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直角坐标系下层状黏弹性地基 Biot 固结解析刚度矩阵解*

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摘要: 基于直角坐标系下 Biot 固结的基本控制方程,并考虑软土土骨架的黏弹性特性,通过 Fourier-Laplace 积分变换、解耦变换、微分方程组理论和矩阵理论,推导了黏弹性地基 Biot 固结三维空间问题和平面应变问题在积分变换域的解析解,进而得到对应问题的单元刚度矩阵.然后根据对号入座原则组装得到层状黏弹性地基 Biot 固结对应问题的总体刚度矩阵.通过求解总体刚度矩阵形成的线性代数方程,得到层状黏弹性地基 Biot 固结对应问题在积分变换域内的解答.最后应用 Fourier-Laplace 逆变换得到其物理域内的解.对比求解黏弹性 Biot 固结问题退化的弹性 Biot 固结问题与已有解答,验证了刚度矩阵计算方法的正确性,为层状黏弹性地基 Biot 固结问题提供了理论基础.

关 键 词: 直角坐标系; 层状黏弹性地基; Biot 固结; 解析刚度矩阵法

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引　　言

我国沿海地区如上海、宁波等存在大量饱和粘土的天然地基,在地质沉积形成中呈现出层状特征.随着市政基础设施的建设,在外部荷载作用过程中,饱和软土内存在土骨架和孔隙水压力相互耦合变化的固结现象.同时,试验研究发现:软土主、次固结之间是耦合不可分的[1-3],固结问题应考虑土骨架具有显著随时间变化的流变特征.

Biot 固结理论能正确地反映孔隙水压力消散和土体骨架变形之间的相互关系[4],现有针对层状地基 Biot 固结的研究大多集中在求解极坐标系下的相关问题[5-15],忽略了软土土骨架的流变特征.同时,地基局部承受外部作用在实际工程中时常发生,因此,研究直角坐标系下三维层状流变地基固结问题十分有必要.同时, 将 Biot 固结三维空间问题简化为平面应变问题同样具有实际应用价值, 如区间隧道开挖对周围环境的影响.现有层状地基 Biot 固结的求解方法主要采用传递矩阵法[6-8], 传递矩阵法在计算指数函数乘积时存在溢出问题[6-8].由于刚度矩阵的对称性及矩阵元素只存在负指数, 从而避免传递矩阵法中正指数存在导致的计算溢出问题[16].

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从直角坐标系下 Biot 固结的基本控制方程入手,并考虑土骨架的黏弹性,推导得到层状黏弹性地基 Biot 固结三维空间问题和平面应变问题均为 6 行 6 列的解析单元刚度矩阵,并集成层状黏弹性地基 Biot 固结对应问题总体刚度矩阵.通过求解总体刚度矩阵形成的代数方程得到对应问题的解答,为实际工程中层状黏弹性地基 Biot 固结问题提供理论基础.

1　黏弹性地基 Biot 固结问题单元刚度矩阵的推导

1.1　三维空间问题单元刚度矩阵

考虑土骨架黏弹性建立 Biot 固结问题的基本方程时,做以下假定:

1) 土体骨架是黏弹性体;

2) 土体变形是微小的;

3) 土颗粒和孔隙水不可压缩;

4) 土体饱和时只考虑水、土两相性;

5) 土体渗流符合 Darcy 定律且渗透系数不随时空变化为常数;

6) 不计土体孔隙中的流体相对于土体骨架运动的惯性力和体力;

7) 土体内任取一微小单元体, z 坐标向下为正,应力和应变以压缩为正,剪应力以使单元体逆时针旋转为正.

黏弹性力学中几何方程、积分型本构方程和不计体力的静力平衡方程分别为[14]

$$\varepsilon_{ij}(t)=-\frac{1}{2}[u_{i,j}(t)+u_{j,i}(t)],\tag{1a}$$

$$\begin{aligned}\sigma'_{ij}(t)&=\delta_{ij}\lambda(t)*\mathrm{d}\varepsilon_{kk}(t)+2G(t)*\mathrm{d}e_{ij}(t)=\\&\quad\mathrm{d}\lambda(t)*\delta_{ij}\varepsilon_{kk}(t)+2\varepsilon_{ij}(t)*\mathrm{d}G(t),\end{aligned}\tag{1b}$$

$$\sigma_{ij,j}=0,\tag{1c}$$

式中, σ_{ij} 为土体应力张量; ε_{ij} 为土体应变张量; ε_{kk} 为体积应变; u_i 为土体位移张量; σ'_{ij} 为土骨架有效应力张量; $\lambda(t)=K(t)-2G(t)/3$, $G(t)$ 为剪切松弛函数, $K(t)$ 为体积松弛函数; $*$ 为广义 Stieltjes 卷积符号; $\mathrm{d}f(t)$ 为 $f(t)$ 对时间 t 的求导, $i,\ j$ 为下标, $i=1,2,3,\ j=1,2,3$; δ_{ij} 为符号函数, $i=j$ 时, $\delta_{ij}=1$, $i\neq j$ 时, $\delta_{ij}=0$.

Biot 固结理论中有效应力原理、渗流连续性方程、土孔隙中流体的平衡方程分别为

$$\sigma_{ij}=\sigma'_{ij}+p\delta_{ij},\tag{1d}$$

$$\dot{\varepsilon}_{ii}=\dot{\omega}_{i,i},\tag{1e}$$

$$\dot{\omega}=-\frac{k_i}{\gamma_{\mathrm{w}}}p_{,i},\tag{1f}$$

式中, p 为孔隙水压力,以压为正; ω_i 为土体中液体相对于土骨架的位移分量,液体流动方向与坐标轴方向相同为正; k_i 为渗透系数; $\gamma_{\mathrm{w}}=\rho_{\mathrm{w}}g$ 为液体重度.

将土孔隙中流体的平衡方程式(1f)代入渗流连续性方程式(1e),得到

$$\dot{\varepsilon}_{ii}=-\frac{k_i}{\gamma_{\mathrm{w}}}p_{,ii}.\tag{1g}$$

将几何方程式(1a)代入积分型本构方程式(1b)得到

$$\sigma'_{ij,j}(t)=\mathrm{d}\lambda(t)*(\varepsilon_{kk})_{,i}-\mathrm{d}G(t)*[u_{i,jj}+(u_{j,j})_{,i}].\tag{2}$$

将式(2)和有效应力原理式(1d)代入平衡方程式(1c),并联合式(1g)展开得到用位移表示的直角坐标系下黏弹性地基渗透各向异性 Biot 固结三维空间问题控制方程,即

$$\mathrm{d}G(t)\nabla^2 u(x,y,z,t) - \mathrm{d}[\lambda(t)+G(t)]\frac{\partial\varepsilon(x,y,z,t)}{\partial x} - \frac{\partial p}{\partial x} = 0, \tag{3a}$$

$$\mathrm{d}G(t)\nabla^2 v(x,y,z,t) - \mathrm{d}[\lambda(t)+G(t)]\frac{\partial\varepsilon(x,y,z,t)}{\partial y} - \frac{\partial p}{\partial y} = 0, \tag{3b}$$

$$\mathrm{d}G(t)\nabla^2 w(x,y,z,t) - \mathrm{d}[\lambda(t)+G(t)]\frac{\partial\varepsilon(x,y,z,t)}{\partial z} - \frac{\partial p}{\partial z} = 0, \tag{3c}$$

$$\frac{\partial\varepsilon}{\partial t} = -\left(\frac{k_x}{\gamma_{\mathrm{w}}}\frac{\partial^2 p}{\partial x^2} + \frac{k_y}{\gamma_{\mathrm{w}}}\frac{\partial^2 p}{\partial y^2} + \frac{k_z}{\gamma_{\mathrm{w}}}\frac{\partial^2 p}{\partial z^2}\right), \tag{3d}$$

式中

$$\varepsilon = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \tag{3e}$$

ε 为体积应变；$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ 为 Laplace 算子；$u(x,y,z,t)$，$v(x,y,z,t)$，$w(x,y,z,t)$ 为 x,y,z 方向的位移；k_x 和 k_y 分别为 x 和 y 方向的渗透系数，根据天然地基沉积性质，令 $k_x = k_y = k_{\mathrm{h}}$。

结合有效应力原理式(1d)和(2)展开得到用位移表示的直角坐标系下三维空间问题的物理方程为

$$\sigma_z(x,y,z,t) - p(x,y,z,t) = \mathrm{d}\lambda(t)\varepsilon - 2\mathrm{d}G(t)\frac{\partial w}{\partial z}, \tag{4a}$$

$$\tau_{xz}(x,y,z,t) = -\mathrm{d}G(t)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{4b}$$

$$\tau_{yz}(x,y,z,t) = -\mathrm{d}G(t)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right). \tag{4c}$$

由土体孔隙中流体的平衡方程式(1f)，得到 0 到 t 时刻 z 方向液体相对于土骨架的单位面积流量为

$$Q = \int_0^t \frac{k_z}{\gamma_{\mathrm{w}}}\frac{\partial p}{\partial z}\mathrm{d}t. \tag{5}$$

对式(3a)、(3b)、(3c)、(3d)和(3e)分别进行 x,y 方向的双重 Fourier 变换和进行时间 t 的 Laplace 变换，得到

$$\frac{\mathrm{d}^2\tilde{\tilde{u}}}{\mathrm{d}z^2} - \xi^2\tilde{\tilde{u}} - \frac{\mathrm{i}(\tilde{\lambda}^* + \tilde{G}^*)\xi_x}{\tilde{G}^*}\tilde{\tilde{\varepsilon}} - \mathrm{i}\xi_x\frac{\tilde{\tilde{p}}}{\tilde{G}^*} = 0, \tag{6a}$$

$$\frac{\mathrm{d}^2\tilde{\tilde{v}}}{\mathrm{d}z^2} - \xi^2\tilde{\tilde{v}} - \frac{\mathrm{i}(\tilde{\tilde{\lambda}}^* + \tilde{G}^*)\xi_y}{\tilde{G}^*}\tilde{\tilde{\varepsilon}} - \mathrm{i}\xi_y\frac{\tilde{\tilde{p}}}{\tilde{G}^*} = 0, \tag{6b}$$

$$\frac{\mathrm{d}^2\tilde{\tilde{w}}}{\mathrm{d}z^2} - \xi^2\tilde{\tilde{w}} - \frac{(\tilde{\lambda}^* + \tilde{G}^*)}{\tilde{G}^*}\frac{\mathrm{d}\tilde{\tilde{\varepsilon}}}{\mathrm{d}z} - \frac{1}{\tilde{G}^*}\frac{\mathrm{d}\tilde{p}}{\mathrm{d}z} = 0, \tag{6c}$$

$$s\tilde{\varepsilon} = \xi_x^2\frac{k_{\mathrm{h}}}{\gamma_{\mathrm{w}}}\tilde{p} + \xi_y^2\frac{k_{\mathrm{h}}}{\gamma_{\mathrm{w}}}\tilde{p} - \frac{k_z}{\gamma_{\mathrm{w}}}\frac{\mathrm{d}^2\tilde{p}}{\mathrm{d}z^2}, \tag{6d}$$

$$\tilde{\tilde{\varepsilon}} = -\mathrm{i}\xi_x\tilde{\tilde{u}} - \mathrm{i}\xi_y\tilde{\tilde{v}} - \frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z}, \tag{6e}$$

式中

$$\tilde{\lambda}^* = s\tilde{\lambda}(s),\ \tilde{G}^* = s\tilde{G}(s),$$

i 为虚数单位,

$$\xi^2 = \xi_x^2 + \xi_y^2,\ \tilde{\bar{f}}(\xi_x, \xi_y, z, s) = \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y, z, t)\mathrm{e}^{-\mathrm{i}\xi_x x}\mathrm{e}^{-\mathrm{i}\xi_y y}\mathrm{e}^{-st}\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

将式(6e)代入式(6a)、(6b)、(6c)和(6d)得到

$$\frac{\mathrm{id}^2\tilde{\bar{u}}}{\mathrm{d}z^2} - \mathrm{i}(\xi^2 + \psi\xi_x^2)\tilde{\bar{u}} - \mathrm{i}\psi\xi_x\xi_y\tilde{\bar{v}} - \psi\xi_x\frac{\mathrm{d}\tilde{\bar{w}}}{\mathrm{d}z} + \xi_x\frac{\tilde{\bar{p}}}{\tilde{G}^*} = 0, \tag{7a}$$

$$\frac{\mathrm{id}^2\tilde{\bar{v}}}{\mathrm{d}z^2} - \mathrm{i}(\xi^2 + \psi\xi_y^2)\tilde{\bar{v}} - \mathrm{i}\psi\xi_x\xi_y\tilde{\bar{u}} - \psi\xi_y\frac{\mathrm{d}\tilde{\bar{w}}}{\mathrm{d}z} + \xi_y\frac{\tilde{\bar{p}}}{\tilde{G}^*} = 0, \tag{7b}$$

$$(1 + \psi)\frac{\mathrm{d}^2\tilde{\bar{w}}}{\mathrm{d}z^2} + \psi\left(\xi_x\frac{\mathrm{id}\tilde{\bar{u}}}{\mathrm{d}z} + \xi_y\frac{\mathrm{id}\tilde{\bar{v}}}{\mathrm{d}z}\right) - \xi^2\tilde{\bar{w}} - \frac{1}{\tilde{G}^*}\frac{\mathrm{d}\tilde{\bar{p}}}{\mathrm{d}z} = 0, \tag{7c}$$

$$\frac{k_z}{\gamma_\mathrm{w}}\frac{\mathrm{d}^2\tilde{p}}{\mathrm{d}z^2} - \xi^2\frac{k_\mathrm{h}}{\gamma_\mathrm{w}}\tilde{p} - \mathrm{i}s\xi_x\tilde{\bar{u}} - \mathrm{i}s\xi_y\tilde{\bar{v}} - s\frac{\mathrm{d}\tilde{\bar{w}}}{\mathrm{d}z} = 0, \tag{7d}$$

式中

$$\psi = \frac{(\tilde{\lambda}^* + \tilde{G}^*)}{\tilde{G}^*}.$$

为了解除以上公式中积分变换后位移量之间的耦合,令

$$M = \frac{\xi_x\tilde{\bar{u}} + \xi_y\tilde{\bar{v}}}{\xi},\ N = \frac{\xi_y\tilde{\bar{u}} - \xi_x\tilde{\bar{v}}}{\xi},\ W = \tilde{\bar{w}},$$

变换得到

$$\tilde{\bar{u}} = \frac{\xi_x M + \xi_y N}{\xi},\ \tilde{\bar{v}} = \frac{\xi_y M - \xi_x N}{\xi}. \tag{8}$$

将式(8)代入式(7a)、(7b)、(7c)和(7d)得到

$$\frac{\mathrm{id}^2 M}{\mathrm{d}z^2} - \xi\psi\frac{\mathrm{d}W}{\mathrm{d}z} - \mathrm{i}\xi^2(1 + \psi)M + \xi\frac{\tilde{\bar{p}}}{\tilde{G}^*} = 0, \tag{9a}$$

$$(1 + \psi)\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} + \xi\psi\frac{\mathrm{id}M}{\mathrm{d}z} - \xi^2 W - \frac{1}{\tilde{G}^*}\frac{\mathrm{d}\tilde{\bar{p}}}{\mathrm{d}z} = 0, \tag{9b}$$

$$\frac{\mathrm{d}^2 N}{\mathrm{d}z^2} - \xi^2 N = 0, \tag{9c}$$

$$\frac{k_z}{\gamma_\mathrm{w}}\frac{\mathrm{d}^2\tilde{\bar{p}}}{\mathrm{d}z^2} - \xi^2\frac{k_\mathrm{h}}{\gamma_\mathrm{w}}\tilde{\bar{p}} - \mathrm{i}s\xi M - s\frac{\mathrm{d}W}{\mathrm{d}z} = 0. \tag{9d}$$

将式(9a)、(9b)和(9d)写成齐次常微分方程组的矩阵形式,即

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\psi & 0 \\ 0 & 0 & \dfrac{k_z}{\gamma_\mathrm{w}} \end{bmatrix}\begin{bmatrix} \dfrac{\mathrm{id}^2 M}{\mathrm{d}z^2} \\ \dfrac{\mathrm{d}^2 W}{\mathrm{d}z^2} \\ \dfrac{\mathrm{d}^2\tilde{\bar{p}}}{\mathrm{d}z^2} \end{bmatrix} + \begin{bmatrix} 0 & -\xi\psi & 0 \\ \xi\psi & 0 & \dfrac{-1}{\tilde{G}^*} \\ 0 & -s & 0 \end{bmatrix}\begin{bmatrix} \dfrac{\mathrm{id}M}{\mathrm{d}z} \\ \dfrac{\mathrm{d}W}{\mathrm{d}z} \\ \dfrac{\mathrm{d}\tilde{\bar{p}}}{\mathrm{d}z} \end{bmatrix} -$$

$$\begin{bmatrix} \xi^2(1+\psi) & 0 & -\dfrac{\xi}{\tilde{G}^*} \\ 0 & \xi^2 & 0 \\ s\xi & 0 & \dfrac{\xi^2 k_{\mathrm{h}}}{\gamma_{\mathrm{w}}} \end{bmatrix} \begin{bmatrix} \mathrm{i}M \\ W \\ \tilde{\tilde{p}} \end{bmatrix} = 0. \tag{10}$$

根据文献[16]提出的运用微分方程组理论和矩阵理论的求解方法,求解矩阵方程(10)得

$$\begin{bmatrix} \mathrm{i}M \\ W \\ \tilde{\tilde{p}} \end{bmatrix} = \begin{bmatrix} -1 & -z-\dfrac{2R}{H} & -1 \\ 1 & z-1/\xi & q/\xi \\ 0 & 2\tilde{G}^*\gamma_{\mathrm{w}}s/H & F/(k_z\xi) \end{bmatrix} \begin{bmatrix} C_1\mathrm{e}^{\xi z} \\ C_2\mathrm{e}^{\xi z} \\ C_5\mathrm{e}^{qz} \end{bmatrix} + \begin{bmatrix} 1 & z-\dfrac{2R}{H} & 1 \\ 1 & z+1/\xi & q/\xi \\ 0 & 2\tilde{G}^*\gamma_{\mathrm{w}}s/H & -F/(k_z\xi) \end{bmatrix} \begin{bmatrix} C_3\mathrm{e}^{-\xi z} \\ C_4\mathrm{e}^{-\xi z} \\ C_6\mathrm{e}^{-qz} \end{bmatrix}, \tag{11}$$

式中

$$q = \sqrt{\frac{k_{\mathrm{h}}}{k_z}\xi^2 + \frac{s\gamma_{\mathrm{w}}}{(1+\psi)k_z}},$$

$$H = \gamma_{\mathrm{w}}s + \psi\tilde{G}^*(k_{\mathrm{h}}-k_z)\xi^2,\ F = \gamma_{\mathrm{w}}s + (1+\psi)\tilde{G}^*(k_{\mathrm{h}}-k_z)\xi^2,$$

$$R = \tilde{G}^*(k_{\mathrm{h}}-k_z)\xi.$$

对式(4a)、(4b)、(4c)和(5)分别进行 x 向的双重 Fourier 变换和进行时间 t 的 Laplace 变换,得到

$$\tilde{\tilde{\sigma}}_z = \tilde{\lambda}^*\tilde{\tilde{\varepsilon}} - 2\tilde{G}^*\frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z} + \tilde{\tilde{p}}, \tag{12a}$$

$$\tilde{\tilde{\tau}}_{xz} = -\tilde{G}^*\left(\frac{\mathrm{d}\tilde{\tilde{u}}}{\mathrm{d}z} + \mathrm{i}\xi_x\tilde{\tilde{w}}\right), \tag{12b}$$

$$\tilde{\tilde{\tau}}_{yz} = -\tilde{G}^*\left(\frac{\mathrm{d}\tilde{\tilde{v}}}{\mathrm{d}z} + \mathrm{i}\xi_y\tilde{\tilde{w}}\right), \tag{12c}$$

$$\tilde{\tilde{Q}} = \frac{k_z}{s\gamma_{\mathrm{w}}}\frac{\mathrm{d}\tilde{\tilde{p}}}{\mathrm{d}z}. \tag{12d}$$

为配合解除积分变换后位移量之间的耦合,令

$$X = \frac{\xi_x\tilde{\tilde{\tau}}_{xz} + \xi_y\tilde{\tilde{\tau}}_{yz}}{\xi},\ Y = \frac{\xi_y\tilde{\tilde{\tau}}_{xz} - \xi_x\tilde{\tilde{\tau}}_{yz}}{\xi},\ Z = \tilde{\tilde{\sigma}}_z.$$

由式(8)、(11)和(12)得到

$$\begin{aligned} Z = \tilde{\tilde{\sigma}}_z = {} & -2\tilde{G}^*\xi C_1\mathrm{e}^{\xi z} + 2\tilde{G}^*C_2\mathrm{e}^{\xi z}(1-\xi z-R\xi/H) + 2\tilde{G}^*\xi C_3\mathrm{e}^{-\xi z} + {} \\ & 2\tilde{G}^*C_4\mathrm{e}^{-\xi z}(1+\xi z-R\xi/H) - 2\tilde{G}^*C_5\xi\mathrm{e}^{qz} + 2\tilde{G}^*C_6\xi\mathrm{e}^{-qz}, \end{aligned} \tag{13a}$$

$$\begin{aligned} \mathrm{i}X = {} & -\mathrm{i}\tilde{G}^*\left(\frac{\mathrm{d}M}{\mathrm{d}z} + \mathrm{i}\xi W\right) = \\ & 2\tilde{G}^*\xi C_1\mathrm{e}^{\xi z} + 2\tilde{G}^*\xi C_2\mathrm{e}^{\xi z}(z+R/H) + 2\tilde{G}^*\xi C_3\mathrm{e}^{-\xi z} + {} \\ & 2\tilde{G}^*\xi C_4\mathrm{e}^{\xi z}(z-R/H) + 2\tilde{G}^*qC_5\mathrm{e}^{qz} + 2\tilde{G}^*qC_6\mathrm{e}^{-qz}, \end{aligned} \tag{13b}$$

$$\tilde{\tilde{Q}} = 2\tilde{G}^*\xi C_2 e^{\xi z} k_z/H - 2\tilde{G}^*\xi C_4 e^{-\xi z} k_z/H + C_5 e^{qz} Fq/(\xi\gamma_w s) + C_6 e^{-qz} Fq/(\xi\gamma_w s), \tag{13c}$$

$$Y = -\tilde{G}^* \frac{dN}{dz}. \tag{13d}$$

由式(11)得到位移分量与待定常数 $C_i(i=1,2,\cdots,6)$ 之间的矩阵关系式,即

$$\boldsymbol{U} = [W(\xi,s,0), iM(\xi,s,0), \tilde{\bar{p}}(\xi,s,0), W(\xi,s,z), iM(\xi,s,z), \tilde{\bar{p}}(\xi,s,z)]^T = \boldsymbol{S}[C_1, C_2, C_5, C_3, C_4, C_6]^T. \tag{14}$$

由式(13a)、(13b)和(13c)得到应力分量与待定常数 $C_i(i=1,2,\cdots,6)$ 之间的矩阵关系式,即

$$\boldsymbol{V} = [-Z(\xi,s,0), -iX(\xi,s,0), -\tilde{\tilde{Q}}(\xi,s,0), Z(\xi,s,z), iX(\xi,s,z), \tilde{\tilde{Q}}(\xi,s,z)]^T = \boldsymbol{P}[C_1, C_2, C_5, C_3, C_4, C_6]^T. \tag{15}$$

由于位移分量矩阵 $\boldsymbol{U}$、应力分量矩阵 $\boldsymbol{V}$ 均可以由待定常数 $C_i(i=1,2,\cdots,6)$ 表达,得到两者之间的关系为

$$\boldsymbol{V} = \boldsymbol{KU},$$

式中, $\boldsymbol{K} = \boldsymbol{PS}^{-1}$ 为直角坐标下黏弹性地基渗透各向异性 Biot 固结三维空间问题在积分变换域内的解析单元刚度矩阵.

当 $k_h = k_z$ 即渗透各向同性时, $R=0$; $H=F=\gamma_w s$.将其代入式(14)和(15), 可以得到直角坐标下黏弹性地基渗透各向同性 Biot 固结三维空间问题在积分变换域内的解析单元刚度矩阵.

1.2　平面应变问题单元刚度矩阵

根据平面应变问题的条件,即 $v=\varepsilon_y=\tau_{yz}=\tau_{xy}=0$, 得到用位移表示的直角坐标系下黏弹性地基渗透各向异性 Biot 固结平面应变问题的控制方程为

$$dG(t)\nabla^2 u(x,z,t) - d[\lambda(t)+G(t)]\frac{\partial\varepsilon(x,z,t)}{\partial x} - \frac{\partial p}{\partial x} = 0, \tag{16a}$$

$$dG(t)\nabla^2 w(x,z,t) - d[\lambda(t)+G(t)]\frac{\partial\varepsilon(x,z,t)}{\partial z} - \frac{\partial p}{\partial z} = 0, \tag{16b}$$

$$\frac{\partial\varepsilon}{\partial t} = -\left(\frac{k_x}{\gamma_w}\frac{\partial^2 p}{\partial x^2} + \frac{k_z}{\gamma_w}\frac{\partial^2 p}{\partial z^2}\right), \tag{16c}$$

式中

$$\varepsilon = -\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right), \tag{16d}$$

ε 为体积应变, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$ 为 Laplace 算子.

用位移表示的直角坐标系下 Biot 固结平面应变问题的物理方程为

$$\sigma_z(x,z,t) - p(x,z,t) = d\lambda(t)\varepsilon - 2dG(t)\frac{\partial w}{\partial z}, \tag{17a}$$

$$\tau_{xz}(x,z,t) = -dG(t)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \tag{17b}$$

对式(16a)、(16b)、(16c)、(16d)、(17a)、(17b)和(5)分别进行 x 方向的 Fourier 变换和

进行时间 t 的 Laplace 变换，得到

$$\frac{\mathrm{d}^2\tilde{\tilde{u}}}{\mathrm{d}z^2}-\xi^2\tilde{\tilde{u}}-\frac{\mathrm{i}(\tilde{\lambda}^*+\tilde{G}^*)\xi}{\tilde{G}^*}\tilde{\tilde{\varepsilon}}-\mathrm{i}\xi\frac{\tilde{\tilde{p}}}{\tilde{G}^*}=0, \tag{18a}$$

$$\frac{\mathrm{d}^2\tilde{\tilde{w}}}{\mathrm{d}z^2}-\xi^2\tilde{\tilde{w}}-\frac{(\tilde{\lambda}^*+\tilde{G}^*)}{\tilde{G}^*}\frac{\mathrm{d}\tilde{\tilde{\varepsilon}}}{\mathrm{d}z}-\frac{1}{\tilde{G}^*}\frac{\mathrm{d}\tilde{\tilde{p}}}{\mathrm{d}z}=0, \tag{18b}$$

$$s\tilde{\tilde{\varepsilon}}=\xi^2\frac{k_x}{\gamma_{\mathrm{w}}}\tilde{\tilde{p}}-\frac{k_z}{\gamma_{\mathrm{w}}}\frac{\mathrm{d}^2\tilde{\tilde{p}}}{\mathrm{d}z^2}, \tag{18c}$$

$$\tilde{\tilde{\varepsilon}}=-\mathrm{i}\xi\tilde{\tilde{u}}-\frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z}, \tag{18d}$$

$$\tilde{\tilde{\sigma}}=-\mathrm{i}\tilde{\lambda}^*\xi\tilde{\tilde{u}}-(\tilde{\lambda}^*+2\tilde{G}^*)\frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z}+\tilde{\tilde{p}}, \tag{18e}$$

$$\mathrm{i}\tilde{\tilde{\tau}}_{xz}=-\tilde{G}^*\left(\frac{\mathrm{id}\tilde{\tilde{u}}}{\mathrm{d}z}-\xi\tilde{\tilde{w}}\right), \tag{18f}$$

$$\tilde{\tilde{Q}}=\frac{k_z}{s\gamma_{\mathrm{w}}}\frac{\mathrm{d}\tilde{\tilde{p}}}{\mathrm{d}z}, \tag{18g}$$

式中

$$\tilde{\lambda}^*=s\tilde{\lambda}(s),\ \tilde{G}^*=s\tilde{G}(s),\ \tilde{\tilde{f}}(\xi,z,s)=\int_0^{+\infty}\int_{-\infty}^{+\infty}f(x,z,t)\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{e}^{-st}\mathrm{d}x\mathrm{d}t.$$

将式(18d)代入式(18a)、(18b)和(18c)得到

$$\frac{\mathrm{id}^2\tilde{\tilde{u}}}{\mathrm{d}z^2}-\xi\psi\frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z}-\mathrm{i}\xi^2(1+\psi)\tilde{\tilde{u}}+\xi\frac{\tilde{\tilde{p}}}{\tilde{G}^*}=0, \tag{19a}$$

$$(1+\psi)\frac{\mathrm{d}^2\tilde{\tilde{w}}}{\mathrm{d}z^2}+\xi\psi\frac{\mathrm{id}\tilde{\tilde{u}}}{\mathrm{d}z}-\xi^2\tilde{\tilde{w}}-\frac{1}{\tilde{G}^*}\frac{\mathrm{d}\tilde{\tilde{p}}}{\mathrm{d}z}=0, \tag{19b}$$

$$\frac{k_z}{\gamma_{\mathrm{w}}}\frac{\mathrm{d}^2\tilde{\tilde{p}}}{\mathrm{d}z^2}-\xi^2\frac{k_x}{\gamma_{\mathrm{w}}}\tilde{\tilde{p}}-\mathrm{i}s\xi\tilde{\tilde{u}}-s\frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z}=0, \tag{19c}$$

式中

$$\psi=\frac{(\tilde{\lambda}^*+\tilde{G}^*)}{\tilde{G}^*}.$$

将式(19a)、(19b)和(19c)写成齐次常微分方程组的矩阵形式，即

$$\begin{bmatrix}1&0&0\\0&1+\psi&0\\0&0&\dfrac{k_z}{\gamma_{\mathrm{w}}}\end{bmatrix}\begin{bmatrix}\dfrac{\mathrm{id}^2\tilde{\tilde{u}}}{\mathrm{d}z^2}\\ \dfrac{\mathrm{d}^2\tilde{\tilde{w}}}{\mathrm{d}z^2}\\ \dfrac{\mathrm{d}^2\tilde{\tilde{p}}}{\mathrm{d}z^2}\end{bmatrix}+\begin{bmatrix}0&-\xi\psi&0\\ \xi\psi&0&\dfrac{-1}{\tilde{G}^*}\\0&-s&0\end{bmatrix}\begin{bmatrix}\dfrac{\mathrm{id}\tilde{\tilde{u}}}{\mathrm{d}z}\\ \dfrac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z}\\ \dfrac{\mathrm{d}\tilde{\tilde{p}}}{\mathrm{d}z}\end{bmatrix}-$$

$$\begin{bmatrix}\xi^2(1+\psi)&0&-\dfrac{\xi}{\tilde{G}^*}\\0&\xi^2&0\\s\xi&0&\dfrac{\xi^2k_x}{\gamma_{\mathrm{w}}}\end{bmatrix}\begin{bmatrix}\mathrm{i}\tilde{\tilde{u}}\\ \tilde{\tilde{w}}\\ \tilde{\tilde{p}}\end{bmatrix}=0. \tag{20}$$

对比矩阵方程(20)和(10),可知两者表达形式是一样的,根据 1.1 小节得到

$$\begin{bmatrix} \mathrm{i}\tilde{\tilde{u}} \\ \tilde{\tilde{w}} \\ \tilde{\tilde{p}} \end{bmatrix} = \begin{bmatrix} -1 & -z-\dfrac{2R}{H} & -1 \\ 1 & z-1/\xi & q/\xi \\ 0 & 2\tilde{G}^{*}\gamma_{\mathrm{w}}s/H & F/(k_z\xi) \end{bmatrix} \begin{bmatrix} C_1\mathrm{e}^{\xi z} \\ C_2\mathrm{e}^{\xi z} \\ C_5\mathrm{e}^{qz} \end{bmatrix} +$$

$$\begin{bmatrix} 1 & z-\dfrac{2R}{H} & 1 \\ 1 & z+1/\xi & q/\xi \\ 0 & 2\tilde{G}^{*}\gamma_{\mathrm{w}}s/H & -F/(k_z\xi) \end{bmatrix} \begin{bmatrix} C_3\mathrm{e}^{-\xi z} \\ C_4\mathrm{e}^{-\xi z} \\ C_6\mathrm{e}^{-qz} \end{bmatrix}, \tag{21}$$

式中

$$H=\gamma_{\mathrm{w}}s+\psi\tilde{G}^{*}(k_x-k_z)\xi^2,\ F=\gamma_{\mathrm{w}}s+(1+\psi)\tilde{G}^{*}(k_x-k_z)\xi^2,$$

$$R=\tilde{G}^{*}(k_x-k_z)\xi.$$

由式(18e)、(18f)、(18g)和(21)得到

$$\begin{aligned}\tilde{\tilde{\sigma}}_z=&-2\tilde{G}^{*}\xi C_1\mathrm{e}^{\xi z}+2\tilde{G}^{*}C_2\mathrm{e}^{\xi z}(1-\xi z-R\xi/H)+2\tilde{G}^{*}\xi C_3\mathrm{e}^{-\xi z}+\\&2\tilde{G}^{*}C_4\mathrm{e}^{-\xi z}(1+\xi z-R\xi/H)-2\tilde{G}^{*}C_5\xi\mathrm{e}^{qz}+2\tilde{G}^{*}C_6\xi\mathrm{e}^{-qz},\end{aligned} \tag{22a}$$

$$\begin{aligned}\mathrm{i}\tilde{\tilde{\tau}}_{xz}=&2\tilde{G}^{*}\xi C_1\mathrm{e}^{\xi z}+2\tilde{G}^{*}\xi C_2\mathrm{e}^{\xi z}(z+R/H)+2\tilde{G}^{*}\xi C_3\mathrm{e}^{-\xi z}+\\&2\tilde{G}^{*}\xi C_4\mathrm{e}^{\xi z}(z-R/H)+2\tilde{G}^{*}qC_5\mathrm{e}^{qz}+2\tilde{G}^{*}qC_6\mathrm{e}^{-qz},\end{aligned} \tag{22b}$$

$$\begin{aligned}\tilde{\tilde{Q}}=&2\tilde{G}^{*}\xi C_2\mathrm{e}^{\xi z}k_z/H-2\tilde{G}^{*}\xi C_4\mathrm{e}^{-\xi z}k_z/H+\\&C_5\mathrm{e}^{qz}Fq/(\xi\gamma_{\mathrm{w}}s)+C_6\mathrm{e}^{-qz}Fq/(\xi\gamma_{\mathrm{w}}s).\end{aligned} \tag{22c}$$

由式(21)得到位移分量与待定常数 $C_i(i=1,2,\cdots,6)$ 之间的矩阵关系式,即

$$\begin{aligned}\boldsymbol{U}'=&[\tilde{\tilde{w}}(\xi,s,0),\mathrm{i}\tilde{\tilde{u}}(\xi,s,0),\tilde{\tilde{p}}(\xi,s,0),\tilde{\tilde{w}}(\xi,s,z),\mathrm{i}\tilde{\tilde{u}}(\xi,s,z),\tilde{\tilde{p}}(\xi,s,z)]^{\mathrm{T}}=\\&\boldsymbol{S}[C_1,C_2,C_5,C_3,C_4,C_6]^{\mathrm{T}}.\end{aligned} \tag{23}$$

由式(18a)和(18b)得到应力分量与待定常数 $C_i(i=1,2,\cdots,6)$ 之间的矩阵关系式,即

$$\begin{aligned}\boldsymbol{V}'=&[-\tilde{\tilde{\sigma}}_z(\xi,s,0),\ -\mathrm{i}\tilde{\tilde{\tau}}_{xz}(\xi,s,0),\ -\tilde{\tilde{Q}}(\xi,s,0),\\&\tilde{\tilde{\sigma}}_z(\xi,s,z),\mathrm{i}\tilde{\tilde{\tau}}_{xz}(\xi,s,z),\tilde{\tilde{Q}}(\xi,s,z)]^{\mathrm{T}}=\boldsymbol{P}[C_1,C_2,C_5,C_3,C_4,C_6]^{\mathrm{T}}.\end{aligned} \tag{24}$$

因 $\boldsymbol{U}',\boldsymbol{V}'$ 均可以由待定常数 $C_i(i=1,2,\cdots,6)$ 表达,可进一步得到两者之间的关系为

$$\boldsymbol{U}'=\boldsymbol{K}'\boldsymbol{V}',$$

式中,$\boldsymbol{K}'=\boldsymbol{P}'\boldsymbol{S}'^{-1}$ 为直角坐标下黏弹性地基渗透各向异性 Biot 固结平面应变问题在变换空间内的解析单元刚度矩阵;同时,平面应变问题和三维空间问题的单元刚度矩阵均为 6 行 6 列的矩阵.

当 $k_x=k_z$,即渗透各向同性时,$R=0$;$H=F=\gamma_{\mathrm{w}}s$.将其代入式(23)和(24)得到渗透各向同性 Biot 固结平面应变问题的解析单元刚度矩阵.

2　层状黏弹性地基 Biot 固结问题刚度矩阵求解

2.1　三维空间问题

层状黏弹性地基 Biot 固结三维空间问题的计算模型如图 1 所示.根据天然地基沉积分层及荷载作用位置将地基土层划分为 n 个计算层,第 i 个计算层底部距地表的距离为 $H_i(i=1,2,\cdots,n)$,第 i 个计算层的厚度为 $h_i=H_i-H_{i-1}$,在第 i 计算层处作用荷载 $q_i(x,y,z,t)$.

假定地基表面是自由排水面，完全渗透,底部固定且不透水,得到以下边界条件：

$$\tilde{\bar{\tau}}_{xz}(\xi,s,0)=\tilde{\bar{\tau}}_{yz}(\xi,s,0)=\tilde{\bar{p}}(\xi,s,0)=0,$$

$$\tilde{\bar{w}}(\xi,s,H_n)=\tilde{\bar{u}}(\xi,s,H_n)=\tilde{\bar{v}}(\xi,s,H_n)=\tilde{\bar{Q}}(\xi,s,H_n)=0.$$

对于层间连续,假定相邻层间完全接触,对于无外力作用的计算地基面的连续条件为

$$W(\xi,s,H_i^+)=W(\xi,s,H_{i+1}^-),\ M(\xi,s,H_i^+)=M(\xi,s,H_{i+1}^-),$$

$$\tilde{\bar{p}}(\xi,s,H_i^+)=\tilde{\bar{p}}(\xi,s,H_{i+1}^-),\ X(\xi,s,H_i^+)=X(\xi,s,H_{i+1}^-),$$

$$Z(\xi,s,H_i^+)=Z(\xi,s,H_{i+1}^-),\ \tilde{\bar{Q}}(\xi,s,H_i^+)=\tilde{\bar{Q}}(\xi,s,H_{i+1}^-).$$

对于外力作用计算地基面 $z=H_i$ 处的连续条件为

$$W(\xi,s,H_i^+)=W(\xi,s,H_{i+1}^-),\ M(\xi,s,H_i^+)=M(\xi,s,H_{i+1}^-),$$

$$\tilde{\bar{p}}(\xi,s,H_i^+)=\tilde{\bar{p}}(\xi,s,H_{i+1}^-),\ X(\xi,s,H_{i+1}^-)=X(\xi,s,H_i^+)+q_X(\xi,s,H_i),$$

$$Z(\xi,s,H_{i+1}^-)=Z(\xi,s,H_i^+)+q_Z(\xi,s,H_i),\ \tilde{\bar{Q}}(\xi,s,H_i^+)=\tilde{\bar{Q}}(\xi,s,H_{i+1}^-),$$

式中，H_i^- 和 H_i^+ 分别表示第 i 层的上下表面深度；$q_X(\xi,s,H_i)$ 和 $q_Z(\xi,s,H_i)$ 为作用在 H_i 深度处的荷载经过 Fourier-Laplace 变换的分量.

将 1.1 小节推导得到的 Biot 固结三维空间问题的单元刚度矩阵应用于各计算层,并结合层间连续条件,按照对号入座原则集成整个层状黏弹性地基的 Biot 固结三维空间问题总体刚度矩阵,同时建立线性矩阵方程,即

$$\begin{bmatrix} -Z(\xi,s,0) \\ -\mathrm{i}X(\xi,s,0) \\ -\tilde{\bar{Q}}(\xi,s,0) \\ \vdots \\ -q_Z(\xi,s,H_i) \\ -\mathrm{i}q_X(\xi,s,H_i) \\ -\tilde{\bar{Q}}(\xi,s,H_i) \\ \vdots \\ Z(\xi,s,H_n) \\ \mathrm{i}X(\xi,s,H_n) \\ \tilde{\bar{Q}}(\xi,s,H_n) \end{bmatrix} = \begin{bmatrix} \boxed{\boldsymbol{K}^{(1)}} & & & & \\ & \ddots & & & \\ & & \boxed{\boldsymbol{K}^{(i)}} & & \\ & & & \ddots & \\ & & & & \boxed{\boldsymbol{K}^{(n)}} \end{bmatrix} \begin{bmatrix} W(\xi,s,0) \\ \mathrm{i}M(\xi,s,0) \\ \tilde{\bar{p}}(\xi,s,0) \\ \vdots \\ W(\xi,s,H_i) \\ \mathrm{i}M(\xi,s,H_i) \\ \tilde{\bar{p}}(\xi,s,H_i) \\ \vdots \\ W(\xi,s,H_n) \\ \mathrm{i}M(\xi,s,H_n) \\ \tilde{\bar{p}}(\xi,s,H_n) \end{bmatrix}, \tag{25}$$

式中，$\boldsymbol{K}^{(i)}$ 为第 $i(i=1,2,\cdots,n)$ 层刚度矩阵.

根据已知边界条件求解线性矩阵方程(25),得到层状黏弹性地基 Biot 固结三维空间问题在积分变换域内的解,采用截断分段 Gauss 积分计算 Fourier 逆变换[17]和 Talbot 提出的 Laplace 逆变换的方法[18]进行 Fourier-Laplace 逆变换即可得到真实物理域内的解.

2.2 平面应变问题求解

图 2 为层状黏弹性地基 Biot 固结平面应变问题的计算模型.将层状黏弹性地基 Biot 固结三维空间问题简化即为平面应变问题,同时两者的解析单元刚度矩阵均为 6 行 6 列矩阵.因此,层状黏弹性地基 Biot 固结平面应变问题的求解同样按照 2.1 小节的过程,不再赘述.

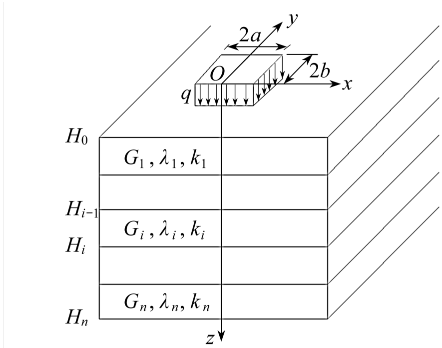

图 1　层状黏弹性地基 Biot 固结三维空间问题

Fig. 1　3D Biot consolidation of multilayered viscoelastic foundation

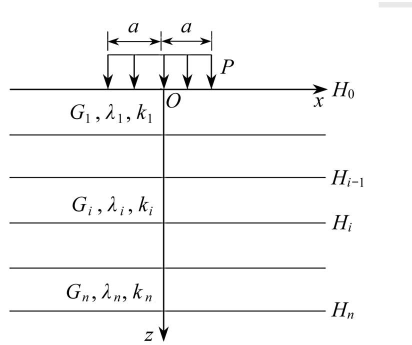

图 2　层状黏弹性地基 Biot 平面应变问题

Fig. 2　Plane strain Biot consolidation of multilayered viscoelastic foundation

3　算 例 验 证

由于研究直角坐标系下黏弹性地基 Biot 固结问题的文献不多，为验证本文解析刚度矩阵方法的正确性，将黏弹性地基 Biot 固结问题退化为弹性地基 Biot 问题。对于 Gibson 地基[6,10-11]，当 $\alpha = 0$ 时为均匀地基。分别采用解析刚度矩阵法计算得到 Biot 固结平面应变和三维空间问题的结果与文献对应的计算结果[10-11]进行对比，如图 3、4 所示。

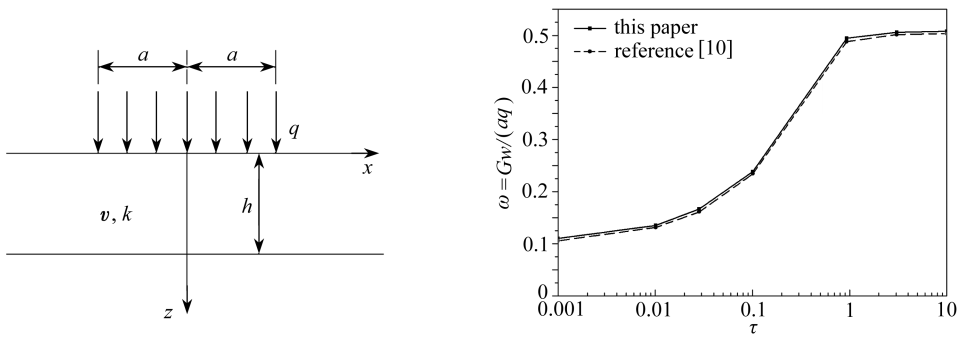

图 3　Biot 固结平面应变问题中心点竖向变形

Fig. 3　Central displacement in the case of plane strain Biot consolidation

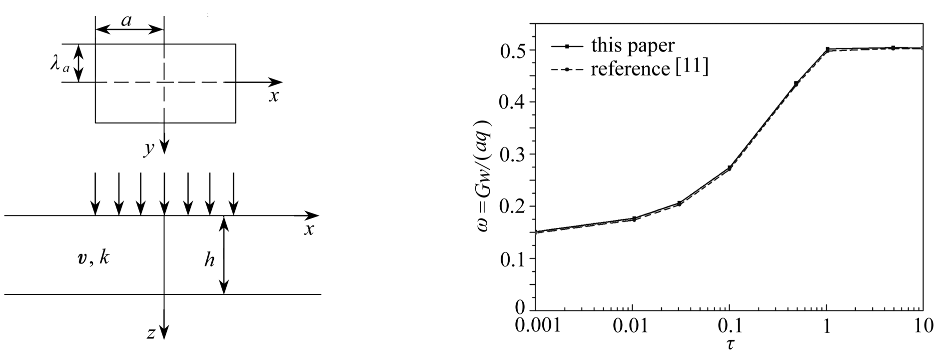

图 4　Biot 固结三维空间问题中心点竖向变形

Fig. 4　Central displacement in the case of 3D Biot consolidation

图中, G 为土体剪切模量, q 为荷载, a 为荷载作用距离, k 为渗透系数, 横坐标为无量纲化的时间因子 $\tau = 2Gkt/\gamma_w b^2$. 从图 3、4 可知, 采用解析刚度矩阵法的计算结果与文献[10-11]的解相当吻合.

4 结语

基于 Biot 固结理论的控制方程, 并考虑土骨架的流变特性, 通过 Fourier-Laplace 变换、解耦变换、矩阵理论和微分方程理论, 推导得到直角坐标系下黏弹性地基 Biot 固结三维空间问题和平面应变问题均为 6 行 6 列的单元刚度矩阵. 根据边界条件和计算层间连续条件求解对应问题的总体刚度矩阵的线性代数方程, 通过求解线性矩阵代数方程和 Fourier-Laplace 积分逆变换得到真实物理域内的解答.

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Analytical Stiffness Matrixes for Biot Consolidation of Multilayered Viscoelastic Foundations in the Cartesian Coordinate System

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Abstract: Based on the basic viscoelastic equations of Biot consolidation in the Cartesian coordinate system and in view of the viscoelasticity of soft soil skeleton, the analytical solutions to 3D problems and plane strain problems of Biot consolidation in the integral transform domain were obtained through the Fourier-Laplace transform and the decoupling transform according to the differential equation theory and the matrix theory, and in turn the corresponding element stiffness matrixes were derived. The global stiffness matrixes for Biot consolidation 3D problems and plane strain problems of multilayered viscoelastic foundations were assembled with the matrix matching method, and the solutions to the corresponding problems of multilayered viscoelastic foundations in the transform domain were obtained in the solution of the algebraic equations for the global stiffness matrixes. The solutions in the physical domain were acquired through the inverse Fourier-Laplace transform. The validity of the proposed method was examined in the comparison of the present results of 2 examples, where viscoelastic Biot consolidation was reduced to elastic Biot consolidation, with the previous reference solutions. The analytical stiffness matrixes provide a theoretical base for Biot consolidation of multilayered viscoelastic foundations.

Key words: Cartesian coordinate system; multilayered viscoelastic foundation; Biot consolidation; analytical stiffness matrix method

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