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五阶双色双向海洋表面波理论*

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摘要: 将经典的"纯波动的三阶单色单向 Stokes 波理论"提升至"可包含环境均匀流效应的有限水深五阶双色双向海洋表面波理论".亦即,在已有三阶双色双波理论的基础上得到了自由表面位移、速度势和非线性振幅色散关系的第四阶、第五阶显式表达式.从中,将居于核心地位的"双色双向波的第五阶非线性振幅色散关系"又推广到"无穷多波中任意两两不同频率不同振幅相互作用波的非线性振幅色散关系".针对双色双向短峰波的典型特性,以若干个图表详加示之.

关 键 词: 五阶理论; 双色双向波; 海洋表面波; 非线性振幅色散关系

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引 言

面对"有限水深抑或无穷水深"的海洋表面波的窄带非线性波动场,近两百年来以 Stokes 波[1]及其发展理论[2-5]为代表.以此,即可对接基本的海洋宽带波湍流模式[4,6-8]以至于整个自然界的波湍流理论[9-11]——这就与普适的 Hamilton 力学[12]紧紧相连了.无疑,Stokes 波在广阔的海洋工程实践中发挥着不可或缺的基础性作用[4,13].

究其经典的 Stokes 波理论[1],却也显现出其本身固有的诸多局限性:"单调低阶"的"单色单向三阶"——这就无法描述一幅实际的物理海洋图景.若是从"阶数"上看,可将"三阶"提升至"五阶"[14],但是"单色单向"未变.若是以"频率和方向"而论,可将 Stokes 波扩展成"二阶双色双向波"[15]或"二阶单向或多向波"[16]或"二阶单色双向短峰波[17]",抑或是"三阶单色双向短峰波"[18-20]甚或更高阶[21],再或是"三阶双色双向深水波"[22-23],以及"三阶三色共线性深水波"[24],那就渐渐显现了一种波浪场格局;如果将 Stokes 波直接推广至"三阶双色双向有限水深波"[25]乃至于"三阶三色三向有限水深波"[26],那就展示了一派波浪场态势;倘若再将其推进到目前尚未达至的状态——"五阶双色双向有限水深波",那就较为真实地刻画出一个大海洋面上波-波相互作用的物理本来面貌和属性.

基于此,本文特为之.

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1 控制方程组、Taylor 级数展开和 Lindstedt-Poincaré 法

假定存在一个无黏、不可压缩的无旋波浪运动,可包含两列不同频率不同方向的双色双向波动：m 波和 n 波,并伴随一个环境均匀水平流动.据此,可建立一个直角坐标系统：x 轴和 y 轴位于平均水平面(the mean water plane,可简称为 MWP)上,z 轴垂直向上.于是,流体运动区域可由海底常水深 $z=-h$ 和自由表面位移 $z=\zeta(x,y,t)\equiv\zeta(\boldsymbol{x},t)$ 所界定,由无旋流动可引入一个速度势函数为 $\boldsymbol{\Phi}(\boldsymbol{x},z,t)$,垂直于 z 轴的均匀流动矢量为 $\boldsymbol{U}$,其中,t 为时间.由此,整个波-流相互作用系统的控制方程组可表示为

$$\nabla^2\boldsymbol{\Phi}+\frac{\partial^2\boldsymbol{\Phi}}{\partial z^2}=0,\qquad -h\leqslant z\leqslant\zeta(\boldsymbol{x},t),\tag{1}$$

$$\frac{\partial\zeta}{\partial t}-\frac{\partial\boldsymbol{\Phi}}{\partial z}+\nabla\boldsymbol{\Phi}\cdot\nabla\zeta=0,\qquad z=\zeta(\boldsymbol{x},t),\tag{2}$$

$$\frac{\partial\boldsymbol{\Phi}}{\partial t}+g\zeta+\frac{1}{2}\left[(\nabla\boldsymbol{\Phi})^2+\left(\frac{\partial\boldsymbol{\Phi}}{\partial z}\right)^2\right]=C,\qquad z=\zeta(\boldsymbol{x},t),\tag{3}$$

$$\frac{\partial\boldsymbol{\Phi}}{\partial z}=0,\qquad z=-h.\tag{4}$$

其中,$\nabla=\partial/\partial\boldsymbol{x}$,$g$ 为重力加速度,C 为保证坐标系原点总是位于 MWP 上而设置的一个总的常数,以此使得该方程在随后进行的摄动展开和 Taylor 展开中出现的各阶常数,均统统匹配于该总常数相应分解的各阶常数,最终相互抵消,以不失于一般性.

为求解该非线性方程组,可对其实施奇异摄动理论中的 Lindstedt-Poincaré 法,以消除高阶长期项：

$$\zeta(\boldsymbol{x},t)=\varepsilon\zeta^{(1)}+\varepsilon^2\zeta^{(2)}+\varepsilon^3\zeta^{(3)}+\varepsilon^4\zeta^{(4)}+\varepsilon^5\zeta^{(5)}+\cdots,\tag{5}$$

$$\boldsymbol{\Phi}(\boldsymbol{x},z,t)=\boldsymbol{U}\cdot\boldsymbol{x}+\varepsilon\boldsymbol{\Phi}^{(1)}+\varepsilon^2\boldsymbol{\Phi}^{(2)}+\varepsilon^3\boldsymbol{\Phi}^{(3)}+\varepsilon^4\boldsymbol{\Phi}^{(4)}+\varepsilon^5\boldsymbol{\Phi}^{(5)}+\cdots,\tag{6}$$

$$\boldsymbol{\tau}=t[1+\varepsilon\boldsymbol{\omega}^{(1)}+\varepsilon^2\boldsymbol{\omega}^{(2)}+\cdots],\ \boldsymbol{\omega}=\boldsymbol{\omega}^{(0)}[1+\varepsilon\boldsymbol{\omega}^{(1)}+\varepsilon^2\boldsymbol{\omega}^{(2)}+\cdots].\tag{7}$$

其中,ε 为引入的名义摄动参数.一旦获得某阶解后,可令 $\varepsilon=1$.

对自由表面速度势在 $z=0$ 处进行 Taylor 级数展开,可将方程(2)和(3)的未知自由表面处化为已知.据此,可得到各阶控制方程组：

$$\nabla^2\boldsymbol{\Phi}^{(i)}+\frac{\partial^2\boldsymbol{\Phi}^{(i)}}{\partial z^2}=0,\qquad -h\leqslant z\leqslant 0,\tag{8}$$

$$\frac{\mathrm{D}\zeta^{(i)}}{\mathrm{D}\boldsymbol{\tau}}-\frac{\partial\boldsymbol{\Phi}^{(i)}}{\partial z}=E^{(i)},\qquad z=0,\tag{9}$$

$$\frac{\mathrm{D}\boldsymbol{\Phi}^{(i)}}{\mathrm{D}\boldsymbol{\tau}}+g\zeta^{(i)}=C^{(i)}+F^{(i)},\qquad z=0,\tag{10}$$

$$\frac{\partial\boldsymbol{\Phi}^{(i)}}{\partial z}=0,\qquad z=-h.\tag{11}$$

其中,$i=1,2,\cdots,5$,$\mathrm{D}/\mathrm{D}\boldsymbol{\tau}=\partial/\partial\boldsymbol{\tau}+\boldsymbol{U}\cdot\nabla$,$C^{(0)}=\boldsymbol{U}^2/2$,$C^{(1)}=0$,$E^{(1)}=F^{(1)}=0$.限于篇幅,$E^{(i)}$ 和 $F^{(i)}(i=2,3,4,5)$ 的表达式不再详述.

2 五 阶 解

求解上述各阶方程组,即可依次得到各阶解.其中,第一、二、三阶解如同文献[25-26]之所

述,不再给出.

2.1 第四阶解

$$\begin{aligned}\zeta^{(4)}=&[G_{4n}(A_{4n}\cos\theta_{4n}+B_{4n}\sin\theta_{4n})+G_{2n}^{(4)}(A_{2n}\cos\theta_{2n}+B_{2n}\sin\theta_{2n})]+(n\to m)+\\&G_{n\pm 3m}(A_{n\pm 3m}\cos\theta_{n\pm 3m}+B_{n\pm 3m}\sin\theta_{n\pm 3m})+(n\leftrightarrow m)+\\&G_{2n\pm 2m}(A_{2n\pm 2m}\cos\theta_{2n\pm 2m}+B_{2n\pm 2m}\sin\theta_{2n\pm 2m})+\\&G_{n\pm m}^{(4)}(A_{n\pm m}\cos\theta_{n\pm m}+B_{n\pm m}\sin\theta_{n\pm m}),\end{aligned}\tag{12}$$

$$\begin{aligned}\Phi^{(4)}=&[F_{4n}(A_{4n}\sin\theta_{4n}-B_{4n}\cos\theta_{4n})\cosh(\kappa_{4n}Z)+\\&F_{2n}^{(4)}(A_{2n}\sin\theta_{2n}-B_{2n}\cos\theta_{2n})\cosh(\kappa_{2n}Z)]+(n\to m)+\\&F_{n\pm 3m}(A_{n\pm 3m}\sin\theta_{n\pm 3m}-B_{n\pm 3m}\cos\theta_{n\pm 3m})\cosh(\kappa_{n\pm 3m}Z)+(n\leftrightarrow m)+\\&F_{2n\pm 2m}(A_{2n\pm 2m}\sin\theta_{2n\pm 2m}-B_{2n\pm 2m}\cos\theta_{2n\pm 2m})\cosh(\kappa_{2n\pm 2m}Z)+\\&F_{n\pm m}^{(4)}(A_{n\pm m}\sin\theta_{n\pm m}-B_{n\pm m}\cos\theta_{n\pm m})\cosh(\kappa_{n\pm m}Z),\end{aligned}\tag{13}$$

$$\begin{aligned}\omega_n=&\omega_n^{(0)}[1+\varepsilon\omega_n^{(1)}+\varepsilon^2\omega_n^{(2)}+\varepsilon^3\omega_n^{(3)}]=\\&\omega_n^{(0)}[1+\varepsilon^2\omega_n^{(2)}]\Rightarrow\omega_n^{(0)}+\varepsilon^2\omega_n^{(2)}=\\&\boldsymbol{k}_n\cdot\boldsymbol{U}+\omega_{1n}\{1+\varepsilon^2[c_n^2\kappa_n^2\Omega_{nn}^{(3)}+c_m^2\kappa_m^2\Omega_{nm}^{(3)}]\},\ (n\leftrightarrow m).\end{aligned}\tag{14}$$

其中,下标 n 和 m 即分别表征 n 波和 m 波;“$\Rightarrow$” 表示以其右端的 $\omega_n^{(2)}$ 取代左端的 $\omega_n^{(0)}\omega_n^{(2)}$,但 $\omega_n^{(0)}$ 保持不变,以此与文献[25-26] 的规定保持一致.在第五阶中存在相应的表示.除非特别明示,后续的 $\omega_n^{(2)}$ 皆指右端的 $\omega_n^{(2)}$.$(n\to m)$ 表示以 m 取代“前项” 或“前面一个或多个方程” 中的 n 而得到的项或方程;$(n\leftrightarrow m)$ 则表示在“前项” 或“前面一个或多个方程” 中进行“n 与 m 的互换” 而得到的项或方程;$G_{2n}^{(4)}$ 表示该项属于第四阶,以有别于 G_{2n},其余以及后续的类似表达式可作相应的解释.在某一完整独立项中出现的若干上下层符号“±,∓”,可以下例表示其意:

$$\pm(F_{n\pm m}\mp F_{n\mp m}+G_{n\pm m})=(F_{n+m}-F_{n-m}+G_{n+m})-(F_{n-m}+F_{n+m}+G_{n-m}).$$

须注意:第二阶的系数 $A_{n\pm m}$ 和 $B_{n\pm m}$,以及第三阶色散关系中的 $\Omega_{nn}^{(3)}$ 和 $\Omega_{nm}^{(3)}$ 可参见文献[25-26].同样,由于篇幅所限,各个传递函数 G,F 和其它符号的详细表达式,以及其第五阶相应的传递函数和其它符号的详细表达式不再赘述.另外,

$$\begin{cases}\theta_n=\omega_n^{(0)}\tau-\boldsymbol{k}_n\cdot\boldsymbol{x}=\omega_n t-\boldsymbol{k}_n\cdot\boldsymbol{x},\ (n\to m),\\\omega_n^{(0)}=\boldsymbol{k}_n\cdot\boldsymbol{U}+\sqrt{gk_n\tanh(\kappa_n h)}\equiv\boldsymbol{k}_n\cdot\boldsymbol{U}+\omega_{1n},\ (n\to m),\\\kappa_n=|\boldsymbol{k}_n|=\sqrt{k_{nx}^2+k_{ny}^2},\ (n\to m),\\Z=z+h,\end{cases}\tag{15}$$

$$\begin{cases}\theta_{2n}=2\theta_n,\ \theta_{4n}=4\theta_n;\ (n\to m);\theta_{n\pm 3m}=\theta_n\pm 3\theta_m,\ (n\leftrightarrow m),\\\theta_{2n\pm 2m}=2\theta_n\pm 2\theta_m,\ \theta_{n\pm m}=\theta_n\pm\theta_m,\end{cases}\tag{16}$$

$$\begin{cases}\kappa_{2n}=|2\boldsymbol{k}_n|=2\kappa_n,\ \kappa_{4n}=|4\boldsymbol{k}_n|=4\kappa_n;\ (n\to m),\\\kappa_{n\pm 3m}=|\boldsymbol{k}_n\pm 3\boldsymbol{k}_m|,\ (n\leftrightarrow m);\ \kappa_{2n\pm 2m}=|2\boldsymbol{k}_n\pm 2\boldsymbol{k}_m|,\ \kappa_{n\pm m}=|\boldsymbol{k}_n\pm\boldsymbol{k}_m|,\end{cases}\tag{17}$$

$$A_{4n}=\frac{1}{h}(A_{2n}^2-B_{2n}^2),\ B_{4n}=\frac{2}{h}A_{2n}B_{2n};\ (n\to m),\tag{18}$$

$$A_{n\pm 3m}=\frac{A_{2m}A_{n\pm m}\mp B_{2m}B_{n\pm m}}{h},\ B_{n\pm 3m}=\frac{A_{2m}B_{n\pm m}\pm B_{2m}A_{n\pm m}}{h};\ (n\leftrightarrow m),\tag{19}$$

$$A_{2n\pm 2m}=\frac{A_{n\pm m}^2-B_{n\pm m}^2}{2h},\ B_{2n\pm 2m}=\frac{A_{n\pm m}B_{n\pm m}}{h},\tag{20}$$

$$c_n^2 = a_n^2 + b_n^2, \ (n \to m). \tag{21}$$

其中, a_n 和 b_n 为自由表面位移的第一阶振幅,式(18)、(19)右端的各个量 A 和 B, 皆为第二阶系数[25-26].

2.2 第五阶解

$$\begin{aligned}\zeta^{(5)} = &[G_{5n}(A_{5n}\cos\theta_{5n} + B_{5n}\sin\theta_{5n}) + G_{3n}^{(5)}(A_{3n}\cos\theta_{3n} + B_{3n}\sin\theta_{3n})] + (n \to m) + \\ &[G_{n\pm4m}(A_{n\pm4m}\cos\theta_{n\pm4m} + B_{n\pm4m}\sin\theta_{n\pm4m}) + \\ &G_{2n\pm3m}(A_{2n\pm3m}\cos\theta_{2n\pm3m} + B_{2n\pm3m}\sin\theta_{2n\pm3m}) + \\ &G_{n\pm2m}^{(5)}(A_{n\pm2m}\cos\theta_{n\pm2m} + B_{n\pm2m}\sin\theta_{n\pm2m})] + (n \leftrightarrow m),\end{aligned} \tag{22}$$

$$\begin{aligned}\Phi^{(5)} = &[F_{5n}(A_{5n}\sin\theta_{5n} - B_{5n}\cos\theta_{5n})\cosh(\kappa_{5n}Z) + \\ &F_n^{(5)}(a_n\sin\theta_n - b_n\cos\theta_n)\cosh(\kappa_n Z) + \\ &F_{3n}^{(5)}(A_{3n}\sin\theta_{3n} - B_{3n}\cos\theta_{3n})\cosh(\kappa_{3n}Z)] + (n \to m) + \\ &[F_{n\pm4m}(A_{n\pm4m}\sin\theta_{n\pm4m} - B_{n\pm4m}\cos\theta_{n\pm4m})\cosh(\kappa_{n\pm4m}Z) + \\ &F_{2n\pm3m}(A_{2n\pm3m}\sin\theta_{2n\pm3m} - B_{2n\pm3m}\cos\theta_{2n\pm3m})\cosh(\kappa_{2n\pm3m}Z) + \\ &F_{n\pm2m}^{(5)}(A_{n\pm2m}\sin\theta_{n\pm2m} - B_{n\pm2m}\cos\theta_{n\pm2m})\cosh(\kappa_{n\pm2m}Z)] + (n \leftrightarrow m),\end{aligned} \tag{23}$$

$$\begin{aligned}\omega_n = \omega_n^{(0)}[1 + \varepsilon^2\omega_n^{(2)} + \varepsilon^4\omega_n^{(4)}] \Rightarrow \omega_n^{(0)} + \varepsilon^2\omega_n^{(2)} + \varepsilon^4\omega_n^{(4)} = \\ \boldsymbol{k}_n \cdot \boldsymbol{U} + \omega_{1n}\{1 + \varepsilon^2[c_n^2\kappa_n^2\Omega_{nn}^{(3)} + c_m^2\kappa_m^2\Omega_{nm}^{(3)}] + \\ \varepsilon^4[c_n^4\kappa_n^4\Omega_{nn}^{(5)} + c_m^4\kappa_m^4\Omega_{mm}^{(5)} + c_n^2c_m^2\Omega_{nm}^{(5)}]\}, (n \leftrightarrow m).\end{aligned} \tag{24}$$

其中, A_{3n} 和 B_{3n}, 以及 $A_{n\pm2m}$ 和 $B_{n\pm2m}$, 皆为第三阶系数[25-26].并且

$$\begin{cases}\theta_{5n} = 5\theta_n, \theta_{3n} = 3\theta_n; (n \to m), \\ \theta_{n\pm4m} = \theta_n \pm 4\theta_m, \ \theta_{2n\pm3m} = 2\theta_n \pm 3\theta_m, \ \theta_{n\pm2m} = \theta_n \pm 2\theta_m; (n \leftrightarrow m),\end{cases} \tag{25}$$

$$\begin{cases}\kappa_{5n} = |5\boldsymbol{k}_n| = 5\kappa_n, \ \kappa_{3n} = |3\boldsymbol{k}_n| = 3\kappa_n; (n \to m), \\ \kappa_{n\pm4m} = |\boldsymbol{k}_n \pm 4\boldsymbol{k}_m|, \ \kappa_{2n\pm3m} = |2\boldsymbol{k}_n \pm 3\boldsymbol{k}_m|, \ \kappa_{n\pm2m} = |\boldsymbol{k}_n \pm 2\boldsymbol{k}_m|; (n \leftrightarrow m),\end{cases} \tag{26}$$

$$A_{5n} = \frac{a_n(a_n^4 - 10a_n^2b_n^2 + 5b_n^4)}{4h^4}, \ B_{5n} = \frac{b_n(5a_n^4 - 10a_n^2b_n^2 + b_n^4)}{4h^4}; \ (n \to m), \tag{27}$$

$$A_{n\pm4m} = \frac{a_n(a_m^4 - 6a_m^2b_m^2 + b_m^4) \pm 4b_na_mb_m(b_m^2 - a_m^2)}{4h^4}, \ (n \leftrightarrow m), \tag{28}$$

$$B_{n\pm4m} = \frac{b_n(a_m^4 - 6a_m^2b_m^2 + b_m^4) \pm 4a_na_mb_m(a_m^2 - b_m^2)}{4h^4}, \ (n \leftrightarrow m), \tag{29}$$

$$A_{2n\pm3m} = \frac{a_m(a_n^4 - 6a_n^2b_n^2 + b_n^4) \pm 4b_ma_nb_n(b_n^2 - a_n^2)}{4h^4}, \ (n \leftrightarrow m), \tag{30}$$

$$B_{2n\pm3m} = \frac{b_m(a_n^4 - 6a_n^2b_n^2 + b_n^4) \pm 4a_ma_nb_n(a_n^2 - b_n^2)}{4h^4}, \ (n \leftrightarrow m), \tag{31}$$

$$\begin{aligned}\Omega_{nn}^{(5)} = &\frac{1}{8\,192[2 + 3\cosh(\kappa_{2n}h)]\sinh^{10}(\kappa_nh)}[13\,704 + \\ &20\,188\cosh(\kappa_{2n}h) + 8\,713\cosh(\kappa_{4n}h) + 1\,464\cosh(\kappa_{6n}h) + \\ &512\cosh(\kappa_{8n}h) + 116\cosh(\kappa_{8n}h) + 15\cosh(\kappa_{12n}h)],\end{aligned} \tag{32}$$

$$\Omega_{mm}^{(5)} = \frac{\cosh(\kappa_{4m}h) + 24\cosh(\kappa_{2m}h) - 13}{256\sinh^4(\kappa_mh)}, \tag{33}$$

$$\Omega_{nm}^{(5)}=\frac{(\boldsymbol{k}_n\cdot\boldsymbol{k}_m)(G_{n+m}+G_{n-m})}{8h^2\omega_{1m}}\left\{\frac{gh}{\omega_{1n}\omega_{1m}}[\omega_{1n}(\kappa_m^2-\kappa_n^2)-\omega_{1m}(\boldsymbol{k}_n\cdot\boldsymbol{k}_m)]-\right.$$
$$[\kappa_{n+m}F_{n+m}\sinh(\kappa_{n+m}h)-\kappa_{n-m}F_{n-m}\sinh(\kappa_{n-m}h)]+$$
$$\frac{g}{\omega_{1m}^2\omega_{1n}}\left[\left(g\boldsymbol{k}_n\cdot\boldsymbol{k}_m+\frac{\omega_{1n}^3\omega_{1m}}{g}\right)(G_{n+m}+G_{n-m})+\right.$$
$$\left.\left.F_{n\pm m}(\omega_{n\pm m}(\boldsymbol{k}_n\cdot\boldsymbol{k}_m\pm\kappa_n^2)\mp\omega_{1n}\kappa_{n\pm m}^2)\cosh(\kappa_{n\pm m}h)\right]\right\}+$$
$$\frac{1}{32h^2\omega_{1n}}\{16\kappa_n^3F_{2n}(G_{n+m}+G_{n-m})\sinh(\kappa_{2n}h)+$$
$$\kappa_n^4hF_n[2G_{2n}+12(G_{n+m}+G_{n-m})+\kappa_nh]\cosh(\kappa_nh)-$$
$$12\kappa_n^2\omega_{1n}[4(G_{n+m}^2+G_{n+m}G_{n-m}+G_{n-m}^2)+2(G_{m+2n}+G_{m-2n})+\kappa_n^2h^2]+$$
$$2\kappa_m^2hF_m(\boldsymbol{k}_n\cdot\boldsymbol{k}_m)(2G_{2n}+3G_{n+m}+3G_{n-m})\cosh(\kappa_mh)-$$
$$2\omega_{1m}(\boldsymbol{k}_n\cdot\boldsymbol{k}_m)[2G_{2n}(G_{n+m}+G_{n-m})+2(G_{m+2n}+G_{m-2n})+h^2\kappa_m^2]\}+$$
$$\frac{1}{16h^3\omega_{1n}}\{[4(2\kappa_{n\pm m}^2-2\kappa_m^2\mp3\boldsymbol{k}_n\cdot\boldsymbol{k}_m)G_{n\pm m}F_{m\pm2n}\cosh(\kappa_{m\pm2n}h)\pm$$
$$2\kappa_{m\pm2n}h(2\kappa_{n\pm m}^2-2\kappa_m^2\mp3\boldsymbol{k}_n\cdot\boldsymbol{k}_m)F_{m\pm2n}\sinh(\kappa_{m\pm2n}h)]+$$
$$2\kappa_{n\pm m}hF_{n\pm m}[(\kappa_{n\mp m}^2-\kappa_m^2\pm3\boldsymbol{k}_n\cdot\boldsymbol{k}_m)G_{2n}+$$
$$4(\kappa_n^2\pm\boldsymbol{k}_n\cdot\boldsymbol{k}_m)G_{n\pm m}+2(\kappa_{n\pm m}^2-\kappa_m^2\mp\boldsymbol{k}_n\cdot\boldsymbol{k}_m)G_{n\mp m}]\sinh(\kappa_{n\pm m}h)+$$
$$F_{n\pm m}[\kappa_{n\pm m}^2h^2(\kappa_{n\pm m}^2+2\kappa_n^2-\kappa_m^2\pm\boldsymbol{k}_n\cdot\boldsymbol{k}_m)+$$
$$4(\kappa_{n\mp m}^2-\kappa_m^2\pm3\boldsymbol{k}_n\cdot\boldsymbol{k}_m)G_{m\pm2n}]\cosh(\kappa_{n\pm m}h)\}. \qquad (34)$$

如前所述,各个传递函数 G 和 F 以及其它符号的详细表达式不再陈述.

由上述第四、五阶解,再加之先前得到的第一、二、三阶解[25-26],即构成双色双向海洋表面波有限水深的五阶解.对其取无限水深极限,即可得相应的深水波五阶解.至于相应的浅水波五阶解,那将有赖于由 3 个水波长度(波幅、波长、水深)组合成的无量纲 U_r 参数的不同而变得丰富多样,这将大大不同于传统的 Airy 理论、Boussinesq 理论[27]及其扩展的高阶情形[4,28],只能以后再和盘托出.

3 非线性振幅色散关系的拓展

水波属于色散波,其核心就在于非线性振幅色散关系.本文的五阶色散关系即由第一阶线性色散关系和随后的第二、三、四、五阶非线性振幅色散关系构成.从中不难发现:第一、三、五阶色散关系存在,第二、四阶色散关系却是不存在,或均归于 0.这是否可推而广之而成为一条惯例或法则呢? 即曰:海洋表面波的奇数阶色散关系存在,但偶数阶色散关系不存在.

如果关注第三、五阶色散关系式(34),体察其分门别类的构造

$$\begin{cases}\omega_n^{(2)}=\omega_{1n}[c_n^2\kappa_n^2\Omega_{nn}^{(3)}+c_m^2\kappa_m^2\Omega_{nm}^{(3)}],\\ \omega_n^{(4)}=\omega_{1n}[c_n^4\kappa_n^4\Omega_{nn}^{(5)}+c_m^4\kappa_m^4\Omega_{mm}^{(5)}+c_n^2c_m^2\Omega_{nm}^{(5)}].\end{cases} \qquad (35)$$

便不难发现:1) 第三阶色散关系 $\omega_n^{(2)}$ 由两项组成,第 1 项的核心乃 $\Omega_{nn}^{(3)}$,表示 n 波的自我相互作用;第 2 项的核心乃 $\Omega_{nm}^{(3)}$,表示 n 波与 m 波的相互作用.那么,n 波与第 3 种波乃至其它千千万万的波又何尝不是呢[25-26]? 2) 第五阶色散关系 $\omega_n^{(4)}$ 由 3 项组成,前两项分别以 $\Omega_{nn}^{(5)}$ 和 $\Omega_{mm}^{(5)}$ 为

核心,自是表示它们各个的自我相互作用.由于均要乘一个因子ω_{1n},便使得第2项$\omega_{1n}\Omega_{mm}^{(5)}$涉及其它波动.至于第3项的核心$\Omega_{nm}^{(5)}$,即告知牵扯到两种波,那就可推想到许许多多波相互作用了.基于此,可将五阶双色双向海洋表面波的非线性振幅色散关系推广至无穷多个波:

$$\omega_n = \boldsymbol{k}_n \cdot \boldsymbol{U} + \omega_{1n}\left\{1 + \varepsilon^2\left[c_n^2\kappa_n^2\Omega_{nn}^{(3)} + \sum_{\substack{m=1\\ m\neq n}}^{\infty} c_m^2\kappa_m^2\Omega_{nm}^{(3)}\right] + \varepsilon^4\left[c_n^4\kappa_n^4\Omega_{nn}^{(5)} + \sum_{\substack{m=1\\ m\neq n}}^{\infty} c_m^4\kappa_m^4\Omega_{mm}^{(5)} + \sum_{\substack{m=1\\ m\neq n}}^{\infty} c_n^2 c_m^2\Omega_{nm}^{(5)}\right]\right\},\ (n\leftrightarrow m). \tag{36}$$

4　图 表 示 例

本文所建理论自成一个由第一阶直至第五阶的完整体系,关乎“一个变量一种阶数一组数据”的精确到位和走向趋势,可以图表明确示之,作为将来的检验凭据.为此,采用一组典型的双色短峰波数据[25]:

$$h = 10\ \mathrm{m},\ \omega_n = 2\pi \times 0.15\ \mathrm{s}^{-1},\ a_n = 1.3\ \mathrm{m},\ \varphi_n = 10°,$$

$$\omega_m = 2\pi \times 0.10\ \mathrm{s}^{-1},\ a_m = 1.0\ \mathrm{m},\ \varphi_m = -10°,$$

其中

$$\boldsymbol{k}_n = \kappa_n(\cos\varphi_n, \sin\varphi_n),\ \boldsymbol{k}_m = \kappa_m(\cos\varphi_m, \sin\varphi_m).$$

据此,可演算出一系列关于“自由表面位移、速度势、波数”的特定数值,如表1所示.由此,可描画出在$t = 0$时如图1所示的由图1(a)~(e)所构成的3种图形:1)图1(a)、(b)、(c)依次给出“一阶、三阶、五阶”的自由表面位移透视图,愈来愈显示出局部的精细特征.因为对于庞大的海洋工程造价而言,这将意味着什么?2)图1(d)描画了沿着中心线($y = 0$)的“一阶、三阶、五阶”的自由表面位移:对于波谷而言,“一阶”处于最低端,“三阶”次之,“五阶”尤次之;对于波峰来说,其情形却恰恰相反,“五阶”上升至最顶端,“一阶”垫底.亦即:阶数越高越是“峰高谷浅”.3)图1(e)给出在中心点处沿着x轴的分量速度u的“一阶、三阶、五阶”剖面图:在同一高度z处,“一阶”最小,“三阶”较大,“五阶”最大.在u的一般定值下,却是“一阶”上升至顶端,“五阶”下降到最低处.对于沿着y轴的分量速度v,同样如此.

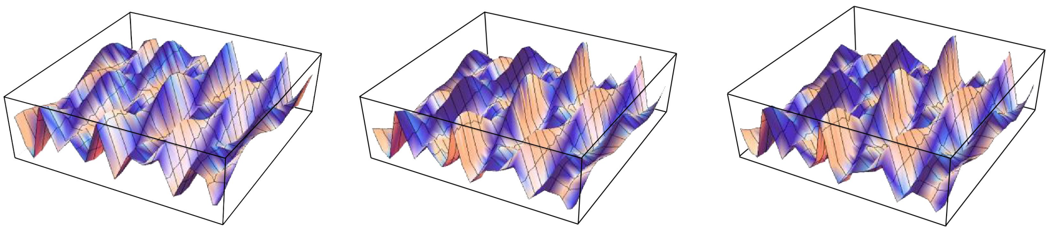
表1　一组典型的“有限水深五阶双色短峰波数值”

Table 1　A set of typical values for the 5th-order bichromatic short-crested waves in finite depth

free surface displacement		velocity potential		wave number	
		F_n	−6.578 14	κ_n	0.107 37
		F_m	−13.295 40	κ_m	0.065 14
G_{2n}	2.577 22	F_{2n}	−2.454 91		
G_{2m}	4.635 46	F_{2m}	−19.063 30		
G_{n+m}	3.132 00	F_{n+m}	−6.449 91	κ_{n+m}	0.170 047
G_{n-m}	−1.406 00	F_{n-m}	32.365 70	κ_{n-m}	0.051 253 9
G_{3n}	3.557 15	F_{3n}	−0.118 115		
G_{3m}	9.370 72	F_{3m}	−10.703 50		
G_{n+2m}	17.633 30	F_{n+2m}	−8.044 55	κ_{n+2m}	0.234 073
G_{n-2m}	−2.075 30	F_{n-2m}	−65.718 61	κ_{n-2m}	0.047 032 4
G_{m+2n}	12.863 60	F_{m+2n}	−1.958 20	κ_{m+2n}	0.276 849
G_{m-2n}	−4.894 60	F_{m-2n}	−14.884 10	κ_{m-2n}	0.155 137

续表 1

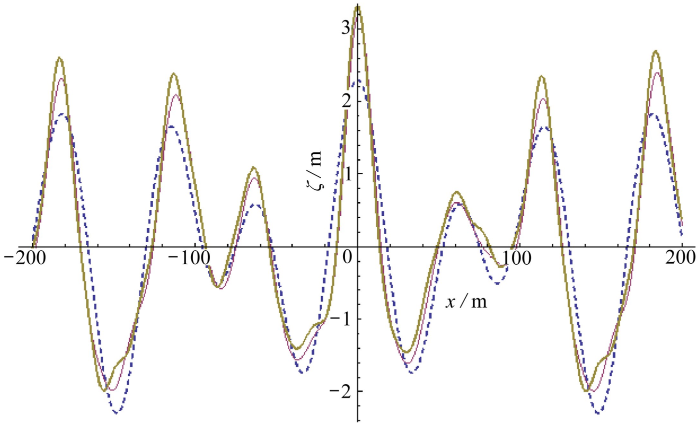
free surface displacement		velocity potential		wave number	
		$F_n^{(3)}$	0.158 40		
		$F_m^{(3)}$	0.426 70		
G_{4n}	9.698 34	F_{4n}	0.161 669		
G_{4m}	47.452 70	F_{4m}	−20.569 00		
G_{n+3m}	25.125 40	F_{n+3m}	−11.324 03	κ_{n+3m}	0.298 582
G_{n-3m}	−8.446 80	F_{n-3m}	−79.741 20	κ_{n-3m}	0.101 408
G_{m+3n}	30.145 60	F_{m+3n}	−7.839 511	κ_{m+3n}	0.383 968
G_{m-3n}	−70.235 70	F_{m-3n}	−16.552 58	κ_{m-3n}	0.261 848
G_{2n+2m}	55.447 30	F_{2n+2m}	−39.584 27		
G_{2n-2m}	−32.773 30	F_{2n-2m}	156.905 21		
$G_{2n}^{(4)}$	0.268 40	$F_{2n}^{(4)}$	0.775 44		
$G_{2m}^{(4)}$	−0.457 90	$F_{2m}^{(4)}$	8.576 51		
$G_{n+m}^{(4)}$	0.588 30	$F_{n+m}^{(4)}$	0.205 36		
$G_{n-m}^{(4)}$	−0.324 20	$F_{n-m}^{(4)}$	1.174 85		
G_{5n}	83.081 90	F_{5n}	0.214 419		
G_{5m}	345.565 00	F_{5m}	7.668 12		
G_{n+4m}	266.054 10	F_{n+4m}	−127.883 15	κ_{n+4m}	0.363 315
G_{n-4m}	−101.599 80	F_{n-4m}	−425.784 29	κ_{n-4m}	0.163 834
G_{m+4n}	200.331 20	F_{m+4n}	−54.043 70	κ_{m+4n}	0.491 197
G_{m-4n}	−411.333 50	F_{m-4n}	−211.705 91	κ_{m-4n}	0.368 942
G_{2n+3m}	118.222 30	F_{2n+3m}	−345.685 20	κ_{2n+3m}	0.403 943
G_{2n-3m}	−326.115 70	F_{2n-3m}	−789.124 45	κ_{2n-3m}	0.073 721 1
G_{2m+3n}	77.025 90	F_{2m+3n}	−101.459 28	κ_{2m+3n}	0.446 761
G_{2m-3n}	−259.123 40	F_{2m-3n}	−419.456 60	κ_{2m-3n}	0.204 598
$G_{3n}^{(5)}$	0.366 20	$F_{3n}^{(5)}$	0.323 62		
$G_{3m}^{(5)}$	0.887 40	$F_{3m}^{(5)}$	0.535 19		
$G_{n+2m}^{(5)}$	2.897 50	$F_{n+2m}^{(5)}$	1.744 01		
$G_{n-2m}^{(5)}$	−0.336 70	$F_{n-2m}^{(5)}$	3.679 42		
$G_{m+2n}^{(5)}$	1.888 10	$F_{m+2n}^{(5)}$	0.216 25		
$G_{m-2n}^{(5)}$	−2.313 90	$F_{m-2n}^{(5)}$	0.546 71		
		$F_n^{(5)}$	0.017 847		
		$F_m^{(5)}$	0.043 563		



(a) 一阶的自由表面位移透视图
(a) A perspective of the 1st-order surface elevations

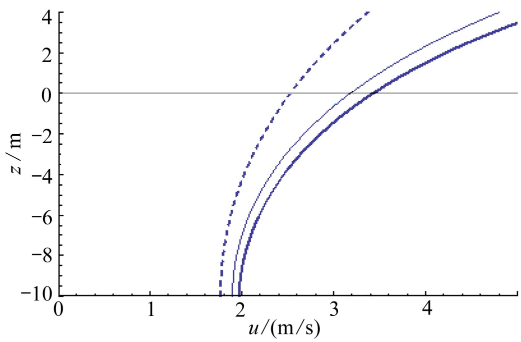
(b) 三阶的自由表面位移透视图
(b) A perspective of the 3rd-order surface elevations

(c) 五阶的自由表面位移透视图
(c) A perspective of the 5th-order surface elevations



（d）沿中心线 $y = 0$ 的自由表面位移：一阶理论（虚线），三阶理论（细实线），五阶理论（粗实线）

（d）Surface elevations along the center line $y = 0$: 1st-order theory (dashed line), 3rd-order theory (thin real line), 5th-order theory (thick real line)

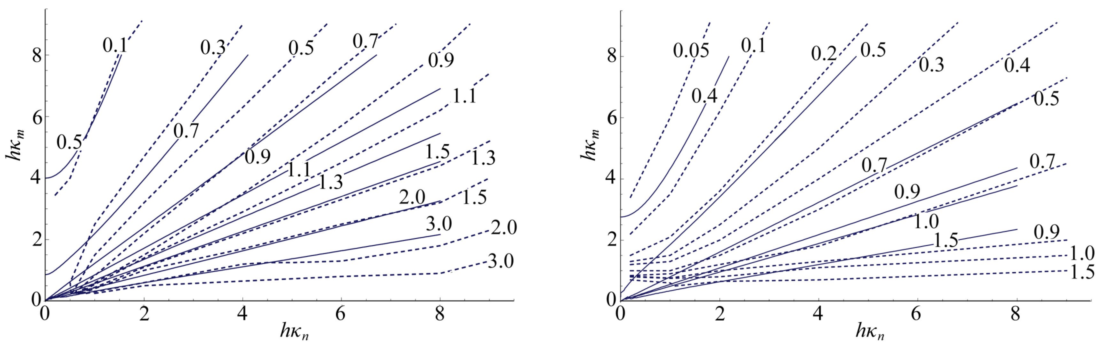


（e）在中心点 $(x,y) = (0,0)$ 处的速度 u 剖面图：一阶理论（虚线），三阶理论（细实线），五阶理论（粗实线）

（e）Velocity u profiles at the center point $(x,y) = (0,0)$: 1st-order theory (dashed line), 3rd-order theory (thin real line), 5th-order theory (thick real line)

图 1　有限水深双色短峰波的五阶解

Fig. 1　The 5th-order solution for a bichromatic short-crested wave in finite depth



（a）共线性相互作用：$\varphi_n = \varphi_m = 0°$

（a）Collinear interaction: $\varphi_n = \varphi_m = 0°$

（b）短峰波相互作用：$\varphi_n = 30°$，$\varphi_m = -30°$

（b）Short-crested wave interaction: $\varphi_n = 30°$, $\varphi_m = -30°$

图 2　双色短峰波相互作用的振幅色散函数 $\Omega_{nm}^{(3)}$（虚线）和 $\Omega_{nm}^{(5)}$（实线）

Fig. 2　The amplitude dispersion functions $\Omega_{nm}^{(3)}$ (dashed line) and $\Omega_{nm}^{(5)}$ (real line) for bichromatic short-crested wave interactions

在第三阶、第五阶色散关系中能够表征“任意两两不同波相互作用”的核心项分别为 $\Omega_{nm}^{(3)}$ 和 $\Omega_{nm}^{(5)}$，其必随着“水深”或“波数”之“$h\kappa_n$ 和 $h\kappa_m$”的不同而发生波及整个波态的演化.为此，同样假设 $\boldsymbol{k}_n=\kappa_n(\cos\varphi_n,\sin\varphi_n)$，$\boldsymbol{k}_m=\kappa_m(\cos\varphi_m,\sin\varphi_m)$，并且选取两种特殊情形：(a) 共线性相互作用：$\varphi_n=\varphi_m=0°$；(b) 短峰波相互作用：$\varphi_n=30°$，$\varphi_m=-30°$.以此构造图 2(以 $h\kappa_n$ 和 $h\kappa_m$ 为自变量(波数矢量的定义：$\boldsymbol{k}_n=\kappa_n(\cos\varphi_n,\sin\varphi_n)$，$\boldsymbol{k}_m=\kappa_m(\cos\varphi_m,\sin\varphi_m)$)).从中不难发现：无论居于何种作用状态下，$\Omega_{nm}^{(3)}$ 和 $\Omega_{nm}^{(5)}$ 的最大值均发生在 $h\kappa_m<h\kappa_n$ 处.此即意味着“浅水波或长波”对 ω_n 的贡献要强于“深水波或短波”.在 $h\kappa_n$ 值一定的情况下，若要 $\Omega_{nm}^{(5)}$ 和 $\Omega_{nm}^{(3)}$ 取同样的值，则前者所要求的 $h\kappa_m$ 值就要比后者的大，即比较短的波.在 $h\kappa_m$ 值一定的情况下，情形却恰恰相反.

5 结　　论

依照目前 Stokes 波理论的研究格局、水准，并考虑到其与普适的波湍流理论[4,6-11]和 Hamilton 力学[12]的内在基本关系，以及反观 Stokes 波理论在全球愈演愈烈的海洋(尤其深水海洋)工程实践活动中所发挥出来的广泛持久的效力，本文特将“经典、纯波的三阶单色单向 Stokes 波理论”大幅度地提升至“现代、可包含环境均匀流效应的五阶双色双向波理论”.亦即给出了最为基本的“五阶之自由表面位移、速度势和非线性振幅色散关系”的显式表达式，又进一步得到处于核心支配地位而不受制于双波运动的“无穷多波中任意两两不同频率不同振幅相互作用的五阶波非线性振幅色散关系”.

凭此，一个新型海洋水波基础理论平台已经搭建和构筑.其效能，其适用性，只得期待于未来的精当物理海洋实验和广泛海洋工程实践的论证和验收了.

可以预测，从海洋波湍流理论出发亦可得出一种类似但更为精致、准确并且最为重要之“对称”[7-11,29]的“五阶双色双向波理论”.这就有赖于又一番漫长繁杂的探索和比较工作.

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A 5th-Order Theory for Bichromatic and Bidirectional Ocean Surface Waves

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Abstract: The classical Stokes wave theory of pure wave motion for the 3rd-order monochromatic and monodirectional waves was expanded to a 5th-order theory for bichromatic and bidirectional ocean surface waves under the ambient uniform current effect in water of finite depth, which, based on the 3rd-order theory for bichromatic and bidirectional waves, comprised the 4th- and the 5th-order explicit expressions for the free surface elevations, the velocity potential and the nonlinear amplitude dispersion relation. The 5th-order nonlinear amplitude dispersion relation playing a key role in the bichromatic and bidirectional wave theory was generalized to one relation of 2 arbitrary interacting waves with different frequencies and ampitudes in pairs out of infinite waves. The typical characteristics of bichromatic and bidirectional short-crested waves were illustrated in detail with diagrams.

Key words: 5th-order theory; bichromatic and bidirectional wave; ocean surface wave; nonlinear amplitude dispersion relation

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