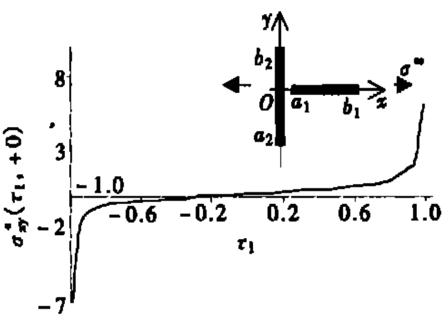

图　6

[参 考 文 献]

[1] JIANG Chi_ping. The plane problem of collinear rigid lines under arbitrary loads[J]. Eng Fracture Mech, 1991, **39**(2): 299—308.

[2] Chen Y Z, Hasebe N. An alternative Fredholm integral equation approach for multiple crack problem and multiple rigid line problem in plane elasticity[J]. Eng Fracture Mech, 1992, **43**(2): 257—268.

[3] Timoshenko S P, Goodier J N. Theory of Elasticity[M]. New York: McGraw Hill, 1970.

[4] Muskhelishvili N I. 数学弹性力学的几个基本问题[M]. 赵惠元 译. 北京: 科学出版社, 1958.

[5] Erdogan F. Mixed boundary_value problem in mechanics[A]. In: S Nemat_Nasser Ed. Mechanics Today[C]. Illinois: Northwestern University, 1978, 71—73.

The Interaction Problem Between the Elastic Line Inclusions

TAO Fang_ming, ZHANG Ming_huan, TANG Ren_ji

(Department of Mechanics Engineering, Shanghai Jiaotong University, Shanghai 200030, P R China)

Abstract: Using the engineering model of elastic line_inclusion and the basic solutions of a single inclusion, the interaction problem between line inclusions in an elastic solid was investigated. A set of standard Cauchy_type singular equations of the problem was presented. The stress intensity factors at points of inclusions and the interface stresses of two sides of the inclusion were calculated. Several numerical examples were given. The results could be regarded as a reference to engineering.

Key words: elastic inclusion; interaction; stress intensity factor; interface stress