

一维六方准晶中带双裂纹的椭圆孔口问题的解析解*

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摘要: 利用复变函数方法, 通过构造保角映射, 研究了一维六方准晶中带双裂纹的椭圆孔口的反平面剪切问题, 给出了 III 型裂纹问题的应力强度因子, 在极限情形下, 不仅可以还原为已有的结果, 而且求得一维六方准晶中带双裂纹的圆形孔口问题、十字裂纹问题在裂纹尖端的应力强度因子。

关键词: 准晶; 带双裂纹的椭圆孔口; 十字裂纹; 应力强度因子; 复变方法

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引 言

准晶是近二十年来被发现的一种新的固体结构和新材料, 其弹性问题的刻画不仅需要描写晶格振动的声子场, 还需要刻画原子准周期排列的相位子场, 而且二者是相互耦合的。自准晶被发现以来, 关于准晶各方面问题的研究已取得了若干重要成果^[1-10]。对于准晶的缺陷, 由于它和准晶材料的应用密切相关, 所以也吸引了许多学者的注意, 如准晶中运动裂纹的研究^[11]、准晶狭长体中单裂纹问题^[12]以及双裂纹问题^[13]的研究、准晶中位错问题的研究^[14]、准晶中裂纹与位错的相互作用^[15]等。由于多裂纹问题的解析解对分析工程断裂问题具有重要意义, 故本文将讨论一维六方准晶中带双裂纹的椭圆孔口问题。

1 基本公式

在一维六方准晶中, 用 u_1, u_2, u_3 表示声子场位移分量, v 表示相位子场位移分量,

$$\begin{aligned} [\varepsilon_j, w_j] &= [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{31}, 2\varepsilon_{12}, w_3, w_1, w_2], \\ [\sigma_j, H_j] &= [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}, H_3, H_1, H_2] \end{aligned}$$

分别表示声子场及相位子场的应变以及应力分量。由文献[12]知, 一维六方准晶中的广义 Hooke 定律为

$$\begin{cases} \sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + R_1w_3, & \sigma_{22} = C_{12}\varepsilon_{11} + C_{11}\varepsilon_{22} + C_{13}\varepsilon_{33} + R_1w_3, \\ \sigma_{33} = C_{13}\varepsilon_{11} + C_{13}\varepsilon_{22} + C_{33}\varepsilon_{33} + R_2w_3, & \sigma_{23} = \sigma_{32} = 2C_{44}\varepsilon_{23} + R_3w_2, \end{cases} \quad (1a)$$

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$$\begin{cases} \sigma_{31} = \sigma_{13} = 2C_{44}\varepsilon_{31} + R_3w_1, & \sigma_{12} = \sigma_{21} = 2C_{66}\varepsilon_{12}, & H_1 = 2R_3\varepsilon_{31} + K_2w_1, \\ H_2 = 2R_3\varepsilon_{23} + K_2w_2, & H_3 = R_1(\varepsilon_{11} + \varepsilon_{22}) + R_2\varepsilon_{33} + K_1w_3. \end{cases} \quad (1b)$$

其中 C_{ij} 与 K_i 分别为声子场与相位子场独立弹性常数, 而 R_i 为耦合的声子场-相位子场弹性常数, 且 $C_{66} = (C_{11} - C_{12})/2$. 几何方程为

$$\varepsilon_j = (\partial_j u_i + \partial_i u_j)/2, \quad w_j = \partial_j v, \quad (2)$$

其中 $\partial_j u_i = \partial u_i / \partial x_j$, $i, j = 1, 2, 3$.

不计体力情况下, 平衡方程为

$$\begin{cases} \partial_1 \sigma_{11} + \partial_2 \sigma_{12} + \partial_3 \sigma_{13} = 0, & \partial_1 \sigma_{21} + \partial_2 \sigma_{22} + \partial_3 \sigma_{23} = 0, \\ \partial_1 \sigma_{31} + \partial_2 \sigma_{32} + \partial_3 \sigma_{33} = 0, & \partial_1 H_1 + \partial_2 H_2 + \partial_3 H_3 = 0. \end{cases} \quad (3)$$

公式(1)~(3)就是一维六方准晶弹性理论的基本方程.

取准周期方向为坐标轴 x_3 方向, 则当裂纹穿透一维六方准晶的准周期方向时, 由准晶体的几何对称性知, 变形不随自变量 x_3 变化, 即

$$\partial_3 u_i = 0, \quad \partial_3 v = 0, \quad \partial_3 \sigma_j = 0, \quad \partial_3 H_j = 0, \quad i, j = 1, 2, 3. \quad (4)$$

把式(4)代入式(1)~(3), 则以上问题可以转化为两个独立的问题, 其中一个问题可以按照经典的弹性理论求解, 这里不再讨论. 另一个问题为

$$\sigma_{23} = \sigma_{32} = 2C_{44}\varepsilon_{23} + R_3w_2, \quad \sigma_{31} = \sigma_{13} = 2C_{44}\varepsilon_{31} + R_3w_1, \quad (5)$$

$$H_1 = 2R_3\varepsilon_{31} + K_2w_1, \quad H_2 = 2R_3\varepsilon_{23} + K_2w_2, \quad (6)$$

$$\partial_1 \sigma_{31} + \partial_2 \sigma_{23} = 0, \quad \partial_1 H_1 + \partial_2 H_2 = 0, \quad (7)$$

$$\varepsilon_j = \varepsilon_j = \partial_j u_j / 2, \quad w_j = \partial_j v, \quad j = 1, 2. \quad (8)$$

这是声子场-相位子场耦合的反平面弹性问题.

把式(8)代入式(5)、(6)后, 再代入式(7)可得

$$C_{44} \cdot^2 u_3 + R_3 \cdot^2 v = 0, \quad R_3 \cdot^2 u_3 + K_2 \cdot^2 v = 0,$$

其中 $\cdot^2 = \partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2$.

当 $C_{44}K_2 - R_3^2 \neq 0$ 时, 有

$$\cdot^2 u_3 = 0, \quad \cdot^2 v = 0. \quad (9)$$

公式(9)说明, 该反平面问题最终可化为两个调和方程求解. 由复变函数理论可知, u_3 和 v 可表为两个解析函数的实部或虚部, 不妨设

$$u_3 = \operatorname{Re} \Phi_1(z), \quad v = \operatorname{Re} \Phi_2(z), \quad (10)$$

其中 Re 代表复变函数的实部, $z = x_1 + ix_2$.

2 一维六方准晶中带双裂纹的椭圆孔口问题的应力场

一维六方准晶椭圆孔的长半轴为 a , 短半轴为 b , 裂纹长为 $c - a$, 建立如图 1 所示的坐标系, 并假设 x_3 轴方向为准周期方向, 且裂纹穿透准周期方向. 由准晶体的几何构型知, 该问题为准晶二维弹性问题, 方程(9)为其控制方程. 由式(3)和(4)知, 此时的平衡方程简化为

$$\partial_1 \sigma_{11} + \partial_2 \sigma_{12} = 0, \quad \partial_2 \sigma_{21} + \partial_2 \sigma_{22} = 0, \quad \partial_1 \sigma_{31} + \partial_2 \sigma_{32} = 0, \quad \partial_1 H_1 + \partial_2 H_2 = 0. \quad (11)$$

图 1 所示反平面问题的带双裂纹的椭圆问题具有如下边界条件:

$$\begin{cases} \sqrt{x_1^2 + x_2^2} \rightarrow \infty: \sigma_{32} = p, \sigma_{31} = H_2 = H_1 = 0, \\ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1: \sigma_{32} = 0, H_2 = 0, \\ -c < x_1 < -a, x_2 = 0: \sigma_{32} = 0, H_2 = 0, \\ a < x_1 < c, x_2 = 0: \sigma_{32} = 0, H_2 = 0. \end{cases} \quad (12)$$

线弹性理论分析的结果表明, 如果准晶在无穷远处不受外应力, 而仅在椭圆孔口及其所带裂纹的表面受 $\sigma_{32} = -p$ 作用, 则这两个问题所得结果除去一个常数项外是等价的. 现在考虑这后一问题, 其边界条件为

$$\begin{cases} \sqrt{x_1^2 + x_2^2} \rightarrow \infty: \sigma_{32} = \sigma_{31} = H_2 = H_1 = 0, \\ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1: \sigma_{32} = -p, H_2 = -q, \\ -c < x_1 < -a, x_2 = 0: \sigma_{32} = -p, H_2 = -q, \\ a < x_1 < c, x_2 = 0: \sigma_{32} = -p, H_2 = -q. \end{cases} \quad (13)$$

下面用复变函数方法求解上述反平面问题的边值问题(9)与式(13).

令 $z = x_1 + ix_2$, $i = \sqrt{-1}$, 若 $f(z)$ 为解析函数, 则有

$$\frac{\partial f}{\partial x_1} = \frac{df}{dz}, \quad \frac{\partial f}{\partial x_2} = i \frac{df}{dz}. \quad (14)$$

若

$$f(z) = P(x_1, x_2) + iQ(x_1, x_2) = \text{Re}f(z) + i\text{Im}f(z). \quad (15)$$

则依 Cauchy-Riemann 关系, 有

$$\frac{\partial P}{\partial x_1} = \frac{\partial Q}{\partial x_2}, \quad \frac{\partial P}{\partial x_2} = -\frac{\partial Q}{\partial x_1}. \quad (16)$$

把式(8)代入式(5)、(6), 再利用式(10)、(14), 有

$$\begin{cases} \sigma_{23} = \sigma_{32} = C_{44} \frac{\partial}{\partial x_2} \text{Re} \varphi_1 + R_3 \frac{\partial}{\partial x_2} \text{Re} \psi_1, \\ \sigma_{31} = \sigma_{13} = C_{44} \frac{\partial}{\partial x_1} \text{Re} \varphi_1 + R_3 \frac{\partial}{\partial x_1} \text{Re} \psi_1, \\ H_1 = K_2 \frac{\partial}{\partial x_1} \text{Re} \varphi_1 + R_3 \frac{\partial}{\partial x_1} \text{Re} \psi_1, \quad H_2 = K_2 \frac{\partial}{\partial x_2} \text{Re} \varphi_1 + R_3 \frac{\partial}{\partial x_2} \text{Re} \psi_1. \end{cases} \quad (17)$$

利用式(16), 上式可写为

$$\sigma_{31} - i\sigma_{32} = C_{44} \dot{\varphi}_1 + R_3 \dot{\psi}_1, \quad H_1 - iH_2 = K_2 \dot{\varphi}_1 + R_3 \dot{\psi}_1, \quad (18)$$

其中 $\dot{\varphi}_1 = d\varphi_1/dz$, $\dot{\psi}_1 = d\psi_1/dz$. 由式(18)有 $\sigma_{23} = \sigma_{32} = -\text{Im}(C_{44} \dot{\varphi}_1 + R_3 \dot{\psi}_1)$, $H_2 = -\text{Im}(K_2 \dot{\varphi}_1 + R_3 \dot{\psi}_1)$, 而对任意的复函数 $f(z)$, 有 $\text{Im}f(z) = (f - \bar{f})/2i$, 从而有

$$\begin{cases} \sigma_{23} = \sigma_{32} = -\frac{1}{2i} [C_{44} (\dot{\varphi}_1 - \overline{\dot{\varphi}_1}) + R_3 (\dot{\psi}_1 - \overline{\dot{\psi}_1})], \\ H_2 = -\frac{1}{2i} [K_2 (\dot{\varphi}_1 - \overline{\dot{\varphi}_1}) + R_3 (\dot{\psi}_1 - \overline{\dot{\psi}_1})]. \end{cases} \quad (19)$$

如果用 L 表示带双裂纹的椭圆孔, 则式(13)可表为

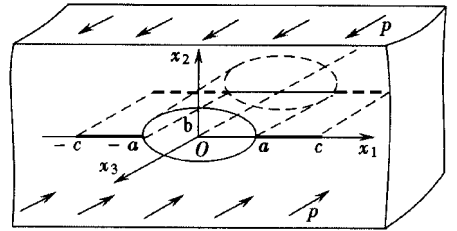


图1 一维六方准晶中带双裂纹的椭圆孔口

$$\begin{cases} C_{44}(\phi_1 - \overline{\phi_1}) + R_3(\phi_1 - \overline{\phi_1}) = 2pi, \\ K_2(\phi_1 - \overline{\phi_1}) + R_3(\phi_1 - \overline{\phi_1}) = 2qi, \end{cases} \quad z \in L. \quad (20)$$

作保角映射

$$z = \omega(\zeta) = \frac{a+b}{2} \frac{2(d^2+1)(1+\zeta^2) + (d^2-1)\sqrt{(1+\zeta)^4 + 2k(1-\zeta^2)^2 + (1-\zeta)^4}}{8d\zeta} + \frac{a-b}{2} \frac{8d\zeta}{2(d^2+1)(1+\zeta^2) + (d^2-1)\sqrt{(1+\zeta)^4 + 2k(1-\zeta^2)^2 + (1-\zeta)^4}}, \quad (21)$$

其中

$$d = \frac{c + \sqrt{c^2 - a^2 + b^2}}{a+b}, \quad k = \frac{d^4 + 6d^2 + 1}{(d^2 - 1)^2}. \quad (22)$$

则该映射把 ζ 平面的单位圆盘保角映射到 z 平面上带双裂纹的椭圆孔外部的区域, 单位圆周 τ 变成了带双裂纹的椭圆孔 L , 且有 $\omega^{-1}(c) \rightarrow 1$, $\omega^{-1}(bi) \rightarrow -i$, $\omega^{-1}(-c) \rightarrow -1$, $\omega^{-1}(-bi) \rightarrow i$, 同时把 a 的上岸映到点 $A_2(2d/(d^2+1), -(d^2-1)/(d^2+1))$, 把 a 的下岸映到点 $A_1(2d/(d^2+1), (d^2-1)/(d^2+1))$, 如图 2 所示.

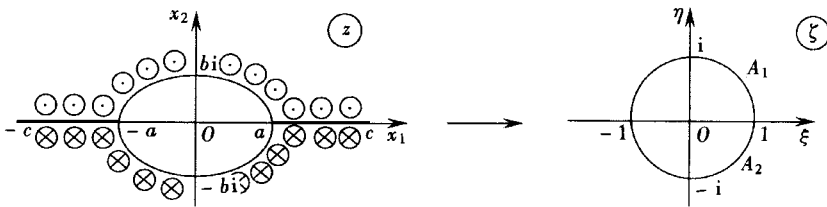


图 2 含带双裂纹的椭圆孔口的无限大平面到单位圆盘的映射

令 $\Phi(\zeta) = \Phi_1(z) = \Phi_1(\omega(\zeta))$, $\phi(\zeta) = \phi_1(z) = \phi_1(\omega(\zeta))$, 则有

$$\phi_1(z) = \phi'(\zeta)/\omega'(\zeta), \quad \phi_1'(z) = \phi''(\zeta)/\omega'(\zeta). \quad (23)$$

将式(23)代入式(20), 整理后将单位圆上的点 $\zeta \equiv \sigma = e^{i\theta}$ 代入, 得

$$\begin{aligned} \phi(\sigma) - \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \overline{\phi(\sigma)} + \frac{R_3}{C_{44}} \left[\phi'(\sigma) - \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \overline{\phi'(\sigma)} \right] &= \frac{2pi}{C_{44}} \omega'(\sigma), \\ \phi'(\sigma) - \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \overline{\phi'(\sigma)} + \frac{R_3}{K_2} \left[\phi(\sigma) - \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \overline{\phi(\sigma)} \right] &= \frac{2qi}{K_2} \omega'(\sigma). \end{aligned}$$

以上两式两端各乘以 $d\sigma/2\pi i(\sigma - \zeta)$ 之后, 沿单位圆周 τ 积分, 得

$$\begin{aligned} \frac{1}{2\pi i} \int_{\tau} \frac{\phi(\sigma)}{\sigma - \zeta} d\sigma - \frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\omega'(\sigma)} \frac{\overline{\phi(\sigma)}}{\sigma - \zeta} d\sigma + \frac{R_3}{C_{44}} \frac{1}{2\pi i} \int_{\tau} \frac{\phi'(\sigma)}{\sigma - \zeta} d\sigma - \\ \frac{R_3}{C_{44}} \frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\omega'(\sigma)} \frac{\overline{\phi'(\sigma)}}{\sigma - \zeta} d\sigma = \frac{2pi}{C_{44}} \frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\sigma - \zeta} d\sigma, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{1}{2\pi i} \int_{\tau} \frac{\phi'(\sigma)}{\sigma - \zeta} d\sigma - \frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\omega'(\sigma)} \frac{\overline{\phi'(\sigma)}}{\sigma - \zeta} d\sigma + \frac{R_3}{K_2} \frac{1}{2\pi i} \int_{\tau} \frac{\phi(\sigma)}{\sigma - \zeta} d\sigma - \\ \frac{R_3}{K_2} \frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\omega'(\sigma)} \frac{\overline{\phi(\sigma)}}{\sigma - \zeta} d\sigma = \frac{2qi}{K_2} \frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\sigma - \zeta} d\sigma. \end{aligned} \quad (25)$$

由于 $\phi(\zeta)$ 与 $\phi'(\zeta)$ 在单位圆 τ 内解析, 从而由复变函数的 Cauchy 积分定理有

$$\frac{1}{2\pi i} \int_{\tau} \frac{\phi(\sigma)}{\sigma - \zeta} d\sigma = \phi(\zeta), \quad \frac{1}{2\pi i} \int_{\tau} \frac{\phi'(\sigma)}{\sigma - \zeta} d\sigma = \phi'(\zeta), \quad |\zeta| < 1. \quad (26)$$

由公式(21), 得

$$\omega'(\zeta) = (1 - \zeta^2) \left[(d^2 + 1) + \frac{(d^2 - 1)(1 + k)(1 + \zeta^2)}{\sqrt{(1 + \zeta)^4 + 2k(1 - \zeta^2)^2 + (1 - \zeta)^4}} \right] \times$$

$$\left\{ -\frac{a+b}{8d\zeta^2} + 8d(a-b) \sqrt{\left[2(d^2+1)(1+\zeta^2) + (d^2-1)\sqrt{(1+\zeta)^4 + 2k(1-\zeta^2)^2 + (1-\zeta)^4} \right]^2} \right\}. \quad (27)$$

由式(27), 并注意到 $\sigma = 1/\zeta$, 得 $\omega'(\sigma)/\overline{\omega'(\sigma)} = -1/\sigma^2$.

由于 $\phi(\zeta)$ 与 $\psi(\zeta)$ 在单位圆内(即 $|\zeta| < 1$) 解析, 故可展成如下的 Taylor 级数

$$\phi(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \quad \psi(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n, \quad |\zeta| < 1.$$

而式(24)与式(25)中的 $\overline{\phi(\sigma)}$ 与 $\overline{\psi(\sigma)}$ 可视为

$$\overline{\phi\left(\frac{1}{\zeta}\right)} = \sum_{n=0}^{\infty} \overline{a_n} \left(\frac{1}{\zeta}\right)^n \quad \text{与} \quad \overline{\psi\left(\frac{1}{\zeta}\right)} = \sum_{n=0}^{\infty} \overline{b_n} \left(\frac{1}{\zeta}\right)^n$$

的边值(在单位圆 τ 上的值), 所以式(24)与式(25)中的

$$\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \overline{\phi(\sigma)} \quad \text{与} \quad \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \overline{\psi(\sigma)}$$

是单位圆外(即 $|\zeta| > 1$) 的解析函数

$$-\frac{1}{\zeta^2} \overline{\phi\left(\frac{1}{\zeta}\right)} \quad \text{与} \quad -\frac{1}{\zeta^2} \overline{\psi\left(\frac{1}{\zeta}\right)}$$

的边值. 由无穷远处的 Cauchy 积分公式, 对于 $|\zeta| < 1$, 有

$$\frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{\phi(\sigma)}}{\sigma - \zeta} d\sigma = \frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{\psi(\sigma)}}{\sigma - \zeta} d\sigma = 0. \quad (28)$$

把式(28)和式(27)代入式(24)以及(25), 可得

$$\begin{cases} \phi(\zeta) + \frac{R_3}{C_{44}} \psi(\zeta) = \frac{2pi}{C_{44}} \frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\sigma - \zeta} d\sigma, \\ \psi(\zeta) + \frac{R_3}{K_2} \phi(\zeta) = \frac{2qi}{K_2} \frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\sigma - \zeta} d\sigma. \end{cases} \quad (29)$$

由式(27)知, $\omega'(\zeta)$ 在单位圆外(即 $|\zeta| > 1$) 解析, 在单位圆外及圆上(即 $|\zeta| \geq 1$) 连续, 所以利用无穷远处的 Cauchy 积分公式有

$$\frac{1}{2\pi i} \int_{\tau} \frac{\omega'(\sigma)}{\sigma - \zeta} d\sigma = \omega'(\infty) = \frac{(d^2+1)(a+b)}{4d} \stackrel{\text{def}}{=} F(\zeta). \quad (30)$$

将式(30)代入式(29)得

$$\phi(\zeta) + \frac{R_3}{C_{44}} \psi(\zeta) = \frac{2pi}{C_{44}} F(\zeta), \quad \psi(\zeta) + \frac{R_3}{K_2} \phi(\zeta) = \frac{2qi}{K_2} F(\zeta). \quad (31)$$

由上式可得

$$\phi(\zeta) = 2i \frac{pK_2 - qR_3}{C_{44}K_2 - R_3^2} F(\zeta), \quad \psi(\zeta) = 2i \frac{qC_{44} - pR_3}{C_{44}K_2 - R_3^2} F(\zeta). \quad (32)$$

利用式(23), 有

$$\begin{cases} \phi_1(z) = \frac{\phi(\zeta)}{\omega'(\zeta)} = \frac{2i(pK_2 - qR_3)}{C_{44}K_2 - R_3^2} \frac{F(\zeta)}{\omega'(\zeta)}, \\ \psi_1(z) = \frac{\psi(\zeta)}{\omega'(\zeta)} = \frac{2i(qC_{44} - pR_3)}{C_{44}K_2 - R_3^2} \frac{F(\zeta)}{\omega'(\zeta)}. \end{cases} \quad (33)$$

将上式代入式(18), 得

$$\sigma_{31} - i\sigma_{32} = 2pi \frac{F(\zeta)}{\omega'(\zeta)}, \quad H_1 - iH_2 = 2qi \frac{F(\zeta)}{\omega'(\zeta)}. \quad (34)$$

这表明应力分布与材料常数无关, 这一点与经典弹性理论一致. 解此方程组, 便可得到应力 σ_{31} 、 σ_{32} 、 H_1 、 H_2 . 由于表达式冗长, 故略去. 为了得到 z 平面上的应力函数, 还需将式(21)的反演 $\zeta = \omega^{-1}(z)$ 代回到(33)式便可得到应力函数.

3 应力强度因子的计算

我们知道, 裂纹端点往往是造成材料破坏的起点, 而应力强度因子可以反映裂纹尖端附近应力的强弱程度, 所以求解和测定材料的应力强度因子是非常重要的. 由文献[12]知, ζ 平面上应力强度因子的计算公式为

$$K_{\text{III}}^{\parallel} = \lim_{\zeta \rightarrow \zeta_1} 2\sqrt{\pi}p \frac{F(\zeta)}{\sqrt{\omega''(\zeta)}}, \quad K_{\text{III}}^{\perp} = \lim_{\zeta \rightarrow \zeta_1} 2\sqrt{\pi}q \frac{F(\zeta)}{\sqrt{\omega''(\zeta)}}, \quad (35)$$

其中 $\zeta = \zeta_1$ 为与裂纹顶端对应的点.

由式(27)可知

$$\sqrt{\omega''(\zeta)} \rightarrow \frac{\sqrt{(d^2+1)[(d^2-1)a + (d^2+1)b]}}{\sqrt{2d(d^2-1)}}, \quad \zeta \rightarrow 1. \quad (36)$$

又由式(30)有

$$F(\zeta) \rightarrow \frac{(d^2+1)(a+b)}{4d}, \quad \zeta \rightarrow 1. \quad (37)$$

从而有

$$\frac{F(\zeta)}{\sqrt{\omega''(\zeta)}} \rightarrow \frac{(a+b)\sqrt{2(d^4-1)}}{4\sqrt{d[(d^2-1)a + (d^2+1)b]}}, \quad \zeta \rightarrow 1. \quad (38)$$

把式(38)代入式(35), 可得在 $\zeta = 1$ 处声子场与相位子场的 III 型应力强度因子分别为

$$K_{\text{III}}^{\parallel} = \lim_{\zeta \rightarrow \zeta_1} 2\sqrt{\pi}p \frac{F(\zeta)}{\sqrt{\omega''(\zeta)}} = \frac{p(a+b)\sqrt{\pi(d^4-1)}}{\sqrt{2d[(d^2-1)a + (d^2+1)b]}}, \quad (39)$$

$$K_{\text{III}}^{\perp} = \lim_{\zeta \rightarrow \zeta_1} 2\sqrt{\pi}q \frac{F(\zeta)}{\sqrt{\omega''(\zeta)}} = \frac{q(a+b)\sqrt{\pi(d^4-1)}}{\sqrt{2d[(d^2-1)a + (d^2+1)b]}}. \quad (40)$$

这就是一维六方准晶中带双裂纹的椭圆孔口在裂纹端点处的应力强度因子.

4 关于应力强度因子的一些讨论与结论

以上我们已求出了一维六方准晶中带双裂纹的椭圆孔口在裂纹尖端处的 III 型应力强度因子. 下面我们对这一结果做进一步讨论:

1) 在式(39)和式(40)中, 令 $b \rightarrow 0$ 时, 则

$$K_{\text{III}}^{\parallel} = p\sqrt{\pi c}, \quad K_{\text{III}}^{\perp} = q\sqrt{\pi c}.$$

这就是一维六方准晶中 Griffith 裂纹在裂尖的应力强度因子, 与文献[12]中的结果完全一致.

2) 在式(39)和式(40), 令 $a \rightarrow 0$ 时, 则

$$K_{\text{III}}^{\parallel} = p\sqrt{\pi c}, \quad K_{\text{III}}^{\perp} = q\sqrt{\pi c}.$$

这是无限大区域内一维六方准晶中十字裂纹在裂尖的应力强度因子, 此结果为本文首次得到.

该结果表明, 水平裂纹中心处具有对称垂直裂纹的准晶体, 如果受纵向剪切载荷时, 垂直裂纹并不影响水平裂纹尖端的应力强度因子, 其结果与 Griffith 裂纹情形相同. 如图 3 所示.

3) 在式(39)和式(40)中, 令 $a \rightarrow b$ 时, 则

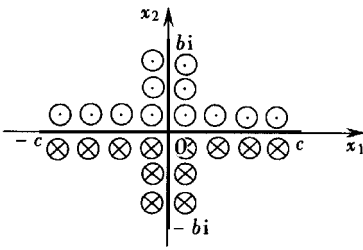


图3 十字裂纹

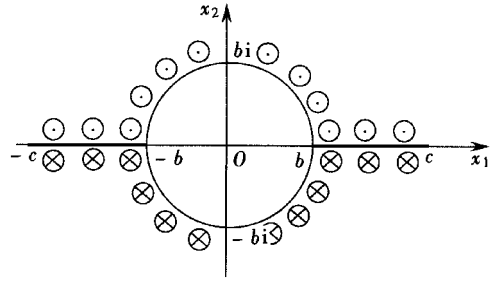


图4 带双裂纹的圆形孔口

$$K_{III}^{\parallel} = p \sqrt{\pi c \left[1 - \frac{b^4}{c^4} \right]}, \quad K_{III}^{\perp} = q \sqrt{\pi c \left[1 - \frac{b^4}{c^4} \right]}.$$

这就是无限大区域内一维六方准晶中带双裂纹的圆形孔口在裂尖的应力强度因子,这也是本文首次得到的一个结果(如图4).从这里可以看出,含具有对称裂纹的圆孔的准晶体,当受纵向剪切载荷时,其裂纹尖端的应力强度因子不仅与圆孔的半径有关,而且与裂纹长度有关.

特别地,当 $b \rightarrow 0$ 时, $K_{III}^{\parallel} = p \sqrt{\pi c}$, $K_{III}^{\perp} = q \sqrt{\pi c}$. 同样得到文献[12]给出的 Griffith 裂纹的情形.

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Analytic Solutions of Problem About an Elliptic Hole With Two Straight Cracks in One-Dimensional Hexagonal Quasicrystals

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Abstract: By means of the complex variable function method and using the technique of conformal mapping, the anti-plane shear problem about an elliptic hole with two straight cracks in one-dimensional hexagonal quasicrystals was investigated and the solution of the stress intensity factor (SIFs) of mode III was found out. Under the condition of limitation, not only the known result can be obtained but also the solutions of the SIFs at the crack tip to a circular hole with two straight cracks and a cross crack in one-dimensional hexagonal quasicrystals are found out.

Key words: quasicrystal; elliptic hole with two straight cracks; cross crack; SIFs; complex variable function method