应用数学和力学, 第 24 卷 第 8 期(2003 年 8 月)
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

文章编号: 1000_0887(2003) 08_0791_08

关于一类五阶常微分方程解的渐近性质*

西密尔· 通兹

(玉珍翠延大学 教育学院 数学系, 凡城 65080, 土耳其)

(钱伟长推荐)

摘要: 给出了一类五阶常微分方程所有解一致有界和当 $t\to\infty$ 时收敛于零的充分条件。 得到的结果包含并改善了 Abou_El_Ela 和 Sadek 1999 年关于非自治微分方程渐近解的结果。

关 键 词: 渐近性质; 有界性; 收敛性
中图分类号: O175 文献标识码: A

引 言

对于三阶和四阶微分方程解的渐近性质, 有很多作者都进行过研究。 文[1] 对这些工作进行了总结。 然而, 据我们所知, 对于五阶方程解的渐近性质仅有很少的结果。 最近 Abou_El_Ela 和 Sadek[2] 讨论了下面微分方程解的一致有界性和收敛性

$$x^{(5)}+a(t)f_1(\ddot x,\dddot x)x^{(4)}+b(t)f_2(\ddot x,\dddot x)+c(t)f_3(\ddot x)+d(t)f_4(\dot x)+e(t)f_5(x)=p(t,x,\dot x,\ddot x,\dddot x,x^{(4)})\text{。}$$

本文的目的旨在研究如下非自治微分方程的解当 $t\to\infty$ 的性质,

$$x^{(5)}+f_1(t,x,\dot x,\ddot x,\dddot x,x^{(4)})x^{(4)}+b(t)f_2(\ddot x,\dddot x)+c(t)f_3(\dot x,\ddot x)+d(t)f_4(\dot x)+e(t)f_5(x)=p(t,x,\dot x,\ddot x,\dddot x,x^{(4)}),\tag{1}$$

或它的等价微分方程组

$$\begin{cases}\dot x=y,\ \dot y=z,\ \dot z=w,\ \dot w=u,\\ \dot u=-f_1(t,x,y,z,w,u)u-b(t)f_2(z,w)-c(t)f_3(y,z)-\\ \qquad d(t)f_4(y)-e(t)f_5(x)+p(t,x,y,z,w,u),\end{cases}\tag{2}$$

这里, $b,\cdots,e,f_1,\cdots,f_5$ 和 p 是连续函数。 $b,\cdots,e$ 是在区间 $\mathbf{R}^+=[0,\infty)$ 上正的可微函数, 导数 $\partial f_2(z,w)/\partial z$、$\partial f_3(y,z)/\partial y$、$f_4'(y)$、$f_5'(x)$ 存在且关于变量 x、y、z 和 w 是连续的。

1 主要结果

我们将证明下面的定理。

定理 如果除了前面对函数 $b,\cdots,e,f_1,\cdots,f_5$ 和 p 的基本假定之外, 还假设存在任意常数

* 收稿日期: 2002_06_24
作者简介: 西密尔 通兹, 博士(E_mail: cemtunc@ yahoo. com)。

$\alpha_1, \cdots, \alpha_5$ 和足够小的正常数 $\varepsilon, \varepsilon_0, \varepsilon_1, \cdots, \varepsilon_5$，并且满足下面的条件:

（i）$b_0, \cdots, e_0$ 和 $B, \cdots, D$ 是常数, 对所有的 $t \in \mathbf{R}^+$, 满足 $B \geqslant b(t) \geqslant b_0 \geqslant 1$, $C \geqslant c(t) \geqslant c_0 \geqslant 1$, $D \geqslant d(t) \geqslant d_0 \geqslant 1$, $E \geqslant e(t) \geqslant e_0 \geqslant 1$•

（ii）对所有的 y 和所有的 $t \in \mathbf{R}^+$，有

$$\alpha_1 > 0,\quad \alpha_1\alpha_2 - \alpha_3 > 0,\quad (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0,$$

$$\delta_0 := (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0,\quad \alpha_5 > 0; \tag{3}$$

$$\Delta_1 := \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - [\alpha_1 d(t) f_4'(y) - \alpha_5] > 2\varepsilon\alpha_2; \tag{4}$$

$$\Delta_2 := \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)}{\alpha_1\alpha_4 - \alpha_5} - \frac{(\alpha_1\alpha_4 - \alpha_5)\gamma d(t)}{\alpha_5(\alpha_1\alpha_2 - \alpha_3)} - \frac{\varepsilon}{\alpha_1} > 0, \tag{5}$$

这里

$$\gamma := \begin{cases} \dfrac{f_4(y)}{y} & y \neq 0, \\ f_4'(0) & y = 0 \end{cases} \tag{6}$$

（iii）对所有的 x、y、z、w、u 和所有的 $t \in \mathbf{R}^+$, $\varepsilon_0 \leqslant f_1(t, x, y, z, w, u) - \alpha_1 \leqslant \varepsilon_1$•

（iv）对所有的 z 和 $w \neq 0$, $f_2(z, 0) = 0$, $0 \leqslant f_2(z, w)/w - \alpha_2 \leqslant \varepsilon_2$; 而对所有的 z 和 w, $\partial f_2(z, w)/\partial z \leqslant 0$•

（v）对所有的 y 和 $z \neq 0$, $f_3(y, 0) = 0$, $0 \leqslant f_3(y, z)/z - \alpha_3 \leqslant \varepsilon_3$; 而对所有的 y 和 z, $\partial f_3(y, z)/\partial y \leqslant 0$•

（vi）对所有的 $y \neq 0$, $f_4(0) = 0$, $f_4(y)/y \geqslant E\alpha_4/d_0$, $f_4'(y) - f_4(y)/y \leqslant \alpha_5\delta_0/(D\alpha_4^2(\alpha_1\alpha_2 - \alpha_3))$; 对所有的 y, $|\alpha_4 - f_4'(y)| \leqslant \varepsilon_4$•

（vii）对所有的 $x \neq 0$, $f_5(0) = 0$, $f_5(x)\operatorname{sgn}x > 0$; 而对所有的 x, 当 $|x| \to \infty$, $F_5(x) \equiv \int_0^x f_5(\xi)\mathrm{d}\xi \to \infty$ 和 $0 \leqslant \alpha_5 - f_5'(x) \leqslant \varepsilon_5$•

（viii）$\int_0^\infty \beta_0(t)\mathrm{d}t < \infty$, 且当 $t \to \infty$ 时, $e'(t) \to 0$, 这里 $\beta_0(t) := b_+'(t) + c_+'(t) + |d'(t)| + |e'(t)|$, $b_+'(t) := \max\{b'(t), 0\}$, 和 $c_+'(t) := \max\{c_+'(t), 0\}$•

（ix）$|p(t, x, y, z, w, u)| \leqslant p_1(t) + p_2(t)[F_5(x) + y^2 + z^2 + w^2 + u^2]^{\sigma/2} + \Delta(y^2 + z^2 + w^2 + u^2)^{1/2}$, 这里 σ、Δ 是常数, 满足 $0 \leqslant \sigma \leqslant 1$, $\Delta \geqslant 0$(足够小), p_1、p_2 是非负的连续函数, 满足

$$\int_0^\infty p_i(t)\mathrm{d}t < \infty \qquad (i = 1, 2)•$$

则(1)的所有解 $x(t)$ 是一致有界的, 且满足

$$x(t),\ \dot{x}(t),\ \ddot{x}(t),\ \dddot{x}(t),\ x^{(4)}(t) \to 0 \qquad 当\ t \to \infty•$$

注　应该指出, 如果在(2)式中令 $f_1(t, x, y, z, w, u) = a(t)f_1(z, w)$, 那么 Abou_El_Ela 和 Sadek[2] 的结论便可在 f_1 更弱的条件下得到•

2　Liapunov 函数 $V_0(t, x, y, z, w, u)$

上述定理的证明主要依赖于连续可微 Liapunov 函数 $V_0 = V_0(t, x, y, z, w, u)$ 的某些定性性质, 该函数由下式定义

$$2V_0=u^2+2\alpha_1uw+\frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}uz+2\delta yu+2b(t)\int_0^w f_2(z,\rho)\,\mathrm{d}\rho+$$
$$\left[\alpha_1^2-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right]w^2+2\left[\alpha_3+\frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\delta\right]wz+$$
$$2\alpha_1\delta wy+2d(t)uf_4(y)+2e(t)uf_5(x)+2\alpha_1c(t)\int_0^z f_3(y,\zeta)\,\mathrm{d}\zeta+$$
$$\left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\alpha_4-\alpha_1\delta\right]z^2+2\delta\alpha_2yz+2\alpha_1d(t)zf_4(y)-$$
$$2\alpha_5yz+2\alpha_1e(t)zf_5(x)+\frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}d(t)\int_0^y f_4(\eta)\,\mathrm{d}\eta+$$
$$(\delta\alpha_3-\alpha_1\alpha_5)y^2+\frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}e(t)yf_5(x)+2\delta e(t)\int_0^x f_5(\xi)\,\mathrm{d}\xi+k, \tag{7}$$

这里

$$\delta:=\frac{\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}+\varepsilon, \tag{8}$$

k 是后面证明中待定的正常数•

V_0 的第一个性质由下面引理表述•

引理 1　定理的假设(i)~(vii) 成立, 则存在正常数 D_7 和 D_8 满足

$$D_7[F_5(x)+y^2+z^2+w^2+u^2+k]\leqslant V_0\leqslant$$
$$D_8[F_5(x)+y^2+z^2+w^2+u^2+k]\,• \tag{9}$$

证明　(7) 式可以写成

$$2V_0=\left[u+\alpha_1w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right]^2+\frac{\alpha_4\delta_0}{(\alpha_1\alpha_4-\alpha_5)^2}\left(z+\frac{\alpha_5}{\alpha_4}y\right)^2+$$
$$\Delta_2(w+\alpha_1z)^2+\frac{\alpha_4(\alpha_1\alpha_4-\alpha_5)}{(\alpha_1\alpha_2-\alpha_3)\sqrt{d(t)}}\left[\frac{\alpha_1\alpha_2-\alpha_3}{\alpha_1\alpha_4-\alpha_5}e(t)f_5(x)+\right.$$
$$\left.\frac{\alpha_1\alpha_2-\alpha_3}{\alpha_1\alpha_4-\alpha_5}\sqrt{d(t)}\,y+\frac{\alpha_1}{\alpha_4}\sqrt{d(t)}\,z+\frac{1}{\alpha_4}\sqrt{d(t)}\,w\right]^2+$$
$$2\varepsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz+k+\sum_{i=1}^{4}S_i, \tag{10}$$

这里

$$S_1:=2\delta e(t)\int_0^x f_5(\xi)\,\mathrm{d}\xi-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5)\sqrt{d(t)}}e^2(t)f_5^2(x),$$
$$S_2:=\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)d(t)}{(\alpha_1\alpha_4-\alpha_5)}\left(2\int_0^y f_4(\eta)\,\mathrm{d}\eta-yf_4(y)\right)+$$
$$\left[\delta\alpha_3-\alpha_1\alpha_5-\frac{\alpha_5^2\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)^2}-\delta^2\right]y^2,$$
$$S_3:=\frac{\varepsilon}{\alpha_1}w^2+2b(t)\int_0^w f_2(z,\rho)\,\mathrm{d}\rho-\alpha_2w^2,$$
$$S_4:=2\alpha_1c(t)\int_0^z f_3(y,\zeta)\,\mathrm{d}\zeta-\alpha_1\alpha_3z^2•$$

如同文[2], 我们可以对函数 S_1、S_2 和 S_3 进行如下估计,

$$S_1\geqslant 2\varepsilon_0\int_0^x f_5(\xi)\,\mathrm{d}\xi$$

$$S_2 \geqslant \frac{\alpha_5 \delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} y^2$$

和

$$S_3 \geqslant \left(\frac{\varepsilon}{\alpha_1}\right) w^2.$$

最后, 利用(ⅰ)和(ⅴ), 且当 $z = 0$ 和 $S_4 = 0$, 我们有

$$S_4 := 2\alpha_1 c(t)\int_0^z f_3(y, \zeta)\,d\zeta - \alpha_1\alpha_3 z^2 \geqslant 2\alpha_1\int_0^z\left[\frac{f_3(y, \zeta)}{\zeta} - \alpha_3\right]\zeta\,d\zeta \geqslant 0,$$

因此我们得出对所有的 y 和 z 有 $S_4 \geqslant 0$.

在(10)中结合所有的估计, 并利用(ⅰ)、(ⅱ)和(ⅵ), 我们得到

$$2V_0 \geqslant \left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right]^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2}\left(z + \frac{\alpha_5}{\alpha_4} y\right)^2 +$$
$$\Delta_2(w + \alpha_1 z)^2 + 2\varepsilon e_0\int_0^x f_5(\xi)\,d\xi + \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} y^2 +$$
$$\left(\frac{\varepsilon}{\alpha_1}\right) w^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right) yz + k. \tag{11}$$

从上面不等式的前 6 项知一定存在足够小的正常数 D_i $(i = 1, 2, 3, 4, 5)$ 满足

$$2V_0 \geqslant D_1 F_5(x) + 2D_2 y^2 + 2D_3 z^2 + D_4 w^2 + D_5 u^2 +$$
$$2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right) yz + k. \tag{12}$$

现在考虑项

$$S_5 = D_2 y^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right) yz + D_3 z^2, \tag{13}$$

它包含在(12)式之中. 考虑到不等式

$$|yz| \leqslant \frac{1}{2}(y^2 + z^2),$$

显然 S_5 满足(由(13)定义)

$$S_5 \geqslant D_2 y^2 + D_3 z^2 - \varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right)(y^2 + z^2) \geqslant D_6(y^2 + z^2),$$

D_6 是某一个正的常数,

$$D_6 = (1/2)\min\{D_2, D_3\}; \tag{14}$$

上式中我们已假定了

$$\varepsilon \leqslant \frac{\alpha_1\alpha_4 - \alpha_5}{2(\alpha_3\alpha_4 - \alpha_2\alpha_5)}\min\{D_2, D_3\}.$$

因此

$$2V_0 \geqslant D_1 F_5(x) + (D_2 + D_6) y^2 + (D_3 + D_6) z^2 + D_4 w^2 + D_5 u^2 + k.$$

显然存在一个正的常数 D_7 使得

$$V_0 \geqslant D_7[F_5(x) + y^2 + z^2 + w^2 + u^2 + k],$$

假定 ε 是足够小的数使得

$$\frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} \geqslant \varepsilon\left[\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_3\right]$$

且(14)式成立.

从定理的假设和恒等式

$$2\alpha_5\int_0^x f_5(\xi)\,\mathrm{d}\xi - f_5^2(x) = 2\int_0^x [\alpha_5 - f_5'(\xi)]f_5(\xi)\,\mathrm{d}\xi - f_5^2(0)$$

知存在一个正的常数 D_8 满足

$$V_0 \leqslant D_8\left[f_5(x) + y^2 + z^2 + w^2 + u^2 + k\right].$$

引理证毕.

引理 2　假设定理的所有条件均满足，则存在正的常数 $D_i(i = 11, 12, 13)$ 使得下式成立

$$\dot{V}_0 \leqslant D_{13}(y^2 + z^2 + w^2 + u^2) + 2D_{12}(y^2 + z^2 + w^2 + u^2)^{1/2}\times$$
$$\left[p_1(t) + p_2(t)\right] + 2D_{12}p_2(t)\left[F_5(x) + y^2 + z^2 + w^2 + u^2\right] + D_{11}\beta_0 V_0. \qquad (15)$$

证明　假定 $y, z, w \neq 0$. 利用条件(iv)、(v)和恒等式

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0 \equiv \frac{\partial V_0}{\partial u}\dot{u} + \frac{\partial V_0}{\partial w}u + \frac{\partial V_0}{\partial z}w + \frac{\partial V_0}{\partial y}z + \frac{\partial V_0}{\partial x}y + \frac{\partial V_0}{\partial t}$$

直接计算得

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0 = -u^2[f_1(t, x, y, z, w, u) - \alpha_1] - w^2\left[\alpha_1\frac{b(t)f_2(z, w)}{w} -\right.$$
$$\left.\left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\}\right] - z^2\left[\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\frac{f_3(y, z)}{z} -\right.$$
$$\left.\left\{\delta\alpha_2 + \alpha_1 d(t)f_4'(y) - \alpha_5\right\}\right] - y^2\left[\delta d(t)\frac{f_4(y)}{y} - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}e(t)f_5'(x)\right] +$$
$$wb(t)\int_0^w \frac{\partial}{\partial z}f_2(z, \rho)\,\mathrm{d}\rho + \alpha_1 c(t)z\int_0^z \frac{\partial}{\partial y}f_3(y, \zeta)\,\mathrm{d}\zeta - \alpha_1 wuf_1(t, x, y, z, w, u) -$$
$$uzc(t)\left[\frac{f_3(y, z)}{z} - \alpha_3\right] - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}uzf_1(t, x, y, z, w, u) -$$
$$\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}wzb(t)\left[\frac{f_2(z, w)}{w} - \alpha_2\right] - \delta yuf_1(t, x, y, z, w, u) -$$
$$ywe(t)[\alpha_5 - f_5'(x)] - \delta ywb(t)\left[\frac{f_2(z, w)}{w} - \alpha_2\right] -$$
$$\alpha_1 yze(t)[\alpha_5 - f_5'(x)] - \delta yzc(t)\left[\frac{f_3(y, z)}{z} - \alpha_3\right] - wzd(t)[\alpha_4 - f_4'(x)] +$$
$$\left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}zw + \delta\alpha_2 yw\right][1 - b(t)] + [\alpha_3 zu + \delta\alpha_3 yz][1 - c(t)] -$$
$$\alpha_4 wz[1 - d(t)] - [\alpha_5 yw + \alpha_1\alpha_5 yz][1 - e(t)] +$$
$$\left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}z + \delta y\right]p(t, x, y, z, w, u) + \frac{\partial V_0}{\partial t}. \qquad (16)$$

由(iii)得

$$f_1(t, x, y, z, w, u) - \alpha_1 \geqslant \varepsilon_0.$$

利用(i)、(ii)、(iv)和(8)式，我们得到(当 $w \neq 0$)

$$\alpha_1\frac{b(t)f_2(z, w)}{w} - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\} \geqslant$$
$$\alpha_1\left[\frac{f_2(z, w)}{w} - \alpha_2\right] + \left[\alpha_1\alpha_2 - \alpha_3 + \delta - \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\right] \geqslant \varepsilon.$$

通过同样的估计，同时应用(i)、(v)、(8)和(4)，我们有(当 $z \neq 0$)

$$\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}c(t)\frac{f_3(y,z)}{z}-[\delta\alpha_2+\alpha_1 d(t)f_4'(y)-\alpha_5]\geqslant$$
$$\frac{(\alpha_3\alpha_4-\alpha_2\alpha_5)(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-[\alpha_1 d(t)f_4'(y)-\alpha_5]-\varepsilon\alpha_2\geqslant\varepsilon\alpha_2.$$

由(ⅰ)、(ⅵ)和(ⅶ),我们得到(令 $y\neq 0$)

$$\delta d(t)\frac{f_4(y)}{y}-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}e(t)f_5'(x)\geqslant$$
$$\varepsilon\alpha_4 E+\frac{\alpha_4 E(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}[\alpha_5-f_5'(x)]\geqslant\varepsilon\alpha_4 E.$$

这样,(16)中含有 u^2、w^2、z^2 和 y^2 的前 4 项主要由下式估计

$$-(\varepsilon_0 u^2+\varepsilon w^2+\varepsilon\alpha_2 z^2+\varepsilon\alpha_4 E y^2).$$

现在令 $R(t,x,y,z,w,u)$ 表示(16)中余下项的和.从定理的假设(ⅰ)、(ⅲ)~(ⅶ),我们可以看出 $R(t,x,y,z,w,u)$ 中的 uw, uz, uy, wz, wy 或 yz 的系数的绝对值不可能超过 $D_9\varepsilon_i$ ($i=1,2,3,4,5$),这里 D_9 是一个正的常数.

这样,再一次利用不等式

$$|uw|\leqslant\frac{1}{2}(u^2+w^2),\ |uz|\leqslant\frac{1}{2}(u^2+z^2),\ |uy|\leqslant\frac{1}{2}(u^2+y^2),$$
$$|wz|\leqslant\frac{1}{2}(w^2+z^2),\ |wy|\leqslant\frac{1}{2}(w^2+y^2),\ |yz|\leqslant\frac{1}{2}(y^2+z^2);$$

对于 $D_9>0$,我们有

$$|R(t,x,y,z,w,u)|\leqslant D_9(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)(y^2+z^2+w^2+u^2)+$$
$$\left|\left\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right\}p(t,x,y,z,w,u)\right|+\frac{\partial V_0}{\partial t},$$

然后我们把上式代到(16)式中,并假设

$$D_9(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)\leqslant\frac{1}{2}\min\left\{\varepsilon_0,\varepsilon,\varepsilon\alpha_2,\varepsilon\alpha_4 E\right\},\tag{17}$$

我们得到

$$\dot{V}\leqslant-(\varepsilon_0 u^2+\varepsilon w^2+\varepsilon\alpha_2 z^2+\varepsilon\alpha_4 E y^2)+$$
$$D_9(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)(y^2+z^2+w^2+u^2)+$$
$$\left|\left\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right\}p(t,x,y,z,w,u)\right|+\frac{\partial V_0}{\partial t}\leqslant$$
$$-\frac{1}{2}\min\left\{\varepsilon_0,\varepsilon,\varepsilon\alpha_2,\varepsilon\alpha_4 E\right\}(y^2+z^2+w^2+u^2)+$$
$$\left|\left\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right\}p(t,x,y,z,w,u)\right|+\frac{\partial V_0}{\partial t}.\tag{18}$$

现假定 D_9 和 ε_1, ε_2, …, ε_5 足够小以致(17)式成立.对于情形 $y,z,w=0$ 一般处理即可.对(7)式进行简单的计算得到

$$\frac{\partial V_0}{\partial t}=b'(t)\int_0^w f_2(z,\rho)\,\mathrm{d}\rho+\alpha_1 c'(t)\int_0^z f_3(y,\zeta)\,\mathrm{d}\zeta+$$
$$d'(t)\left[uf_4(y)+\alpha_1 zf_4(y)+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\int_0^y f_4(\eta)\,\mathrm{d}\eta\right]+$$
$$e'(t)\left[wf_5(x)+\alpha_1 zf_5(x)+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}yf_5(x)+\delta\int_0^x f_5(\xi)\,\mathrm{d}\xi\right].\tag{19}$$

根据定理的假设和利用(9)式,我们很容易得到

$$\frac{\partial V_0}{\partial t} \leqslant D_{10}\left\{ b'_+(t) + c'_+(t) + |d'(t)| + |e'(t)| \right\}\left\{ F_5(x) + y^2 + z^2 + w^2 \right\} \leqslant D_{11}\beta_0 V_0, \tag{20}$$

这里 D_{10} 是正的常数,而且 $D_{11} = D_{10}/D_7$. 正象文[2]中所示的一样,我们从(18)、(20)并利用(ix)得到

$$\dot{V}_0 \leqslant -D_{13}(y^2 + z^2 + w^2 + u^2) + 2D_{12}(y^2 + z^2 + w^2 + u^2)^{1/2} \times \left\{ p_1(t) + p_2(t) \right\} + 2D_{12}p_2(t)\left\{ F_5(x) + y^2 + z^2 + w^2 + u^2 \right\} + D_{11}\beta_0 V_0.$$

引理 2 证毕.

3 定理的证明

考虑如下定义的函数 $V(t, x, y, z, w, u)$:

$$V(t, x, y, z, w, u) = e^{-\int_0^t \gamma(s)\,ds} V_0(t, x, y, z, w, u), \tag{21}$$

这里

$$\gamma(t) := D_{11}\beta_0 + \frac{4D_{12}}{D_7}\left\{ p_1(t) + p_2(t) \right\}.$$

容易看出存在两个函数 ϕ_1 和 ϕ_2, 对所有的 $x \in \mathbf{R}^5$ 和 $t \in \mathbf{R}^+$, 满足

$$\phi_1(\| x \|) \leqslant V(t, x, y, z, w, u) \leqslant \phi_2(\| x \|), \tag{22}$$

这里 ϕ_1 是一个确定的正的连续单增函数,且当 $r \to \infty$ 时, $\phi_1(r) \to \infty$; ϕ_2 是连续的单增函数.

对于(2)的任意解 (x, y, z, w, u) 我们有

$$\begin{aligned}
\dot{V} = {} & e^{-\int_0^t \gamma(\tau)\,d\tau}\left\{ \dot{V}_0 + \gamma(t) V_0 \right\} \leqslant \\
& e^{-\int_0^t \gamma(\tau)\,d\tau}\Big[-D_{13}(y^2 + z^2 + w^2 + u^2) + \\
& 2D_{12}(y^2 + z^2 + w^2 + u^2)^{1/2}\left\{ p_1(t) + p_2(t) \right\} - \\
& 2D_{12}\left\{ p_1(t) + p_2(t) \right\}\left\{ F_5(x) + y^2 + z^2 + w^2 + u^2 + 2k \right\} \Big] \leqslant \\
& e^{-\int_0^t \gamma(\tau)\,d\tau}\Big[-D_{13}(y^2 + z^2 + w^2 + u^2) - 2D_{12}\left\{ p_1(t) + p_2(t) \right\} \times \\
& \left\{ \left(\sqrt{y^2 + z^2 + w^2 + u^2} - \frac{1}{2} \right)^2 - \frac{1}{4} + 2k \right\} \Big].
\end{aligned}$$

设 $k \geqslant 1/8$, 我们能找到一个正常数 D_{14} 使得

$$\dot{V} \leqslant -D_{14}(y^2 + z^2 + w^2 + u^2). \tag{23}$$

从不等式(22)和(23),我们能得到方程(2)所有解 (x, y, z, w, u) 的一致有界性[3, 定理 10.2].

剩下的讨论类同于文[2],这里略去. 定理证毕.

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On the Asymptotic Behaviour of Solutions of Certain Fifth_Order Ordinary Differential Equations

Cemil Tun,,

(Department of Mathematics, Faculty of Education,

Y z nc YIL University 65080, Van, Turkey)

Abstract: The sufficient conditions are given for all solutions of certain non_autonomous differential equation to be uniformly bounded and convergence to zero as $t \to \infty$. The result given includes and improves that result obtained by Abou_El_Ela & Sadek.

Key words: asymptotic behaviour; boundedness; convergence