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具非线性边界条件的半线性时滞微分方程边值问题奇摄动*

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(我刊原编委林宗池推荐)

摘要: 利用微分不等式理论研究了一类具非线性边界条件的半线性时滞微分方程边值问题· 采用新的方法构造上下解, 得到了此边值问题解的存在性的充分条件, 并给出了解的一致有效渐近展开式·

关 键 词: 奇摄动; 时滞微分方程; 边值问题; 一致有效渐近展开式
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引 言

本文研究具非线性边界条件的半线性时滞微分方程边值问题:

$$\varepsilon x''(t)=f(t,x(t),x(t-\varepsilon),\varepsilon),\qquad t\in(0,1),\tag{1}$$

$$x(t)=\varphi(t,\varepsilon),\quad t\in[-\varepsilon_0,0],\quad h(x(1),x'(1),\varepsilon)=A(\varepsilon),\tag{2}$$

其中 $\varepsilon>0$ 是小参数, ε_0 为充分小的正数· 关于这类问题的研究目前已有很多结果[1~5], 但它们有一个共同特点, 即方程(1) 与 $x'(t)$ 有关, 且边界条件是线性的, 此时过去所采用的边界层函数构造方法将不再适用, 因而本文将采用新的方法构造边值问题(1) ~ (2) 的上下解, 利用微分不等式理论证明边值问题解的存在性, 并给出了解的一致有效渐近展开式·

本文作如下假设:

[H1] $f(t,x,y,\varepsilon)$ 在 $[0,1]\times R^2\times[0,\varepsilon_0]$ 上连续且关于 x,y,ε 可微· $\varphi(t,\varepsilon)$ 于 $[-\varepsilon_0,0]\times[0,\varepsilon_0]$ 上连续, 且关于 ε 适当阶可微, $A(\varepsilon)$ 在 $[0,\varepsilon_0]$ 上连续, $h(x,y,\varepsilon)$ 在 $R^2\times[0,\varepsilon_0]$ 上连续可微, 其中 ε_0 为某个非常小的正数·

[H2] 退化问题

$$f(t,x(t),x(t),0)=0$$

在 $[-\varepsilon_0,1]$ 上存在唯一解 $x(t)$ 满足 $x(t)\in C^{2+N}_{[-\varepsilon_0,1]}$, 其中 N 为一正数·

[H3] 令

$$\Omega=\Big\{(t,x,y,\varepsilon)\mid t\in[0,1],\ \mid x-x(t)\mid\leqslant d,$$

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$$|y - x(t)| \leqslant d,\ \varepsilon \in [0, \varepsilon_0]\Big\},$$

其中 $d = |\varphi(0, 0) - x(0)| + \delta$, $\delta > 0$ 为充分小的正数, 且存在两常数 $m > l > 0$ 和 $l_0 > 0$ 使得当 $(t, x, y, \varepsilon) \in \Omega$ 时,

$$f'_x(t, x, y, \varepsilon) \geqslant m,\ 0 \leqslant f'_y(t, x, y, \varepsilon) \leqslant -l,$$

且当 $(x, y, \varepsilon) \in R^2 \times [0, \varepsilon_0]$ 时,

$$h'_x(x, y, \varepsilon) \geqslant l_0,\ h'_y(x, y, \varepsilon) > 0.$$

1 边值问题解的存在性

本节考虑一般的泛函微分方程边值问题

$$x''(t) = F(t, x(t), x(t - \tau)), \qquad t \in (0, 1), \tag{3}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0],\ h(x(1), x'(1)) = A. \tag{4}$$

引理[6] 假设

1) $F(t, x, y)$ 于 $[0, 1] \times R^2$ 上连续, 且关于 y 单调不增;

2) 存在函数 $\alpha(t), \beta(t) \in C_{[-\tau, 1]} \cap C^2_{[0, 1]}$ 满足

$$\alpha(t) \leqslant \beta(t), \qquad t \in [-\tau, 1],$$

$$\alpha''(t) \geqslant F(t, \alpha(t), \alpha(t - \tau)), \qquad t \in [0, 1],$$

$$\beta''(t) \leqslant F(t, \beta(t), \beta(t - \tau)), \qquad t \in [0, 1];$$

则对 $\forall \varphi(t) \in C_{[-\tau, 0]}$ 和 $A \in \mathbf{R}$, 当

$$\alpha(t) \leqslant \varphi(t) \leqslant \beta(t), \qquad t \in [-\tau, 0],$$

$$\alpha(1) \leqslant A \leqslant \beta(1)$$

时, 边值问题

$$x''(t) \geqslant F(t, x(t), x(t - \tau)), \qquad t \in [0, 1], \tag{5}$$

$$x(t) = \varphi(t), \qquad t \in [-\tau, 0],\ x(1) = A \tag{6}$$

存在解 $x(t)$ 满足

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \qquad t \in [0, 1].$$

定理 1 假设

1) $F(t, x, y)$ 于 $[0, 1] \times R^2$ 上连续, 且关于 y 单调不增;

2) 存在函数 $\alpha(t), \beta(t) \in C_{[-\tau, 1]} \cap C^2_{[0, 1]}$ 满足

$$\alpha(t) \leqslant \beta(t), \qquad t \in [-\tau, 1],$$

$$\alpha''(t) \geqslant F(t, \alpha(t), \alpha(t - \tau)), \qquad t \in [0, 1],$$

$$\beta''(t) \leqslant F(t, \beta(t), \beta(t - \tau)), \qquad t \in [0, 1],$$

$$\alpha(1) < \beta(1);$$

3) $h(x, y)$ 关于 y 不减, 且

$$h(\alpha(1), \alpha'(1)) \leqslant A,\ h(\beta(1), \beta'(1)) \geqslant A;$$

则对 $\forall \varphi(t) \in C_{[-\tau, 0]}$ 和 $A \in \mathbf{R}$, 当

$$\alpha(t) \leqslant \varphi(t) \leqslant \beta(t), \qquad t \in [-\tau, 0],$$

$$\alpha(1) \leqslant A \leqslant \beta(1)$$

时, 边值问题(3) ~ (4) 存在解 $x(t)$ 满足

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \qquad t \in [0, 1].$$

证明 $\forall c\in[\alpha(1),\beta(1)]$，由引理得边值问题

$$x''(t)=F(t,x(t),x(t-\tau)),\qquad t\in(0,1),\tag{7}$$

$$x(t)=\varphi(t),\quad t\in[-\tau,0],\ x(1)=c\tag{8}$$

存在解 $x_c(t)$ 满足

$$\alpha(t)\leqslant x_c(t)\leqslant\beta(t),\qquad t\in[-\tau,1].$$

如果 $c=\alpha(1)$，则 $x_c'(1)\leqslant\alpha'(1)$。因此

$$h(x_c(1),x_c'(1))=h(\alpha(1),x_c'(1))\leqslant h(\alpha(1),\alpha'(1))\leqslant A.\tag{9}$$

若 $c=\beta(1)$，同理可得

$$h(x_c(1),x_c'(1))\geqslant h(\beta(1),\beta'(1))\geqslant A.\tag{10}$$

令

$$\Omega_1=\left\{c\mid c\in[\alpha(1),\beta(1)],\,h(x_c(1),x_c'(1))<A\right\},$$
$$\Omega_2=\left\{c\mid c\in[\alpha(1),\beta(1)],\,h(x_c(1),x_c'(1))>A\right\}.$$

如果定理结论不正确，则得 $\Omega_1\cup\Omega_2=[\alpha(1),\beta(1)]$，则由(9)、(10)知 Ω_1、Ω_2 均非空。可以证明 Ω_1 是闭的，事实上，令 $c_n\in\Omega_1$，且 $\lim\limits_{n\to\infty}c_n=c_0$，并设 $x_n(t)=x_{c_n}(t)$，易得 $h(x_n(1),x_n'(1))<A$。通过计算可得序列 $\left\{x_n(t)\right\}$ 是等度连续的。因此存在序列 $\left\{x_n(t)\right\}$ 中的一子列，其于 $[-\tau,1]$ 上一致收敛于函数 $x_0(t)$，其中 $x_0(t)$ 满足下列条件

$$\begin{cases}x_0''(t)=F(t,x_0(t),x_0(t-\tau)),\\ x_0(t)=\varphi(t),\qquad t\in[-\tau,0],\ x_0(1)=c_0\end{cases}$$

且

$$h(x_0(1),x_0'(1))\leqslant A.\tag{11}$$

根据假设，(11)式中的等号不可能成立，故 $h(x_0(1),x_0'(1))<A$，即 $c_0\in\Omega_1$。因而 Ω_1 是闭的。由此得 Ω_2 是开的。但用同样的方法可得 Ω_2 也是闭的，由此得矛盾。此矛盾说明了定理的结论是正确的。

2 奇摄动边值问题

令

$$x(t,\varepsilon)=x_0(t)+x_1(t)\varepsilon+\cdots+x_N(t,\varepsilon)\varepsilon^N+\cdots\tag{12}$$

为边值问题(1)~(2)的外部解，由 Taylor 公式得到

$$x_i(t-\varepsilon)=x_i(t)-x_i'(t)\varepsilon+\cdots+(-1)^N\frac{x_i^{(N)}(t)}{N!}\varepsilon^N+O(\varepsilon^{N+1}).\tag{13}$$

将(12)、(13)代入(1)并按 ε 幂展开比较同次幂系数得

$$f(t,x_0(t),x_0(t),0)=0\tag{14$_0$}$$

和

$$x_{i-1}''(t)=[f_x'(t,x_0(t),x_0(t),0)+f_y'(t,x_0(t),x_0(t),0)]x_i(t)+p_i(t),\tag{14$_i$}$$

其中 $p_i(t)$ 为

$$x_0(t),x_1(t),\cdots,x_{i-1}(t),x_0'(t),x_1'(t),\cdots,x_{i-1}'(t),x_0''(t),x_1''(t),\cdots,$$
$$x_{i-1}''(t),\cdots,x_0^{(i-1)}(t),x_1^{(i-1)}(t),\cdots,x_{i-1}^{(i-1)}(t),x_0^{(i)}(t)$$

的已知函数。由题设条件知(14)$_0$ 存在唯一解

$$x_0(t)=x(t),\qquad t\in[-\varepsilon_0,1].\tag{15}$$

经逐步迭代得 $(14)_i$ 存在唯一解 $x_i(t) \in C^{N+2}_{[-\varepsilon_0, 1]}$。 令

$$X_N(t, \varepsilon) = \sum_{i=0}^{N} x_i(t)\varepsilon^i,$$

则得 $X_N(t, \varepsilon) \in C^2_{[0,1]}$，且满足

$$|\varepsilon X_N''(t, \varepsilon) - f(t, X_N(t, \varepsilon), X_N(t-\varepsilon, \varepsilon), \varepsilon)| \leqslant D\varepsilon^{N+1}, \qquad t \in [0, 1], \tag{16}$$

其中 D 为某个与 ε 无关的常数。 由题设条件 $m > l$ 知，存在常数 $\delta > 0$，使得 $m > l + 2\delta$。

令

$$F(\lambda) = \varepsilon\lambda^2 - m\lambda + l\mathrm{e}^{-\lambda\varepsilon}, \quad F_1(\lambda) = \varepsilon\lambda^2 - m\lambda + l\mathrm{e}^{\lambda\varepsilon},$$

易证

$$\lambda_0, \lambda_1 \in \left(-\sqrt{\frac{m-l-\delta}{\varepsilon}} + 1, -\sqrt{\frac{m}{\varepsilon}} + 1\right)$$

使得

$$F(\lambda_0) = F_1(\lambda_1) = 0.$$

设

$$h(x, X_N'(1, \varepsilon), \varepsilon) = A(\varepsilon). \tag{17}$$

由于 $h_x'(x, y, \varepsilon) \geqslant l_0 > 0$，故由(17)知存在唯一函数 $\omega(\varepsilon)$ 使得

$$h(\omega(\varepsilon), X_N'(1, \varepsilon), \varepsilon) = A(\varepsilon).$$

令

$$w(t, \varepsilon) = |\omega(\varepsilon) - X_N(1, \varepsilon)|\mathrm{e}^{\lambda_1(1-t)}, \qquad t \in [-\varepsilon_0, 1], \tag{18}$$

$$\Gamma(t, \varepsilon) = |\varphi(0, \varepsilon) - X_N(0, \varepsilon)|\mathrm{e}^{\lambda_0 t}, \qquad t \in [-\varepsilon_0, 1]. \tag{19}$$

由时滞微分方程理论[7]知，$w(t, \varepsilon)$、$\Gamma(t, \varepsilon)$ 满足

$$\varepsilon w''(t, \varepsilon) - mw(t, \varepsilon) + lw(t-\varepsilon, \varepsilon) = 0, \qquad t \in \mathbf{R} \tag{20}$$

和

$$\varepsilon\Gamma''(t, \varepsilon) - m\Gamma(t, \varepsilon) + l\Gamma(t-\varepsilon, \varepsilon) = 0, \qquad t \in \mathbf{R}. \tag{21}$$

定理2 若条件[H1] ~ [H2]满足，且

$$|X_N(t, \varepsilon) - \varphi(t, \varepsilon)| \leqslant |X_N(0, \varepsilon) - \varphi(0, \varepsilon)|, \qquad t \in [-\varepsilon_0, 0],$$

则当 $\varepsilon > 0$ 充分小时，边值问题(1) ~ (2) 存在解 $x(t, \varepsilon)$ 满足

$$|x(t, \varepsilon) - X_N(t, \varepsilon)| \leqslant w(t, \varepsilon) + \Gamma(t, \varepsilon) + M\varepsilon^{N+1}, \qquad t \in [0, 1],$$

其中 M 为与 ε 无关的正常数。

证明 令

$$\alpha(t, \varepsilon) = X_N(t, \varepsilon) - w(t, \varepsilon) - \Gamma(t, \varepsilon) - r\varepsilon^{N+1}, \qquad t \in [-\varepsilon_0, 1],$$

$$\beta(t, \varepsilon) = X_N(t, \varepsilon) + w(t, \varepsilon) + \Gamma(t, \varepsilon) + r\varepsilon^{N+1}, \qquad t \in [-\varepsilon_0, 1],$$

其中

$$\Gamma(t, \varepsilon) = \begin{cases} |X_N(t, \varepsilon) - \varphi(t, \varepsilon)|, & t \in [-\varepsilon_0, 0], \\ \Gamma(t, \varepsilon), & t \in [0, 1], \end{cases}$$

$r > D/(m-l)$ 为常数。 易得

$$\alpha(t, \varepsilon), \beta(t, \varepsilon) \in C_{[-\varepsilon_0, 1]} \cap C^2_{[0,1]},$$

并有

$$\alpha(t, \varepsilon) \leqslant \varphi(t, \varepsilon) \leqslant \beta(t, \varepsilon), \quad t \in [-\varepsilon_0, 0], \quad \alpha(1, \varepsilon) \leqslant \beta(1, \varepsilon). \tag{22}$$

又

$$\begin{aligned}&h(\beta(1,\varepsilon),\beta'(1,\varepsilon),\varepsilon)-A(\varepsilon)\geqslant\\&\quad h(\beta(1,\varepsilon),X_N'(1,\varepsilon),\varepsilon)-A(\varepsilon)=\\&\quad h(X_N(1,\varepsilon)+|\omega(\varepsilon)-X_N(1,\varepsilon)|+\Gamma(1,\varepsilon),X_N'(1,\varepsilon),\varepsilon)-\\&\quad h(\omega(\varepsilon),X_N'(1,\varepsilon),\varepsilon)+h(\omega(\varepsilon),X_N'(1,\varepsilon),\varepsilon)-A(\varepsilon)=\\&\quad\int_0^1\Big\{h_x'(\omega(\varepsilon)+\theta(X_N(1,\varepsilon)+|\omega(\varepsilon)-X_N(1,\varepsilon)|+\\&\quad\Gamma(1,\varepsilon)-\omega(\varepsilon)),X_N'(1,\varepsilon),\varepsilon)\Big\}\mathrm{d}\theta\times\\&\quad[X_N(1,\varepsilon)+|\omega(\varepsilon)-X_N(1,\varepsilon)|+\Gamma(1,\varepsilon)-\omega(\varepsilon)]\geqslant\\&\quad l_0\Gamma(1,\varepsilon)>0.\end{aligned}\tag{23}$$

同理可证

$$h(\alpha(1,\varepsilon),\alpha'(1,\varepsilon),\varepsilon)-A(\varepsilon)\leqslant 0.\tag{24}$$

当 $t\in[\varepsilon_0,1]$，应用中值定理易得

$$\begin{aligned}&f(t,\alpha(t,\varepsilon),\alpha(t-\varepsilon,\varepsilon),\varepsilon)-\varepsilon\alpha''(t,\varepsilon)=\\&\quad f(t,\alpha(t,\varepsilon),\alpha(t-\varepsilon,\varepsilon),\varepsilon)-f(t,X_N(t,\varepsilon),\alpha(t-\varepsilon,\varepsilon),\varepsilon)+\\&\quad f(t,X_N(t,\varepsilon),\alpha(t-\varepsilon,\varepsilon),\varepsilon)-f(t,X_N(t,\varepsilon),X_N(t-\varepsilon,\varepsilon),\varepsilon)+\\&\quad f(t,X_N(t,\varepsilon),X_N(t-\varepsilon,\varepsilon),\varepsilon)-\varepsilon X_N''(t,\varepsilon)+\\&\quad \varepsilon\Gamma''(t,\varepsilon)+\varepsilon w''(t,\varepsilon)\leqslant\\&\quad -m(\Gamma''(t,\varepsilon)+w(t,\varepsilon)+r\varepsilon^{N+1})+l\Gamma(t-\varepsilon,\varepsilon)+lw(t-\varepsilon,\varepsilon)+\\&\quad lr\varepsilon^{N+1}+D\varepsilon^{N+1}+D\varepsilon^{N+1}+\varepsilon\Gamma''(t,\varepsilon)+\varepsilon w''(t,\varepsilon)=\\&\quad \varepsilon\Gamma''(t,\varepsilon)-m\Gamma(t,\varepsilon)+l\Gamma(t-\varepsilon,\varepsilon)+\varepsilon w''(t,\varepsilon)-\\&\quad mw(t,\varepsilon)+lw(t-\varepsilon,\varepsilon)+lr\varepsilon^{N+1}-mr\varepsilon^{N+1}+D\varepsilon^{N+1}\leqslant 0.\end{aligned}\tag{25}$$

当 $t\in[0,\varepsilon_0]$，类似可得

$$\begin{aligned}&f(t,\alpha(t,\varepsilon),\alpha(t-\varepsilon,\varepsilon),\varepsilon)-\varepsilon\alpha''(t,\varepsilon)\leqslant\\&\quad \varepsilon\Gamma''(t,\varepsilon)-m\Gamma(t,\varepsilon)+l\Gamma(t-\varepsilon,\varepsilon)+\varepsilon w''(t,\varepsilon)-\\&\quad mw(t,\varepsilon)+lw(t-\varepsilon,\varepsilon)+lr\varepsilon^{N+1}-mr\varepsilon^{N+1}+D\varepsilon^{N+1}\leqslant\\&\quad \varepsilon\Gamma''(t,\varepsilon)-m\Gamma(t,\varepsilon)+l\Gamma(t-\varepsilon,\varepsilon)+\\&\quad l(\Gamma(t-\varepsilon,\varepsilon)-w(t-\varepsilon,\varepsilon))=\\&\quad l(\Gamma(t-\varepsilon,\varepsilon)-w(t-\varepsilon,\varepsilon))=\\&\quad l(|X_N(t,\varepsilon)-\varphi(t,\varepsilon)|-|X_N(0,\varepsilon)-\varphi(0,\varepsilon)|\,\mathrm{e}^{\lambda_0(t-\varepsilon)})\leqslant 0.\end{aligned}\tag{26}$$

综合(25)、(26)得

$$f(t,\alpha(t,\varepsilon),\alpha(t-\varepsilon,\varepsilon),\varepsilon)-\varepsilon\alpha''(t,\varepsilon)\leqslant 0,\qquad t\in[0,1].\tag{27}$$

同理可得

$$f(t,\beta(t,\varepsilon),\beta(t-\varepsilon,\varepsilon),\varepsilon)-\varepsilon\beta''(t,\varepsilon)\geqslant 0.\tag{28}$$

故由定理 1 知边值问题存在解 $x(t,\varepsilon)$ 满足

$$\alpha(t,\varepsilon)\leqslant x(t,\varepsilon)\leqslant\beta(t,\varepsilon),\qquad t\in[0,1],$$

即

$$|x(t,\varepsilon)-X_N(t,\varepsilon)|\leqslant w(t,\varepsilon)+M\varepsilon^{N+1},\qquad t\in[0,1],$$

其中 $M=r$ 为与 ε 无关的常数.

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Singularly Perturbed Boundary Value Problems for Semi_Linear Retarded Differential Equations With Nonlinear Boundary Conditions

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Abstract: A boundary value problems for functional differenatial equations, with nonlinear boundary condition, is studied by the theorem of differential inequality. Using new method to construct the upper solution and lower solution, sufficient conditions for the existence of the problems' solution are established. A uniformly valid asymptotic expansions of the solution is also given.

Key words: singular perturbation; functional differential equation; boundary value problem; uniformly valid asymptotic expansion