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汽车车体振动系统的对称性与守恒量研究*

翟晓洋, 傅景礼

(浙江理工大学 数学物理研究所, 杭州 310018)

摘要: 用 Lie 群方法研究汽车车体振动系统的对称性,寻找其存在的守恒量.以汽车车体做上下垂直振动和绕其质心的前后俯仰振动,采用 Lagrange 函数的方法,构建汽车车体振动系统.以此系统为对象,引入 Lie 群方法,给出该振动系统的 Noether 对称性理论与 Lie 对称性理论;由此推导该汽车系统存在的 Noether 对称性与 Lie 对称性,并得到系统相应的的守恒量.该方法对车体振动问题提出了新的对称性解法,同时扩大了 Lie 群方法的应用范围.

关 键 词: 汽车车体振动系统; Noether 对称性; Lie 对称性; 守恒量
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引 言

分析力学是从能量的角度阐述力学的基本原理,并由这些原理导出基本运动微分方程,研究这些方程以及它们的积分方法[1-2].我们知道,动力学系统的方程主要有 Lagrange 方程和 Hamilton 正则方程等.Lagrange 方程是从能量的观点上统一建立起来的系统能量与功之间的标量关系,它回避了系统未知的约束反力,成为研究约束系统动力学问题的普遍而有效的方法;Hamilton 正则方程和 Hamilton 原理继承了 Lagrange 力学方法,采用广义坐标和广义动量,引入 Hamilton 函数,开辟了解决受约束物体,以及更复杂的物体系的运动和平衡问题的新途径.

对称性原理是近代分析力学中的一个更高层次的法则,动力学系统中的守恒量更能揭示深刻的物理规律.动力学系统的对称性与守恒量紧密地联系在一起,二者之间有着潜在的关系.寻求动力学系统的对称性和守恒量,已成为近代分析力学的一大热点问题.寻求动力学系统守恒量的方法主要有:Noether 对称性法[1,3-6]、Lie 对称性法[7-12]、Mei 对称性法[13-16]等.Noether 对称性是基于系统的 Hamilton 作用量泛函在无限小变换下的一种不变性,Lie 对称性是基于微分方程在无限小变换下的不变性,Mei 对称性是一种动力学方程在动力学函数无限小变换下的形式不变性.通过这些方法可以得到系统的 Noether 守恒量、Hojman 守恒量或 Mei 守恒量.近几十年来,对称性与守恒量之间的研究在分析力学领域得到快速发展,已取得许多重要成果.对称性与守恒量的研究方法已经用到力学、物理学、机械振动等各个领域.然而,对于机械振动系统的对称性与守恒量的研究却比较少.将对称性方法引入机械振动系统,通过寻找系统的 Lagrange 函数,进一步得出系统的运动方程,再引进系统相关变量的无限小变换,给

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作者简介: 翟晓洋(1991—),男,江苏东台人,硕士生(通讯作者. E-mail:jszxy@foxmail.com).

出该系统的 Noether 对称性和 Lie 对称性,寻找出系统存在的守恒量,是一种解决机械振动问题的有效方法.

我们知道,汽车车体振动系统是一常见的机械振动系统.研究汽车车体振动系统,导出系统的对称性和守恒量,具有重要的应用价值.利用得到的守恒量,不仅可以了解到系统的性质及物理意义,还可以进一步求出系统的解.本文通过寻找汽车车体振动系统的动能与势能,并构造系统的 Lagrange 函数,由 Lagrange 方程可以方便地得到其振动方程,然后用 Lie 群方法研究该汽车系统的 Noether 对称性和 Lie 对称性,以发现系统存在的守恒量.

1 汽车车体振动系统的构造及其振动方程

将汽车车体视为一个系统,将车轮部件(包括轮胎和悬挂弹簧)视为无质量的弹簧,不考虑零部件的振动和车体的左右振动,研究车体在其对称平面内的振动,图 1 为汽车车体振动的简化动力学模型[17].其中,车体作上下垂直振动和绕其质心的前后俯仰振动.设车体质量为 m,对其质心的转动惯量为 J,前后车轮的刚度分别是 k_1,k_2,质心和前后车轮的距离分别为 l_1,l_2,系统阻尼不计.

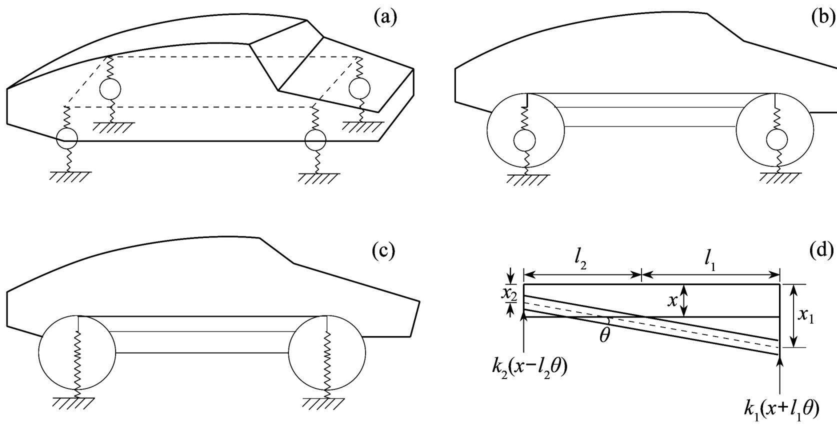

图 1 汽车车体振动系统

Fig. 1 The vehicle body vibration system

取质心的垂直位移 x 和车体绕质心转动的角位移 θ 为两个广义坐标,则车体的动能为

$$E_k = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J\dot{\theta}^2. \tag{1}$$

不计汽车重量的影响,势能为存储于变形的弹簧中的势能,即

$$E_p = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2, \tag{2}$$

其中,x_1 和 x_2 分别为前后车轮处垂直位移.考虑到车体为微幅振动,有如下的约束关系:

$$x_1 = x + l_1\theta, \tag{3}$$

$$x_2 = x - l_2\theta, \tag{4}$$

那么势能可以写为

$$E_p = \frac{1}{2} k_1 (x + l_1\theta)^2 + \frac{1}{2} k_2 (x - l_2\theta)^2. \tag{5}$$

故系统的 Lagrange 函数可表示为

$$L=E_{\mathrm{k}}-E_{\mathrm{p}}=\frac{1}{2}m\dot{x}^{2}+\frac{1}{2}J\dot{\theta}^{2}-\frac{1}{2}k_{1}(x+l_{1}\theta)^{2}-\frac{1}{2}k_{2}(x-l_{2}\theta)^{2}. \tag{6}$$

由 Lagrange 方程给出系统的微分方程为

$$m\ddot{x}+k_{1}(x+l_{1}\theta)+k_{2}(x-l_{2}\theta)=0, \tag{7}$$

$$J\ddot{\theta}+k_{1}l_{1}(x+l_{1}\theta)-k_{2}l_{2}(x-l_{2}\theta)=0, \tag{8}$$

系统(7)和(8)即为汽车车体振动系统.

2 系统的 Noether 对称性与守恒量

引进群关于时间、坐标的无限小变换:

$$\begin{cases} t^{*}=t+\varepsilon\xi_{0}(t,x,\dot{x},\theta,\dot{\theta}),\\ x^{*}(t^{*})=x(t)+\varepsilon\xi_{1}(t,x,\dot{x},\theta,\dot{\theta}),\\ \theta^{*}(t^{*})=\theta(t)+\varepsilon\xi_{2}(t,x,\dot{x},\theta,\dot{\theta}),\end{cases} \tag{9}$$

其中, ε 为无限小参数, ξ_0,ξ_1,ξ_2 为无限小变换的生成元.

对于汽车车体振动系统,如果存在规范函数 $G=G(t,x,\dot{x},\theta,\dot{\theta})$, 那么系统的 Noether 等式:

$$\begin{aligned}&[-k_{1}(x+l_{1}\theta)-k_{2}(x-l_{2}\theta)]\xi_{1}+[-k_{1}l_{1}(x+l_{1}\theta)-k_{2}l_{2}(x-l_{2}\theta)]\xi_{2}+\\&\quad m\dot{x}\dot{\xi}_{1}+J\dot{\theta}\dot{\xi}_{2}+\left[-\frac{1}{2}m\dot{x}^{2}-\frac{1}{2}J\dot{\theta}^{2}-\frac{1}{2}k_{1}(x+l_{1}\theta)^{2}-\right.\\&\quad\left.\frac{1}{2}k_{2}(x-l_{2}\theta)^{2}\right]\dot{\xi}_{0}=-\dot{G}.\end{aligned} \tag{10}$$

Noether 等式(10)可以写成广义 Killing 方程的形式,即

$$\begin{cases}\dfrac{\partial L}{\partial t}\xi_{0}+\dfrac{\partial L}{\partial x}\xi_{1}+\dfrac{\partial L}{\partial \theta}\xi_{2}+\left[L-\dfrac{\partial L}{\partial \dot{x}}\dot{x}-\dfrac{\partial L}{\partial \dot{\theta}}\dot{\theta}\right]\left(\dfrac{\partial \xi_{0}}{\partial t}+\dfrac{\partial \xi_{0}}{\partial x}\dot{x}+\dfrac{\partial \xi_{0}}{\partial \theta}\dot{\theta}\right)+\\ \quad\dfrac{\partial L}{\partial \dot{x}}\left(\dfrac{\partial \xi_{1}}{\partial t}+\dfrac{\partial \xi_{1}}{\partial x}\dot{x}+\dfrac{\partial \xi_{1}}{\partial \theta}\dot{\theta}\right)+\dfrac{\partial L}{\partial \dot{\theta}}\left(\dfrac{\partial \xi_{2}}{\partial t}+\dfrac{\partial \xi_{2}}{\partial x}\dot{x}+\dfrac{\partial \xi_{2}}{\partial \theta}\dot{\theta}\right)=\\ \quad-\dfrac{\partial G}{\partial t}-\dfrac{\partial G}{\partial x}\dot{x}-\dfrac{\partial G}{\partial \theta}\dot{\theta},\\ \left(L-\dfrac{\partial L}{\partial \dot{x}}\dot{x}-\dfrac{\partial L}{\partial \dot{\theta}}\dot{\theta}\right)\dfrac{\partial \xi_{0}}{\partial \dot{x}}+\dfrac{\partial L}{\partial \dot{x}}\dfrac{\partial \xi_{1}}{\partial \dot{x}}+\dfrac{\partial L}{\partial \dot{\theta}}\dfrac{\partial \xi_{2}}{\partial \dot{x}}=-\dfrac{\partial G}{\partial \dot{x}},\\ \left(L-\dfrac{\partial L}{\partial \dot{x}}\dot{x}-\dfrac{\partial L}{\partial \dot{\theta}}\dot{\theta}\right)\dfrac{\partial \xi_{0}}{\partial \dot{\theta}}+\dfrac{\partial L}{\partial \dot{x}}\dfrac{\partial \xi_{1}}{\partial \dot{\theta}}+\dfrac{\partial L}{\partial \dot{\theta}}\dfrac{\partial \xi_{2}}{\partial \dot{\theta}}=-\dfrac{\partial G}{\partial \dot{\theta}}.\end{cases} \tag{11}$$

将式(6)代入广义 Killing 方程(11)中,得到如下 3 个方程:

$$\begin{aligned}&[-k_{1}(x+l_{1}\theta)-k_{2}(x-l_{2}\theta)]\xi_{1}+[-k_{1}l_{1}(x+l_{1}\theta)+k_{2}l_{2}(x-l_{2}\theta)]\xi_{2}+\\&\quad\left[-\frac{1}{2}m\dot{x}^{2}-\frac{1}{2}J\dot{\theta}^{2}-\frac{1}{2}k_{1}(x+l_{1}\theta)^{2}-\frac{1}{2}k_{2}(x-l_{2}\theta)^{2}\right]\times\\&\quad\left(\frac{\partial \xi_{0}}{\partial t}+\frac{\partial \xi_{0}}{\partial x}\dot{x}+\frac{\partial \xi_{0}}{\partial \theta}\dot{\theta}\right)+m\dot{x}\left(\frac{\partial \xi_{1}}{\partial t}+\frac{\partial \xi_{1}}{\partial x}\dot{x}+\frac{\partial \xi_{1}}{\partial \theta}\dot{\theta}\right)+\\&\quad J\dot{\theta}\left(\frac{\partial \xi_{2}}{\partial t}+\frac{\partial \xi_{2}}{\partial x}\dot{x}+\frac{\partial \xi_{2}}{\partial \theta}\dot{\theta}\right)=-\frac{\partial G}{\partial t}-\frac{\partial G}{\partial x}\dot{x}-\frac{\partial G}{\partial \theta}\dot{\theta},\end{aligned} \tag{12}$$

$$\left[-\frac{1}{2}m\dot{x}^{2}-\frac{1}{2}J\dot{\theta}^{2}-\frac{1}{2}k_{1}(x+l_{1}\theta)^{2}-\frac{1}{2}k_{2}(x-l_{2}\theta)^{2}\right]\frac{\partial \xi_{0}}{\partial \dot{x}}+$$

$$\frac{\partial L}{\partial \dot{x}}\frac{\partial \xi_1}{\partial \dot{x}}+\frac{\partial L}{\partial \dot{\theta}}\frac{\partial \xi_2}{\partial \dot{x}}=-\frac{\partial G}{\partial \dot{x}}, \tag{13}$$

$$\left[-\frac{1}{2}m\dot{x}^2-\frac{1}{2}J\dot{\theta}^2-\frac{1}{2}k_1(x+l_1\theta)^2-\frac{1}{2}k_2(x-l_2\theta)^2\right]\frac{\partial \xi_0}{\partial \dot{\theta}}+$$

$$\frac{\partial L}{\partial \dot{x}}\frac{\partial \xi_1}{\partial \dot{\theta}}+\frac{\partial L}{\partial \dot{\theta}}\frac{\partial \xi_2}{\partial \dot{\theta}}=-\frac{\partial G}{\partial \dot{\theta}}。 \tag{14}$$

假设生成函数有形式：

$$\begin{cases}\xi_0=c_0,\\ \xi_1=g_1(x,\theta)+f_1(x,\theta)\dot{x}+f_2(x,\theta)\dot{\theta},\\ \xi_2=g_2(x,\theta)+f_3(x,\theta)\dot{x}+f_4(x,\theta)\dot{\theta}。\end{cases} \tag{15}$$

将式(15)代入式(13)、(14)，得

$$m\dot{x}f_1+J\dot{\theta}f_3=-\frac{\partial G}{\partial \dot{x}}, \tag{16}$$

$$m\dot{x}f_2+J\dot{\theta}f_4=-\frac{\partial G}{\partial \dot{\theta}}。 \tag{17}$$

由式(16)、(17)可以得到

$$G=-\frac{1}{2}m\dot{x}f_1-\frac{1}{2}J\dot{\theta}f_4-J\dot{\theta}\dot{x}f_3+f_5, \tag{18}$$

$$Jf_3=mf_2。 \tag{19}$$

将式(15)、(18)代入方程式(12)，得

$$\begin{aligned}&[-k_1(x+l_1\theta)-k_2(x-l_2\theta)](g_1+f_1\dot{x}+f_2\dot{\theta})+\\&[-k_1l_1(x+l_1\theta)+k_2l_2(x-l_2\theta)](g_2+f_3\dot{x}+f_4\dot{\theta})+\\&m\dot{x}\left(\frac{\partial g_1}{\partial x}\dot{x}+\frac{\partial f_1}{\partial x}\dot{x}^2+\frac{\partial f_2}{\partial x}\dot{x}\dot{\theta}+\frac{\partial g_1}{\partial \theta}\dot{\theta}+\frac{\partial f_1}{\partial \theta}\dot{\theta}\dot{x}+\frac{\partial f_2}{\partial \theta}\dot{\theta}^2\right)+\\&J\dot{\theta}\left(\frac{\partial g_2}{\partial x}\dot{x}+\frac{\partial f_3}{\partial x}\dot{x}^2+\frac{\partial f_4}{\partial x}\dot{x}\dot{\theta}+\frac{\partial g_2}{\partial \theta}\dot{\theta}+\frac{\partial f_3}{\partial \theta}\dot{\theta}\dot{x}+\frac{\partial f_4}{\partial \theta}\dot{\theta}^2\right)=\\&\frac{1}{2}m\dot{x}^3\frac{\partial f_1}{\partial x}+\frac{1}{2}J\dot{\theta}^2\dot{x}\frac{\partial f_4}{\partial x}+J\dot{\theta}\dot{x}^2\frac{\partial f_3}{\partial x}-\dot{x}\frac{\partial f_6}{\partial x}+\\&\frac{1}{2}m\dot{x}^2\dot{\theta}\frac{\partial f_1}{\partial \theta}+\frac{1}{2}J\dot{\theta}^3\frac{\partial f_4}{\partial \theta}+J\dot{\theta}^2\dot{x}\frac{\partial f_3}{\partial \theta}-\dot{\theta}\frac{\partial f_6}{\partial \theta}。\end{aligned} \tag{20}$$

分开含 $\dot{x}^3,\dot{\theta}^2\dot{x},\dot{\theta}\dot{x}^2,\dot{\theta}^3,\dot{x}^2,\dot{\theta}\dot{x},\dot{\theta}^2$ 的项，以及不含 $\dot{x},\dot{\theta}$ 的项，得到以下等式：

$$m\frac{\partial f_1}{\partial x}=\frac{1}{2}m\frac{\partial f_1}{\partial x}, \tag{21}$$

$$m\frac{\partial f_2}{\partial \theta}+J\frac{\partial f_4}{\partial x}+J\frac{\partial f_3}{\partial \theta}=\frac{1}{2}J\frac{\partial f_4}{\partial x}+J\frac{\partial f_3}{\partial \theta}, \tag{22}$$

$$m\frac{\partial f_2}{\partial x}+m\frac{\partial f_1}{\partial \theta}+J\frac{\partial f_3}{\partial x}=J\frac{\partial f_3}{\partial x}+\frac{1}{2}m\frac{\partial f_1}{\partial \theta}, \tag{23}$$

$$J\frac{\partial f_4}{\partial \theta}=\frac{1}{2}J\frac{\partial f_4}{\partial \theta}, \tag{24}$$

$$m\frac{\partial g_1}{\partial x}=0, \tag{25}$$

$$m\frac{\partial g_1}{\partial \theta}+J\frac{\partial g_2}{\partial x}=0, \tag{26}$$

$$J\frac{\partial g_2}{\partial \theta}=0, \tag{27}$$

$$[-k_1(x+l_1\theta)-k_2(x-l_2\theta)]f_1+ [-k_1l_1(x+l_1\theta)+k_2l_2(x-l_2\theta)]f_3=-\frac{\partial f_5}{\partial x}, \tag{28}$$

$$[-k_1(x+l_1\theta)-k_2(x-l_2\theta)]f_2+ [-k_1l_1(x+l_1\theta)+k_2l_2(x-l_2\theta)]f_4=-\frac{\partial f_5}{\partial \theta}, \tag{29}$$

$$[-k_1(x+l_1\theta)-k_2(x-l_2\theta)]g_1+ [-k_1l_1(x+l_1\theta)+k_2l_2(x-l_2\theta)]g_2=0. \tag{30}$$

由式(25)、(26)、(27)、(29)得

$$g_1=0,\ g_2=0. \tag{31}$$

由式(21)可知

$$f_1=f_1(\theta). \tag{32}$$

由式(22)、(23)、(24)得

$$\frac{\partial^2 f_2}{\partial x\partial\theta}=-\frac{1}{2}\frac{\partial^2 f_1}{\partial\theta^2}=-\frac{1}{2}\frac{J}{m}\frac{\partial^2 f_4}{\partial x^2}=c_2. \tag{33}$$

由式(32)、(33)可以得到

$$\begin{cases} f_1=-c_2\theta^2-2c_3\theta+c_6, \\ f_2=c_2x\theta+c_3x+c_4\theta+c_5=\dfrac{J}{m}f_3, \\ f_4=\dfrac{m}{J}(-c_2x^2-2c_4x+c_7). \end{cases} \tag{34}$$

由式(28)、(30)得

$$\begin{aligned} -\frac{\partial^2 f_5}{\partial x\partial\theta}&=(-k_1l_1+k_2l_2)f_1+[-k_1(x+l_1\theta)-k_2(x-l_2\theta)]\frac{\partial f_1}{\partial\theta}+ \\ &\quad(-k_1l_1^2+k_2l_2^2)f_3+[-k_1l_1(x+l_1\theta)+k_2l_2(x-l_2\theta)]\frac{\partial f_3}{\partial\theta}= \\ &\quad(-k_1-k_2)f_2+[-k_1(x+l_1\theta)-k_2(x-l_2\theta)]\frac{\partial f_2}{\partial x}+ \\ &\quad(-k_1l_1+k_2l_2)f_4+[-k_1l_1(x+l_1\theta)+k_2l_2(x-l_2\theta)]\frac{\partial f_4}{\partial x}. \end{aligned} \tag{35}$$

将式(34)代入上式(35)中,并比较 x,θ 的同次幂系数,从而得出

$$c_2=c_3=c_4=c_5=0,\ c_6=\frac{m}{J}c_7,$$

$$f_2=f_3=0.$$

因此

$$\begin{cases}\xi_0 = c_0 ,\\ \xi_1 = c_6\dot{x} ,\\ \xi_2 = \dfrac{m}{J} c_7\dot{\theta} = c_6\dot{\theta} .\end{cases} \tag{36}$$

由式(28)、(30)积分得

$$f_5 = c_6\left(\frac{k_1 + k_2}{2}x^2 + \frac{k_1 l_1^2 + k_2 l_2^2}{2}\theta^2 + k_1 l_1\theta x - k_2 l_2\theta x\right). \tag{37}$$

再由式(18)得

$$G = c_6\left(-\frac{1}{2}m\dot{x}^2 - \frac{1}{2}J\dot{\theta}^2 + \frac{k_1 + k_2}{2}x^2 + \frac{k_1 l_1^2 + k_2 l_2^2}{2}\theta^2 + k_1 l_1\theta x - k_2 l_2\theta x\right). \tag{38}$$

式(36)、(38)就是一般形式下的生成函数和规范函数.当 c_0,c_6,c_7 取特殊值时,可以得到以下 3 种对称性:

1) $$c_0 = 1,\ c_6 = 0,\ \xi_0 = 1,\ \xi_1 = \xi_2 = 0,\ G = 0; \tag{39}$$

2) $$\begin{cases}c_0 = 1,\ c_6 = 1,\\ \xi_0 = 1,\ \xi_1 = \dot{x},\ \xi_2 = \dot{\theta},\\ G = -\dfrac{1}{2}m\dot{x}^2 - \dfrac{1}{2}J\dot{\theta}^2 + \dfrac{1}{2}k_1(x + l_1\theta)^2 + \dfrac{1}{2}k_2(x - l_2\theta)^2;\end{cases} \tag{40}$$

3) $$\begin{cases}c_0 = 0,\ c_6 = 1,\\ \xi_0 = 0,\ \xi_1 = \dot{x},\ \xi_2 = \dot{\theta},\\ G = -\dfrac{1}{2}m\dot{x}^2 - \dfrac{1}{2}J\dot{\theta}^2 + \dfrac{1}{2}k_1(x + l_1\theta)^2 + \dfrac{1}{2}k_2(x - l_2\theta)^2 .\end{cases} \tag{41}$$

汽车车体振动系统的 Noether 定理 假设给定的有限群 G_r 的无限小变换是汽车车体振动系统的准对称变换,那么系统存在 r 个线性独立的第一积分,形如

$$I = L\xi_0 + \frac{\partial L}{\partial \dot{x}}(\xi_1 - \dot{x}\xi_0) + \frac{\partial L}{\partial \dot{\theta}}(\xi_2 - \dot{\theta}\xi_0) + G = \mathrm{const}. \tag{42}$$

可以得到以下个 3 个守恒量:

$$I_1 = -\frac{1}{2}m\dot{x}^2 - \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}k_1(x + l_1\theta)^2 - \frac{1}{2}k_2(x - l_2\theta)^2, \tag{43}$$

$$I_2 = 0, \tag{44}$$

$$I_3 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}k_1(x + l_1\theta)^2 + \frac{1}{2}k_2(x - l_2\theta)^2. \tag{45}$$

这些守恒量不是相互独立的,其中, $I_1 = -I_3$, 并且式(40)对应的对称性只能给出平凡解.

3 系统的 Lie 对称性与守恒量

由汽车车体振动系统的运动方程(7)、(8)可得

$$\begin{cases}\ddot{x} = \dfrac{1}{m}[-k_1(x + l_1\theta) - k_2(x - l_2\theta)] = \alpha_1 ,\\ \ddot{\theta} = \dfrac{1}{J}[-k_1 l_1(x + l_1\theta) + k_2 l_2(x - l_2\theta)] = \alpha_2 .\end{cases} \tag{46}$$

用无限小变换(9)构造无限小生成元向量为

$$X^{(0)} = \varepsilon_0 \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial \theta}. \tag{47}$$

一次扩展为

$$X^{(1)} = \varepsilon_0 \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial \theta} + (\dot{\xi}_1 - \dot{x}\dot{\xi}_0)\frac{\partial}{\partial \dot{x}} + (\dot{\xi}_2 - \dot{\theta}\dot{\xi}_0)\frac{\partial}{\partial \dot{\theta}}; \tag{48}$$

二次扩展为

$$X^{(2)} = X^{(1)} + \left[\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\xi}_1 - \dot{x}\dot{\xi}_0) - \ddot{x}\dot{\xi}_1\right]\frac{\partial}{\partial \ddot{x}} + \left[\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\xi}_2 - \ddot{\theta}\dot{\xi}_0) - \ddot{\theta}\dot{\xi}_2\right]\frac{\partial}{\partial \ddot{\theta}}. \tag{49}$$

汽车车体振动系统的 Lie 对称性的确定方程为

$$\ddot{\xi}_1 - \dot{x}\dot{\xi}_0 - 2\dot{\xi}_0\alpha_1 = X^{(1)}(\alpha_1), \tag{50}$$

$$\ddot{\xi}_2 - \dot{\theta}\dot{\xi}_0 - 2\dot{\xi}_0\alpha_2 = X^{(2)}(\alpha_2). \tag{51}$$

对汽车车体振动系统,我们能证明存在

Lie 定理 如果存在规范函 $G = G(t, x, \dot{x}, \theta, \dot{\theta})$, 满足如下结构方程:

$$L\dot{\xi}_0 + X^{(1)}(L) + \dot{G} = 0, \tag{52}$$

则系统存在对应于 Lie 对称性变换的守恒量

$$I = L\xi_0 + \frac{\partial L}{\partial \dot{x}}(\xi_1 - \dot{x}\xi_0) + \frac{\partial L}{\partial \dot{\theta}}(\xi_2 - \dot{\theta}\xi_0) + G = \mathrm{const}. \tag{53}$$

由式(50)、(51)、(52)可以找到系统的生成元和规范函数.

1) $$\xi_0 = 1,\ \xi_1 = \xi_2 = 0,\ X^{(0)} = -\frac{\partial}{\partial t},\ G = 0. \tag{54}$$

将式(47)代入式(46),得到相应的守恒量为

$$I_4 = -\frac{1}{2}m\dot{x}^2 - \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}k_1(x + l_1\theta)^2 - \frac{1}{2}k_2(x - l_2\theta)^2. \tag{55}$$

2) $$\begin{cases} \xi_0 = 1,\ \xi_1 = \dot{x},\ \xi_2 = \dot{\theta},\ X^{(0)} = x\dfrac{\partial}{\partial x} + \theta\dfrac{\partial}{\partial \theta}, \\ \dot{G} = (k_1 + k_2)x\dot{x} + 2(k_1 l_1 - k_2 l_2)\theta x + (k_1 l_1^2 + k_2 l_2^2)\theta\dot{\theta} - m\dot{x}^2 - J\dot{\theta}^2. \end{cases} \tag{56}$$

找不到相应的规范函数.

3) $$\begin{cases} \xi_0 = 1,\ \xi_1 = \dot{x},\ \xi_2 = \dot{\theta},\ X^{(0)} = \dot{x}\dfrac{\partial}{\partial x} + \dot{\theta}\dfrac{\partial}{\partial \theta}, \\ G = -\dfrac{1}{2}m\dot{x}^2 - \dfrac{1}{2}J\dot{\theta}^2 + \dfrac{1}{2}k_1(x + l_1\theta)^2 + \dfrac{1}{2}k_2(x - l_2\theta)^2. \end{cases} \tag{57}$$

将式(50)代入式(46),得到相应的守恒量为

$$I_5 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{k_1 + k_2}{2}x^2 + \frac{k_1 l_1^2 + k_2 l_2^2}{2}\theta^2 + k_1 l_1\theta x - k_2 l_2\theta x. \tag{58}$$

由式(55)、(58)可以看出这 2 个积分不是独立的,有 $I_4 = -I_5$.对于式(56),由于找不到相关的规范函数 G, 因此没有相应的守恒量.

4 结　　论

本文给出了汽车车体上下垂直振动和绕其质心的前后俯仰振动系统的 Noether 对称性和

Lie 对称性，找到了系统存在的守恒量.这种方法可以推广应用到其它机械振动系统.

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Study on Symmetries and Conserved Quantities of Vehicle Body Vibration Systems

ZHAI Xiao-yang, FU Jing-li

(*Institute of Mathematical Physics*, *Zhejiang Sci-Tech University*, *Hangzhou* 310018, *P.R.China*)

Abstract: Symmetries and conserved quantities of vehicle body vibration systems were studied with the Lie group method. The vertical translational vibration and the pitching vibration around the mass center were addressed by means of the Lagrangian functions to construct the vehicle body vibration model. According to this vibration system, the Noether symmetry theory and the Lie symmetry theory were derived via the introduction of the Lie group method. The existence of the Noether symmetries and the Lie symmetries of the system were proved with the corresponding conserved quantities obtained. This work provides a new symmetry solution to the vehicle body vibration problem, and meanwhile expands the application scope of the Lie group method.

Key words: vehicle body vibration system; Noether symmetry; Lie symmetry; conserved quantity

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