

一端固支一端简支变厚度梁的弹性力学解^{*}

徐业鹏¹, 周 叮², 张佑³

(1. 南京理工大学 理学院, 南京 210094;

2. 南京工业大学 土木工程学院, 南京 210009;

3. 香港大学 土木工程系, 香港)

(本刊编委张佑 来稿)

摘要: 研究了一端固支另一端简支连续变厚度梁在静力荷载作用下的应力和位移分布. 通过引入单位脉冲函数和 Dirac 函数, 将固支边等效为简支边与未知水平反力的叠加, 利用平面应力问题的基本方程, 导出满足控制微分方程及左右两端边界条件的位移函数的一般解, 对上下表面的边界方程作 Fourier 级数展开, 结合固支边位移为 0 的条件确定待定系数, 得到的解是高精度的. 数值结果与商业有限元软件 ANSYS 进行了比较, 显示出很好的精度.

关键词: 梁; 固支边; 变厚度; Fourier 展开; 弹性力学解

中图分类号: O343.1 **文献标识码:** A

引 言

众所周知, 在航空、机械、土木工程等领域, 梁式结构得到了广泛的应用. 在实际工程中, 除了等厚度梁外, 变厚度梁也常常得到使用. 因此, 研究变厚度梁在静力荷载下的力学特性有重要意义.

通常, 有两种理论可用于梁的受力分析, 一种是经典的材料力学理论, 另一种是更为精确的弹性力学理论. 在材料力学理论中, 通过引入假定, 使得描述问题的微分方程得到简化, 相应地使得求解也较为容易, 典型的有适用于长细梁的 Euler 梁理论和适用于中厚度梁的 Timoshenko 梁理论. 然而, 由于采用了人为假定, 材料力学理论不可避免地带来误差, 应力分布的计算精度往往较低, 尤其以厚梁更为显著. 现代工业中, 特别是在一些高科技领域, 如航空航天工程和微型机械的设计, 经常要求对梁的应力分布做精细分析, 此时, 材料力学理论的精度往往不够, 需使用更为精确的二维弹性力学理论进行分析.

平面应力问题的弹性力学解早就受到科学家和工程设计人员的关注^[1], 对于等厚度梁, 对各向同性和各向异性材料以及各种不同边界条件下的弹性力学解, 已有大量的研究成果^[2-9]. 而对于变厚度梁, 研究仍基于 Euler 梁理论和 Timoshenko 梁理论, 其中传递矩阵法^[10]是一种较为常见的分析方法. 到目前为止, 尚未发现有人采用弹性力学理论分析变厚度梁的应力和位

* 收稿日期: 2007-08-24; 修订日期: 2008-01-25

作者简介: 徐业鹏(1982—), 男, 湖北人, 博士生;

周叮(1957—), 男, 江苏人, 教授, 博士, 博士生导师(联系人. Tel: + 86-25-84316695; E-mail: dingzhou57@yahoo.com).

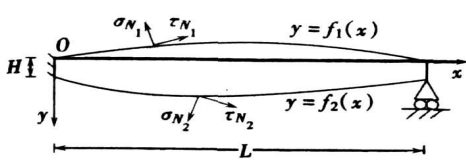
移分布.

本文根据变厚度梁的特点, 研究梁的一端为固支的情况, 引入单位脉冲函数和 Dirac 函数, 将固支端等效为简支端加上未知的水平反力, 取位移函数使其直接满足左右两端的简支边界条件, 对上下表面的边界方程作 Fourier 正弦级数展开, 并利用固支端位移为 0 的条件共同确定待定系数, 从而使问题得到解决.

1 基本方程

考虑如图 1 所示的变厚度梁. 梁的左端固支, 右端简支. 梁长 L , 梁的左端厚度为 H , 梁的上下表面用连续函数 $f_1(x)$ 和 $f_2(x)$ 表示. 上边界承受的水平载荷和竖向载荷分别为 $q_1(x)$ 和 $q_2(x)$, 下边界承受的水平载荷和竖向载荷分别为 $q_3(x)$ 和 $q_4(x)$.

首先引入单位脉冲函数 $H(x)$ 和 Dirac 函数(即 δ -函数):



$$\left\{ \begin{array}{l} H(x) = \begin{cases} 1, & \text{当 } x = 0 \text{ 时,} \\ 0, & \text{当 } x \neq 0 \text{ 时,} \end{cases} \\ \delta(x) = -\frac{dH(x)}{dx} = \begin{cases} \infty, & \text{当 } x = 0 \text{ 时,} \\ 0, & \text{当 } x \neq 0 \text{ 时.} \end{cases} \end{array} \right. \quad (1)$$

图 1 一端固支一端简支的连续变厚度梁

令水平应力为

$$\alpha_x = \alpha_x + H(x)p^{(0)}(y), \quad (2)$$

其中, α_x 是两端简支梁的水平应力, $p^{(0)}(y)$ 是固支端的水平应力. 忽略体力, 将式(1)和式(2)代入平面应力问题的平衡方程得到

$$\frac{\partial \alpha_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \delta(x)p^{(0)}(y), \quad \frac{\partial \alpha_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \quad (3)$$

上式中, 如果把 $\delta(x)p^{(0)}(y)$ 看作是沿 x 方向的体力, 则对应于 x 方向的正应力为 α_x . 此时, 梁域内的本构关系为

$$\left\{ \begin{array}{l} \alpha_x = \frac{E}{1-\mu^2} \left(\frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y} \right), \quad \alpha_y = \frac{E}{1-\mu^2} \left(\frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \right), \\ \tau_{xy} = \frac{E}{2(1+\mu)} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right). \end{array} \right. \quad (4)$$

将式(4)带入式(3), 得到用位移表示的平衡微分方程

$$\frac{E}{1-\mu^2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2} \frac{\partial^2 u}{\partial y^2} + \frac{1+\mu}{2} \frac{\partial^2 v}{\partial x \partial y} \right) = \delta(x)p^{(0)}(y), \quad (5)$$

$$\frac{E}{1-\mu^2} \left(\frac{\partial^2 v}{\partial y^2} + \frac{1-\mu}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1+\mu}{2} \frac{\partial^2 u}{\partial x \partial y} \right) = 0. \quad (6)$$

取位移函数为

$$u = \sum_{m=1}^{\infty} U_m(y) \cos \frac{m\pi x}{L} + U_0(y), \quad (7)$$

$$v = \sum_{m=0}^{\infty} V_m(y) \sin \frac{m\pi x}{L}. \quad (8)$$

这时下列边界条件可直接得到满足:

$$x = 0: \quad \alpha_x = 0, \quad v = 0, \quad (9)$$

$$x = L: \quad \alpha_x = 0, \quad v = 0. \quad (10)$$

在固支端 $x = 0$ 处, $\alpha_x = p^{(0)}(y)$. 作 Fourier 余弦展开, 得到:

$$p^{(0)}(y) = \frac{EL}{2(1+\mu)} \sum_{k=0}^M R_k \cos \frac{k\pi y}{H}. \quad (11)$$

将 δ 函数展开成级数函数

$$\delta(x) = \frac{1}{L} + \frac{2}{L} \sum_{m=1}^{\infty} \cos \frac{m\pi x}{L}. \quad (12)$$

将式(7)、(8)、(11)、(12)带入式(5)和(6), 联立求解 $u(x, y)$ 和 $v(x, y)$, 最后得到:

$$\begin{aligned} u(x, y) = & \sum_{m=1}^{\infty} \left\{ \left(A_m + \frac{3-\mu}{1+\mu} \frac{L}{m\pi} D_m \right) \cosh \frac{m\pi y}{L} + \right. \\ & \left. \left(B_m + \frac{3-\mu}{1+\mu} \frac{L}{m\pi} C_m \right) \sinh \frac{m\pi y}{L} + C_m y \cosh \frac{m\pi y}{L} + D_m y \sinh \frac{m\pi y}{L} + \right. \\ & \left. \sum_{k=1}^M \left[\frac{(\mu-1)m^2\pi^2/L^2 - 2k^2\pi^2/H^2}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} R_k \cos \frac{k\pi y}{H} \right] \cos \frac{m\pi x}{L} - \right. \\ & \left. \frac{H^2}{k^2\pi^2} \sum_{k=1}^M R_k \cos \frac{k\pi y}{H} \right\}, \quad (13) \end{aligned}$$

$$\begin{aligned} v(x, y) = & \sum_{m=1}^{\infty} \left[A_m \sinh \frac{m\pi y}{L} + B_m \cosh \frac{m\pi y}{L} + C_m y \sinh \frac{m\pi y}{L} + D_m y \cosh \frac{m\pi y}{L} - \right. \\ & \left. \sum_{k=1}^M \frac{(1+\mu)mk\pi^2/(LH)}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} R_k \sin \frac{k\pi y}{H} \right] \sin \frac{m\pi x}{L}. \quad (14) \end{aligned}$$

以上 2 式中, 前 4 项 (含有 A_m, B_m, C_m, D_m) 为方程的齐次解, 其他项 (含有对 k 的求和) 为方程的特解.

由边界条件(9)和(10)可知, 在梁的两端, 竖直位移为 0. 因此, $R_0 = 0$. 将式(13)和式(14)带回到式(4)中得到

$$\begin{aligned} \alpha_x = & \frac{E}{1+\mu} \sum_{m=1}^{\infty} \left\{ - \left[\frac{m\pi}{L} \cosh \frac{m\pi y}{L} A_m + \frac{m\pi}{L} \sinh \frac{m\pi y}{L} B_m + \left(\frac{3+\mu}{1+\mu} \sinh \frac{m\pi y}{L} + \right. \right. \right. \\ & \left. \left. \frac{m\pi}{L} y \cosh \frac{m\pi y}{L} \right) C_m + \left(\frac{3+\mu}{1+\mu} \cosh \frac{m\pi y}{L} + \frac{m\pi}{L} y \sinh \frac{m\pi y}{L} \right) D_m \right] + \\ & \left. \sum_{k=1}^M \frac{(m\pi/L) [\mu m^2\pi^2/L^2 + (\mu+2)k^2\pi^2/H^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} R_k \cos \frac{k\pi y}{H} \right\} \sin \frac{m\pi y}{L}, \quad (15) \end{aligned}$$

$$\begin{aligned} \alpha_y = & \frac{E}{1+\mu} \sum_{m=1}^{\infty} \left\{ \left[\frac{m\pi}{L} \cosh \frac{m\pi y}{L} A_m + \frac{m\pi}{L} \sinh \frac{m\pi y}{L} B_m + \left(\frac{1-\mu}{1+\mu} \sinh \frac{m\pi y}{L} + \right. \right. \right. \\ & \left. \left. \frac{m\pi}{L} y \cosh \frac{m\pi y}{L} \right) C_m + \left(\frac{1-\mu}{1+\mu} \cosh \frac{m\pi y}{L} + \frac{m\pi}{L} y \sinh \frac{m\pi y}{L} \right) D_m \right] + \\ & \left. \sum_{k=1}^M \frac{(m\pi/L) [\mu m^2\pi^2/L^2 - k^2\pi^2/H^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} R_k \cos \frac{k\pi y}{H} \right\} \sin \frac{m\pi y}{L}, \quad (16) \end{aligned}$$

$$\begin{aligned} \tau_{xy} = & \frac{E}{1+\mu} \sum_{m=1}^{\infty} \left\{ \left[\frac{m\pi}{L} \sinh \frac{m\pi y}{L} A_m + \frac{m\pi}{L} \cosh \frac{m\pi y}{L} B_m + \left(\frac{2}{1+\mu} \cosh \frac{m\pi y}{L} + \right. \right. \right. \\ & \left. \left. \frac{m\pi}{L} y \sinh \frac{m\pi y}{L} \right) C_m + \left(\frac{2}{1+\mu} \sinh \frac{m\pi y}{L} + \frac{m\pi}{L} y \cosh \frac{m\pi y}{L} \right) D_m \right] + \\ & \left. \sum_{k=1}^M \frac{(k\pi/H) [k^2\pi^2/H^2 - \mu m^2\pi^2/L^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} R_k \sin \frac{k\pi y}{H} \right\} \cos \frac{m\pi y}{L} + \end{aligned}$$

$$\frac{E}{1+\mu} \frac{H}{2k\pi} \sum_{k=1}^M \sin \frac{k\pi y}{H} R_k, \quad (17)$$

其中 A_m, B_m, C_m, D_m, R_k 为待定系数.

2 上下表面边界条件

上表面 $y = f_1(x)$ 的边界条件为

$$l_1(x) \sigma_x + m_1(x) \tau_{xy} = q_1(x), \quad (18)$$

$$m_1(x) \sigma_y + l_1(x) \tau_{xy} = q_2(x), \quad (19)$$

其中

$$\begin{cases} l_1(x) = \cos(N_1, x) = - \frac{df_1(x)}{dx} \sqrt{1 + \left(\frac{df_1(x)}{dx}\right)^2}, \\ m_1(x) = \cos(N_1, y) = - 1 \sqrt{1 + \left(\frac{df_1(x)}{dx}\right)^2}. \end{cases} \quad (20)$$

下表面 $y = f_2(x)$ 的边界条件为

$$l_2(x) \sigma_x + m_2(x) \tau_{xy} = q_3(x), \quad (21)$$

$$m_2(x) \sigma_y + l_2(x) \tau_{xy} = q_4(x), \quad (22)$$

其中

$$\begin{cases} l_2(x) = \cos(N_2, x) = - \frac{df_2(x)}{dx} \sqrt{1 + \left(\frac{df_2(x)}{dx}\right)^2}, \\ m_2(x) = \cos(N_2, y) = 1 \sqrt{1 + \left(\frac{df_2(x)}{dx}\right)^2}. \end{cases} \quad (23)$$

在式(7)和式(8)的基础上, 固支端的水平方向位移为 0, 即

$$\begin{aligned} & \sum_{m=1}^{\infty} \left\{ A_m + \frac{3-\mu}{1+\mu} \frac{L}{m\pi} D_m \right\} \cosh \frac{m\pi y}{L} + \\ & \left\{ B_m + \frac{3-\mu}{1+\mu} \frac{L}{m\pi} C_m \right\} \sinh \frac{m\pi y}{L} + C_{my} \cosh \frac{m\pi y}{L} + D_{my} \sinh \frac{m\pi y}{L} + \\ & \sum_{k=1}^M \left\{ \frac{L(\mu-1)m^2\pi^2/L^2 - 2k^2\pi^2/H^2}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} - \frac{H^2}{k^2\pi^2} \right\} R_k \cos \frac{k\pi y}{H} = 0. \end{aligned} \quad (24)$$

将固支端的边界条件近似为: 在 $x = 0$ 处, 沿 y 方向上有 M 个点的水平位移为 0. 为简便起见, 不妨设这些点在固支边上等距分布, 其坐标值分别为, $H/(M+1), 2H/(M+1), \dots, MH/(M+1)$, 这样, 由式(24) 得到 M 个代数方程.

3 系数矩阵

联立式(15)~ 式(23), 对每个方程作 Fourier 正弦展开, 即

$$\int_0^L \sin \frac{n\pi x}{L} [l_1(x) \sigma_x + m_1(x) \tau_{xy}] dx = \int_0^L \sin \frac{n\pi x}{L} q_1(x) dx, \quad (25)$$

$$\int_0^L \sin \frac{n\pi x}{L} [m_1(x) \sigma_y + l_1(x) \tau_{xy}] dx = \int_0^L \sin \frac{n\pi x}{L} q_2(x) dx, \quad (26)$$

$$\int_0^L \sin \frac{n\pi x}{L} [l_2(x) \sigma_x + m_2(x) \tau_{xy}] dx = \int_0^L \sin \frac{n\pi x}{L} q_3(x) dx, \quad (27)$$

$$\int_0^L \sin \frac{n\pi x}{L} [m_2(x) \sigma_y + l_2(x) \tau_{xy}] dx = \int_0^L \sin \frac{n\pi x}{L} q_4(x) dx. \quad (28)$$

截断 $N + 1$ 阶以上的级数项(即取 $n = 1, 2, 3, \dots, N$), 并结合式(24), 得到如下的矩阵方程:

$$\begin{bmatrix}
 A_{11}^{(1)} & \dots & A_{1N}^{(1)} & B_{11}^{(1)} & \dots & B_{1N}^{(1)} & C_{11}^{(1)} & \dots & C_{1N}^{(1)} & D_{11}^{(1)} & \dots & D_{1N}^{(1)} & R_{11}^{(1)} & \dots & R_{1M}^{(1)} \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 A_{N1}^{(1)} & \dots & A_{NN}^{(1)} & B_{N1}^{(1)} & \dots & B_{NN}^{(1)} & C_{N1}^{(1)} & \dots & C_{NN}^{(1)} & D_{N1}^{(1)} & \dots & D_{NN}^{(1)} & R_{N1}^{(1)} & \dots & R_{NM}^{(1)} \\
 A_{11}^{(2)} & \dots & A_{1N}^{(2)} & B_{11}^{(2)} & \dots & B_{1N}^{(2)} & C_{11}^{(2)} & \dots & C_{1N}^{(2)} & D_{11}^{(2)} & \dots & D_{1N}^{(2)} & R_{11}^{(2)} & \dots & R_{1M}^{(2)} \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 A_{N1}^{(2)} & \dots & A_{NN}^{(2)} & B_{N1}^{(2)} & \dots & B_{NN}^{(2)} & C_{N1}^{(2)} & \dots & C_{NN}^{(2)} & D_{N1}^{(2)} & \dots & D_{NN}^{(2)} & R_{N1}^{(2)} & \dots & R_{NM}^{(2)} \\
 A_{11}^{(3)} & \dots & A_{1N}^{(3)} & B_{11}^{(3)} & \dots & B_{1N}^{(3)} & C_{11}^{(3)} & \dots & C_{1N}^{(3)} & D_{11}^{(3)} & \dots & D_{1N}^{(3)} & R_{11}^{(3)} & \dots & R_{1M}^{(3)} \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 A_{N1}^{(3)} & \dots & A_{NN}^{(3)} & B_{N1}^{(3)} & \dots & B_{NN}^{(3)} & C_{N1}^{(3)} & \dots & C_{NN}^{(3)} & D_{N1}^{(3)} & \dots & D_{NN}^{(3)} & R_{N1}^{(3)} & \dots & R_{NM}^{(3)} \\
 A_{11}^{(4)} & \dots & A_{1N}^{(4)} & B_{11}^{(4)} & \dots & B_{1N}^{(4)} & C_{11}^{(4)} & \dots & C_{1N}^{(4)} & D_{11}^{(4)} & \dots & D_{1N}^{(4)} & R_{11}^{(4)} & \dots & R_{1M}^{(4)} \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 A_{N1}^{(4)} & \dots & A_{NN}^{(4)} & B_{N1}^{(4)} & \dots & B_{NN}^{(4)} & C_{N1}^{(4)} & \dots & C_{NN}^{(4)} & D_{N1}^{(4)} & \dots & D_{NN}^{(4)} & R_{N1}^{(4)} & \dots & R_{NM}^{(4)} \\
 A_{11}^{(5)} & \dots & A_{1N}^{(5)} & B_{11}^{(5)} & \dots & B_{1N}^{(5)} & C_{11}^{(5)} & \dots & C_{1N}^{(5)} & D_{11}^{(5)} & \dots & D_{1N}^{(5)} & R_{11}^{(5)} & \dots & R_{1M}^{(5)} \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 A_{M1}^{(5)} & \dots & A_{MN}^{(5)} & B_{M1}^{(5)} & \dots & B_{MN}^{(5)} & C_{M1}^{(5)} & \dots & C_{MN}^{(5)} & D_{M1}^{(5)} & \dots & D_{MN}^{(5)} & R_{M1}^{(5)} & \dots & R_{MM}^{(5)}
 \end{bmatrix}
 \begin{bmatrix}
 A_1 \\
 \vdots \\
 A_N \\
 B_1 \\
 \vdots \\
 B_N \\
 C_1 \\
 \vdots \\
 C_N \\
 D_1 \\
 \vdots \\
 D_N \\
 R_1 \\
 \vdots \\
 R_M
 \end{bmatrix}
 =
 \begin{bmatrix}
 S_1^{(1)} \\
 \vdots \\
 S_N^{(1)} \\
 S_1^{(2)} \\
 \vdots \\
 S_N^{(2)} \\
 S_1^{(3)} \\
 \vdots \\
 S_N^{(3)} \\
 S_1^{(4)} \\
 \vdots \\
 S_N^{(4)} \\
 0 \\
 \vdots \\
 0
 \end{bmatrix},
 \tag{29}$$

矩阵中每个元素的表达式参见附录, 由 Gauss 数值积分给出数值. 式(29)为 $(4N + M) \times (4N + M)$ 的线性代数方程组, 可以唯一地解出 $A_1, \dots, A_N, B_1, \dots, B_N, C_1, \dots, C_N, D_1, \dots, D_N, R_1, \dots, R_M$ 这 $4N + M$ 个未知系数. 将这些系数代回到方程式(13)~(17), 则可以求得梁上任意一点处的应力和位移.

4 收敛性和算例

在下面的算例中, 均取弹性模量 $E = 2.2 \times 10^{11}$ Pa, Poisson 比 $\mu = 0.3$. 首先考察如图 2 所示的一端固支另一端简支的等厚度梁, 梁的厚跨比为 $H/L = 0.10$, 梁承受均布的垂直压力 q . 取 3 个不同的水平零位移约束点数 $M = 5, 10, 20$ 以及 4 个不同的级数项 $N = 40, 60, 80, 100$, 计算梁在 $x = L/2, y = 0$ 处的应力和位移, 表 1 给出了它们的收敛性. 从表 1 中看到, 当 $M = 10$ 和 $N = 80$ 时, 本文方法可以保证 3 位有效数字的精度.

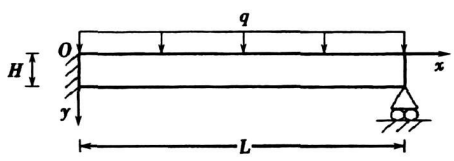


图 2 一端固支一端简支的等厚度梁

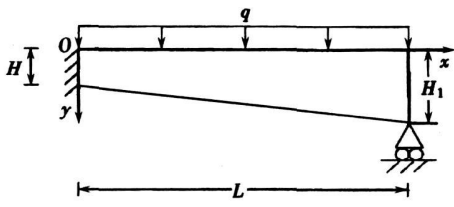


图 3 一端固支一端简支下表面线性变化的楔形梁

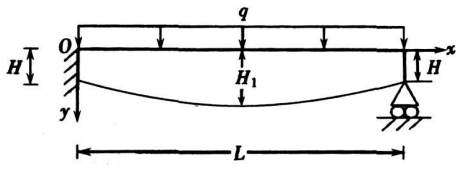


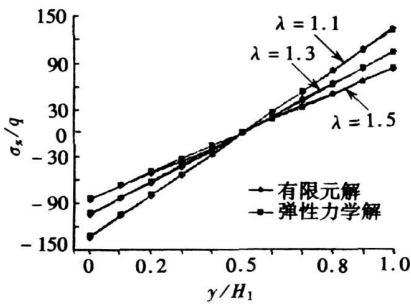
图 4 一端固支一端简支下表面抛物线变化的外凸梁

表1 一端固支一端简支等厚度梁 ($H/L = 0.10$) 在 $x = L/2, y = 0$ 处应力和位移的收敛性

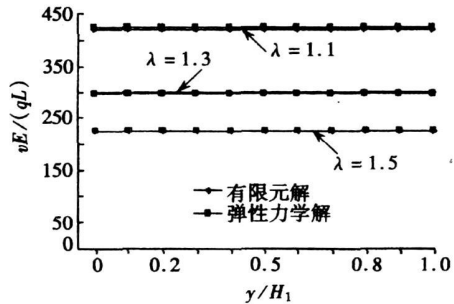
固定点数 M	级数项 N	σ_x/q	σ_y/q	τ_{xy}/q	$uE/(qL)$	$vE/(qL)$
5	40	-38.4	-0.984	0.000	2.95	66.1
	60	-38.4	-0.989	0.000	2.94	66.6
	80	-38.4	-0.993	0.000	2.93	66.8
	100	-38.4	-0.993	0.000	2.93	66.8
10	40	-38.6	-0.984	0.000	2.96	65.5
	60	-38.5	-0.989	0.000	2.95	65.9
	80	-38.3	-0.993	0.000	2.94	66.2
	100	-38.3	-0.993	0.000	2.94	66.2
20	40	-38.8	-0.984	0.000	2.96	65.3
	60	-38.6	-0.989	0.000	2.95	65.8
	80	-38.3	-0.993	0.000	2.94	66.2
	100	-38.3	-0.993	0.000	2.94	66.2

表2 一端固支一端简支下表面线性变化楔形梁 ($H/L = 0.05$) 在 $x = L/2, y = 0$ 处应力和位移的收敛性

	H_1/H	$N = 40$	$N = 60$	$N = 80$	$N = 100$
$\frac{\sigma_x}{q}$	1.00	-152	-152	-152	-152
	1.25	-129	-130	-131	-131
	1.50	-111	-111	-112	-112
$\frac{uE}{qL}$	1.00	12.3	12.2	12.2	12.2
	1.25	7.71	7.65	7.59	7.59
	1.50	4.81	4.70	4.63	4.63
$\frac{vE}{qL}$	1.00	508	510	513	513
	1.25	386	386	387	387
	1.50	303	300	303	303



(a) σ_x/q



(b) $vE/(qL)$

图5 一端固支一端简支下表面抛物线变化外凸梁弹性力学解与有限元解的比较

考察如图3所示的一端固支一端简支下表面线性变化的楔形梁, 梁的最小厚跨比为 $H/L = 0.05$, 梁的上表面水平且承受均布的垂直压力 q . 取 $M = 10$ 和 4 个不同的级数项 $N = 40, 60, 80, 100$, 计算 3 个不同厚度比 $H_1/H = 1, 1.25, 1.5$ 时的应力和位移. 表2 给出了梁在 $x =$

$L/2, y = 0$ 处应力和位移的收敛性. 从表2中看出, 当级数项分别取 $N = 80$ 和 $N = 100$ 时, 各力学量收敛一致, 精度可以达到3位有效数字, 由此说明本文的解法有着很好的收敛性, 在本文以后的算例中, 均取 $M = 10, N = 80$.

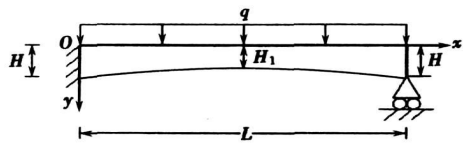


图6 一端固支一端简支下表面抛物线变化的内凹梁

下面计算2种典型结构的弹性力学解, 同时给出 ANSYS 有限元解以供比较, ANSYS 求解时采用的单

元类型为 Solid 8node82 单元. 图4为一下表面抛物线变化的外凸梁, 梁的最小厚跨比为 $H/L = 0.05$. 梁的上表面水平且承受均布压力荷载 q , 设梁的中截面厚度为 H_1 , 从上到下依次取11个等距点分析应力和位移沿高度的分布情况, 图5给出了3种不同厚度比 $\lambda = H_1/H = 1.1, 1.3, 1.5$ 时梁的中截面上水平应力 σ_x 和垂直位移 v 的分布情况. 从图5可以看出本文弹性力学解和 ANSYS 解十分吻合. 位移 v 随坐标 y 的变化很小, 应力的最大值和纵向位移的最大值均随厚度比 λ 的增大而减小.

图6为一下表面抛物线变化的内凹梁, 梁的最大厚跨比为 $H/L = 0.05$, 梁的上表面水平且承受均布的竖直荷载 q , 表3给出了3种不同厚度比 $\lambda = H_1/H = 0.7, 0.8, 0.9$ 时中截面上水平应力、剪切应力和垂直位移的分布情况. 从表3可以看出本文采用的弹性力学解与 ANSYS 解非常接近, 再次证明了本文方法的正确性和高精度.

表3 一端固支一端简支下表面抛物线内凹梁 ($H/L = 0.05, \lambda = H_1/H$) 截面 $x = L/2$ 处的应力和位移

$x = L/2$		σ_x/q		τ_{xy}/q		$vE/(qL)$	
λ	y/H_1	本文解	ANSYS 解	本文解	ANSYS 解	本文解	ANSYS 解
0.9	0.00	-176	-173	0.000	-0.230	626	622
	0.25	-87.9	-87.0	3.26	3.54	627	622
	0.50	1.07	0.000	4.36	4.57	626	622
	0.75	88.2	87.0	3.32	3.54	627	622
	1.00	175	173	0.000	0.23	627	622
0.8	0.00	-207	-203	0.000	-0.294	784	773
	0.25	-103	-102	3.93	4.24	784	774
	0.50	2.57	-0.060	5.25	5.47	784	774
	0.75	106	102	4.06	4.25	784	774
	1.00	207	203	0.000	0.295	784	773
0.7	0.00	-240	-242	0.000	-0.360	921	943
	0.25	-127	-121	4.15	4.20	922	944
	0.50	-1.16	-0.093	5.50	5.68	922	944
	0.75	122	121	4.18	4.21	922	944
	1.00	240	242	0.000	0.360	922	943

5 结 论

本文研究一端固支另一端简支连续变厚度梁在静力荷载作用下的弹性力学解, 通过引入

单位脉冲函数和 Dirac 函数, 将固支端等效为简支端加上未知的水平反力, 取位移函数使其直接满足左右两端简支的边界条件, 对上下表面的边界方程作 Fourier 正弦级数展开, 利用固支端位移为 0 的条件共同确定待定系数. 收敛性和比较研究证明了本文方法的正确性和高精度, 本文方法可直接应用于对应力分析要求较高的工程问题, 如航空航天工程和微型机械的设计等.

附录

式(29)中各元素的表达式为

$$A_{mn}^{(1)} = \frac{m\pi}{L} \int_0^L \left[-l_1(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \cosh \frac{m\mathcal{I}f_1(x)}{L} + m_1(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \sinh \frac{m\mathcal{I}f_1(x)}{L} \right] dx,$$

$$B_{mn}^{(1)} = \frac{m\pi}{L} \int_0^L \left[-l_1(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \sinh \frac{m\mathcal{I}f_1(x)}{L} + m_1(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \cosh \frac{m\mathcal{I}f_1(x)}{L} \right] dx,$$

$$C_{mn}^{(1)} = \int_0^L \left\{ -l_1(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \left[\frac{3+\mu}{1+\mu} \sinh \frac{m\mathcal{I}f_1(x)}{L} + \frac{m\mathcal{I}f_1(x)}{L} \cosh \frac{m\mathcal{I}f_1(x)}{L} \right] + m_1(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \left[\frac{2}{1+\mu} \cosh \frac{m\mathcal{I}f_1(x)}{L} + \frac{m\mathcal{I}f_1(x)}{L} \sinh \frac{m\mathcal{I}f_1(x)}{L} \right] \right\} dx,$$

$$D_{mn}^{(1)} = \int_0^L \left\{ -l_1(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \left[\frac{3+\mu}{1+\mu} \cosh \frac{m\mathcal{I}f_1(x)}{L} + \frac{m\mathcal{I}f_1(x)}{L} \sinh \frac{m\mathcal{I}f_1(x)}{L} \right] + m_1(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \left[\frac{2}{1+\mu} \sinh \frac{m\mathcal{I}f_1(x)}{L} + \frac{m\mathcal{I}f_1(x)}{L} \cosh \frac{m\mathcal{I}f_1(x)}{L} \right] \right\} dx,$$

$$R_{nk}^{(1)} = \sum_{m=1}^{\infty} \int_0^L \left\{ l_1(x) \frac{(m\mathcal{I}/L)[m^2\pi^2/L^2 + (\mu+2)k^2\pi^2/H^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} \cos \frac{k\mathcal{I}f_1(x)}{H} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} + m_1(x) \frac{(k\mathcal{I}/H)[k^2\pi^2/H^2 - \mu m^2\pi^2/L^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} \sin \frac{k\mathcal{I}f_1(x)}{H} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L} + \frac{H}{2k\pi} m_1(x) \sin \frac{k\mathcal{I}f_1(x)}{H} \sin \frac{n\pi x}{L} \right\} dx,$$

$$A_{mn}^{(2)} = \frac{m\pi}{L} \int_0^L \left[m_1(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \cosh \frac{m\mathcal{I}f_1(x)}{L} + l_1(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \sinh \frac{m\mathcal{I}f_1(x)}{L} \right] dx,$$

$$B_{mn}^{(2)} = \frac{m\pi}{L} \int_0^L \left[m_1(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \sinh \frac{m\mathcal{I}f_1(x)}{L} + l_1(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \cosh \frac{m\mathcal{I}f_1(x)}{L} \right] dx,$$

$$C_{mn}^{(2)} = \int_0^L \left\{ m_1(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \left[\frac{1-\mu}{1+\mu} \sinh \frac{m\mathcal{I}f_1(x)}{L} + \frac{m\mathcal{I}f_1(x)}{L} \cosh \frac{m\mathcal{I}f_1(x)}{L} \right] + l_1(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \left[\frac{2}{1+\mu} \cosh \frac{m\mathcal{I}f_1(x)}{L} + \frac{m\mathcal{I}f_1(x)}{L} \sinh \frac{m\mathcal{I}f_1(x)}{L} \right] \right\} dx,$$

$$D_{mn}^{(2)} = \int_0^L \left\{ m_1(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \left[\frac{1-\mu}{1+\mu} \cosh \frac{m\mathcal{I}f_1(x)}{L} + \frac{m\mathcal{I}f_1(x)}{L} \sinh \frac{m\mathcal{I}f_1(x)}{L} \right] + l_1(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \left[\frac{2}{1+\mu} \sinh \frac{m\mathcal{I}f_1(x)}{L} + \frac{m\mathcal{I}f_1(x)}{L} \cosh \frac{m\mathcal{I}f_1(x)}{L} \right] \right\} dx,$$

$$R_{nk}^{(2)} = \sum_{m=1}^{\infty} \int_0^L \left\{ m_1(x) \frac{(m\mathcal{I}/L)[\mu m^2\pi^2/L^2 - k^2\pi^2/H^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} \cos \frac{k\mathcal{I}f_1(x)}{H} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} + l_1(x) \frac{(k\mathcal{I}/H)[k^2\pi^2/H^2 - \mu m^2\pi^2/L^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} \sin \frac{k\mathcal{I}f_1(x)}{H} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L} + \frac{H}{2k\pi} l_1(x) \sin \frac{k\mathcal{I}f_1(x)}{H} \sin \frac{n\pi x}{L} \right\} dx,$$

$$A_{mn}^{(3)} = \frac{m\pi}{L} \int_0^L \left[-l_2(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \cosh \frac{m\mathcal{I}f_2(x)}{L} + m_2(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \sinh \frac{m\mathcal{I}f_2(x)}{L} \right] dx,$$

$$B_{mn}^{(3)} = \frac{m\pi}{L} \int_0^L \left[-l_2(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \sinh \frac{m\mathcal{I}f_2(x)}{L} + m_2(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \cosh \frac{m\mathcal{I}f_2(x)}{L} \right] dx,$$

$$C_{mn}^{(3)} = \int_0^L \left\{ -l_2(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \left[\frac{3+\mu}{1+\mu} \sinh \frac{m\mathcal{I}f_2(x)}{L} + \frac{m\mathcal{I}f_2(x)}{L} \cosh \frac{m\mathcal{I}f_2(x)}{L} \right] + m_2(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \left[\frac{2}{1+\mu} \cosh \frac{m\mathcal{I}f_2(x)}{L} + \frac{m\mathcal{I}f_2(x)}{L} \sinh \frac{m\mathcal{I}f_2(x)}{L} \right] \right\} dx,$$

$$\begin{aligned}
 & m_2(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \left[\frac{2}{1+\mu} \cosh \frac{m\mathcal{J}f_2(x)}{L} + \frac{m\mathcal{J}f_2(x)}{L} \sinh \frac{m\mathcal{J}f_2(x)}{L} \right] \Bigg\} dx, \\
 D_{mn}^{(3)} = & \int_0^L \left\{ -l_2(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \left[\frac{3+\mu}{1+\mu} \cosh \frac{m\mathcal{J}f_2(x)}{L} + \frac{m\mathcal{J}f_2(x)}{L} \sinh \frac{m\mathcal{J}f_2(x)}{L} \right] + \right. \\
 & \left. m_2(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \left[\frac{2}{1+\mu} \sinh \frac{m\mathcal{J}f_2(x)}{L} + \frac{m\mathcal{J}f_2(x)}{L} \cosh \frac{m\mathcal{J}f_2(x)}{L} \right] \right\} dx, \\
 R_{nk}^{(3)} = & \sum_{m=1}^N \int_0^L \left\{ l_2(x) \frac{(m\mathcal{J}L/L) [m^2\pi^2/L^2 + (\mu+2)k^2\pi^2/H^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} \cos \frac{k\mathcal{J}f_2(x)}{H} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} + \right. \\
 & m_2(x) \frac{(k\mathcal{J}H) [k^2\pi^2/H^2 - \mu m^2\pi^2/L^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} \sin \frac{k\mathcal{J}f_2(x)}{H} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L} + \\
 & \left. \frac{H}{2k\pi} m_2(x) \sin \frac{k\mathcal{J}f_2(x)}{H} \sin \frac{n\pi x}{L} \right\} dx, \\
 A_{mn}^{(4)} = & \frac{m\pi}{L} \int_0^L \left[m_2(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \cosh \frac{m\mathcal{J}f_2(x)}{L} + l_2(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \sinh \frac{m\mathcal{J}f_2(x)}{L} \right] dx, \\
 B_{mn}^{(4)} = & \frac{m\pi}{L} \int_0^L \left[m_2(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \sinh \frac{m\mathcal{J}f_2(x)}{L} + l_2(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \cosh \frac{m\mathcal{J}f_2(x)}{L} \right] dx, \\
 C_{mn}^{(4)} = & \int_0^L \left\{ m_2(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \left[\frac{1-\mu}{1+\mu} \sinh \frac{m\mathcal{J}f_2(x)}{L} + \frac{m\mathcal{J}f_2(x)}{L} \cosh \frac{m\mathcal{J}f_2(x)}{L} \right] + \right. \\
 & \left. l_2(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \left[\frac{2}{1+\mu} \cosh \frac{m\mathcal{J}f_2(x)}{L} + \frac{m\mathcal{J}f_2(x)}{L} \sinh \frac{m\mathcal{J}f_2(x)}{L} \right] \right\} dx, \\
 D_{mn}^{(4)} = & \int_0^L \left\{ m_2(x) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \left[\frac{1-\mu}{1+\mu} \cosh \frac{m\mathcal{J}f_2(x)}{L} + \frac{m\mathcal{J}f_2(x)}{L} \sinh \frac{m\mathcal{J}f_2(x)}{L} \right] + \right. \\
 & \left. l_2(x) \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \left[\frac{2}{1+\mu} \sinh \frac{m\mathcal{J}f_2(x)}{L} + \frac{m\mathcal{J}f_2(x)}{L} \cosh \frac{m\mathcal{J}f_2(x)}{L} \right] \right\} dx, \\
 R_{nk}^{(4)} = & \sum_{m=1}^N \int_0^L \left\{ m_2(x) \frac{(m\mathcal{J}L/L) [\mu m^2\pi^2/L^2 - k^2\pi^2/H^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} \cos \frac{k\mathcal{J}f_2(x)}{H} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} + \right. \\
 & l_2(x) \frac{(k\mathcal{J}H) [k^2\pi^2/H^2 - \mu m^2\pi^2/L^2]}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} \sin \frac{k\mathcal{J}f_2(x)}{H} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L} + \\
 & \left. \frac{m\mathcal{J}}{L} l_2(x) \sin \frac{k\mathcal{J}f_2(x)}{H} \sin \frac{n\pi x}{L} \right\} dx, \\
 A_{kn}^{(5)} = & \cosh \left(\frac{n\pi}{L} \frac{kH}{M+1} \right), \quad B_{kn}^{(5)} = \sinh \left(\frac{n\pi}{L} \frac{kH}{M+1} \right), \\
 C_{kn}^{(5)} = & \frac{3-\mu}{1+\mu} \frac{L}{m\pi} \sinh \left(\frac{n\pi}{L} \frac{kH}{M+1} \right) + \frac{kH}{M+1} \cosh \left(\frac{n\pi}{L} \frac{kH}{M+1} \right), \\
 D_{kn}^{(5)} = & \frac{3-\mu}{1+\mu} \frac{L}{m\pi} \cosh \left(\frac{n\pi}{L} \frac{kH}{M+1} \right) + \frac{kH}{M+1} \sinh \left(\frac{n\pi}{L} \frac{kH}{M+1} \right), \\
 R_{sk}^{(5)} = & \sum_{m=1}^N \left[\frac{(\mu-1)m^2\pi^2/L^2 - 2k^2\pi^2/H^2}{(k^2\pi^2/H^2 + m^2\pi^2/L^2)^2} \cos \left(\frac{sk}{M+1} \pi \right) - \frac{H^2}{k^2\pi^2} \cos \left(\frac{sk}{M+1} \pi \right) \right], \\
 S_n^{(1)} = & \int_0^L \sin \frac{n\pi x}{L} q_1(x) dx, \quad S_n^{(2)} = \int_0^L \sin \frac{n\pi x}{L} q_2(x) dx, \\
 S_n^{(3)} = & \int_0^L \sin \frac{n\pi x}{L} q_3(x) dx, \quad S_n^{(4)} = \int_0^L \sin \frac{n\pi x}{L} q_4(x) dx, \quad S_n^{(5)} = 0, \\
 & (m = 1, 2, \dots, N; n = 1, 2, \dots, N; s = 1, 2, \dots, M; k = 1, 2, \dots, M).
 \end{aligned}$$

[参 考 文 献]

[1] Timoshenko S P, Goodier J C. Theory of Elasticity [M]. New York: McGraw-Hill, 1970.
 [2] 黄德进, 丁浩江, 王惠明. 均布荷载作用下正交各向异性固支梁的解析解[J]. 浙江大学学报(工学版), 2006, 40(3): 511-514.
 [3] Hashin Z. Plane anisotropic beams[J]. Journal of Applied Mechanics, 1967, 34(1): 257-263.
 [4] 梅甫良, 曾德顺. 深梁的精确解[J]. 力学与实践, 2002, 24(3): 58-60.

- [5] 梅甫良. 两端固支深梁弯曲问题的解析解[J]. 强度与环境, 2003, 30(3): 23-28.
- [6] 江爱民, 邱洪林, 林定远. 均布荷载作用下简支磁电弹性梁的解析解[J]. 浙江工业大学学报, 2004, 32(2): 239-244.
- [7] 江爱民, 林定远, 邱洪林. 均布荷载作用下悬臂磁电弹性梁的解析解[J]. 应用力学学报, 2004, 21(4): 106-109.
- [8] 林启荣, 刘正兴, 金占礼. 均布荷载作用下的两端简支压电梁的解析解[J]. 应用数学和力学, 2000, 21(6): 617-623.
- [9] Ding H J, Huang D J, Chen W Q. Elasticity solutions for plane anisotropic functionally graded beams[J]. International Journal of Solids and Structures, 2007, 44(1): 176-196.
- [10] 刘庆潭. 含楔形变截面梁静力分析的传递矩阵法求解[J]. 力学与实践, 1993, 15(1): 64-66.

Elasticity Solution of Clamped-Simply Supported Beams With Variable Thickness

XU Ye-peng¹, ZHOU Ding², Y. K. Cheung³

(1. School of Science, Nanjing University of Science and Technology,
Nanjing 210094, P. R. China ;

2. College of Civil Engineering, Nanjing University of Technology,
Nanjing 210009, P. R. China ;

3. Department of Civil Engineering, Faculty of Engineering,
The University of Hong Kong, Hong Kong, P. R. China)

Abstract: The stress and displacement distributions of continuously varying thickness beams with one end clamped and the other end simply supported under static loads are studied. By introducing the unit pulse functions and Dirac functions, the clamped edge can be made equivalent to the simply supported one by adding the unknown horizontal reactions. According to the governing equations of plane stress problem, the general expressions of displacements, which satisfy the governing differential equations and the boundary conditions at two ends of the beam, can be deduced. The unknown coefficients in the general expressions were then determined by using the Fourier sinusoidal series expansion along the upper and lower boundaries of the beams and using the condition of zero displacements at the clamped edge. The solution obtained has excellent convergence property. The numerical results being compared with those obtained from the commercial software ANSYS, excellent accuracy of the present method is demonstrated.

Key words: beam; clamped edge; variable thickness; Fourier expansion; elasticity solution