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受迫振动的超临界输液管 Galerkin 数值模拟*

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(我刊编委陈立群来稿)

摘要: 当流速超过临界值,输液管的直线平衡位形会发生失稳,但是系统会重新稳定在新的曲线平衡位置.通过引入坐标变换的方法,动力学模型转变为含有变系数的偏微分控制方程.采用 4 阶 Galerkin 截断的方法,使控制方程转变为常微分方程.给出具体的数值算例,发现 4 阶截断的固有频率要比 2 阶截断的固有频率更精确.同时,计算出前两阶固有频率出现可公度的情况,从而激发 2∶1 内共振现象.利用 Runge-Kutta 数值模拟的方法,在发生内共振流速范围的特定区域进行大量数值运算,结果表明高维系统的条件下,管道的不同径向坐标点的横向位置处,均出现软硬特性,而在内外共振完全调谐时,出现双跳跃现象.

关 键 词: 超临界; Galerkin 方法; 内共振; 软硬特性; 双跳跃
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引 言

管内存在流体的运动,会导致管结构在速度垂直的横向产生复杂的动力学行为.流速越大引起的横向振动就越明显,产生的振动特性越来越难于准确地预测.输液管的非线性液固耦合振动问题可能导致各种谐波共振或者内共振现象.

随着科学技术的飞跃发展,对输液管的非线性振动问题的研究有了长足的进步,一些代表性的成果有:输液管的非线性运动方程的建立[1-2],定常流和振荡流作用下的受约束输液管的非线性动力学行为[3],两端支承输液管非线性振动的稳定性问题[4],以及矩形输液管中 Maxwell 流动[5]等等,探求出了输液管非线性振动的频幅曲线特性变化的规律.Ibrahim 教授在 2010 年撰写的综述文章[6-7]表明输液管的物理模型简明,并且易于理解,其简单形式的控制方程却蕴涵着丰富而复杂的动力学内容,同时也表明对这样问题的非线性动力学研究的必要性.

当流速超过临界值时,引起管道的静力屈曲,即两端支承管在临界流速下发生屈曲失稳,系统会重新稳定在曲线平衡位形.超临界问题已经成为研究的主流,特别是在轴向运动连续体

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中的研究成果较多.Ghayesh 等[8-9]用数值方法模拟了超临界轴向运动梁受迫激励下的非线性动力学问题.Ding 和 Chen 等[10-14]对超临界轴向运动梁做了更加深入的研究,在超临界轴向运动梁的振动特性做了很多工作,但是研究没有发现任何内共振现象.管道也属于轴向运动连续体,但与梁的振动在一些方面有着本质区别,例如管道系统内液体流速的改变,从而导致多种内共振的存在,产生更为复杂的动力学现象.徐鉴和杨前彪[15]研究了亚临界流速下输液管的内共振和模态转换,发现存在多种分岔行为.基于 2 阶 Galerkin 截断,Zhang 和 Chen[16-18]指出超临界条件下输液管系统存在 2∶1 内共振,并用多尺度近似分析了输液管自由和受迫振动的动力学特性.车小玉等[19]对双层管道进行了屈曲实验研究,并发现与数值模拟的结果相吻合.

调研表明,在超临界流速条件下,输液管振动的成果依然较少,特别是对非线性现象解释与内共振的机理研究仍处于起步阶段.采用高阶 Galerkin 截断的方法,结合数值计算的技巧,对超临界输液管的受迫振动进行数值仿真,发现了软硬特性现象,以及双跳跃现象.

1　数 学 模 型

1.1　扰动方程

理论模型如图 1 所示.

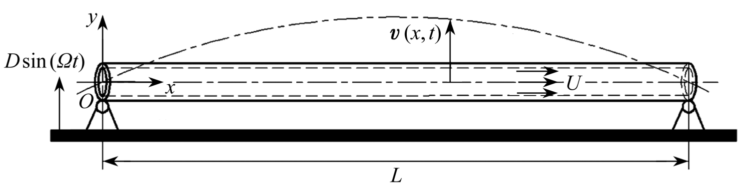


图 1　水平放置的输液管道

Fig.1　A horizontal pipe conveying fluid

输液管道水平放置在基座上,并且基座自身作简谐运动 ($v(x,t)$ 为横向位移,D 为激励振幅,Ω 为激励频率,U 为流体流速),其运动方向垂直于管道的轴线.利用 d' Alambert 原理分别对管道和流体单元分析[18],同时根据轴线可伸长理论,考虑管道横向弯曲时引起的附加非线性轴向力,可以得到系统运动方程,然后经过无量纲化后得到方程(1)[18]:

$$\eta_{,\tau\tau} + 2M_{\mathrm{r}}u\eta_{,\tau\xi} + \left(u^2 - P - \frac{\kappa^2}{2}\int_0^1(\eta_{,\xi})^2\mathrm{d}\xi - \alpha\kappa^2\int_0^1(\eta_{,\xi}\eta_{,\tau\xi})\mathrm{d}\xi\right)\eta_{,\xi\xi} + \eta_{,\xi\xi\xi\xi} + \alpha\eta_{,\tau\xi\xi\xi\xi} = f\omega^2\sin(\omega\tau). \qquad (1)$$

方程中无量纲 α 为管道材料的 Kelvin-Voigt 黏弹性阻尼系数; η 为管道轴线相对平衡位置的横向变形;ξ 为管道横截面位置坐标;τ 为时间;u 为流体流速;P 为轴向预紧力;M_{r} 为质量比;κ 为非线性因数; f 和 ω 分别为无量纲外激励振幅和频率;下标逗号表示对无量纲时间和空间的导数.系统基座为简单支承,其边界条件为

$$\begin{cases}\eta(0,\tau) = 0,\ \eta(1,\tau) = 0;\\ \eta_{,\xi\xi}(0,\tau) = 0,\ \eta_{,\xi\xi}(1,\tau) = 0.\end{cases} \qquad (2)$$

若流速超过临界值后,输液管的静平衡位形失稳,则新稳定的曲线平衡位形 $\hat{\eta}(\xi)$, 可以通过用 Wickert 方法[20]计算出

$$\hat{\eta}_k^{\pm}(\xi) = \pm\frac{2}{k\pi\kappa}\sqrt{u^2 - u_{(k)}^2}\sin(k\pi\xi), \qquad k = 1,2,3,\cdots, \qquad (3)$$

其中 $u_{(k)}$ 为第 k 阶临界流速.引入坐标变换 $\eta(\xi,\tau)\leftrightarrow\hat{\eta}_k^{\pm}(\xi) + \eta(\xi,\tau)$,同时考虑重刻度 $f\leftrightarrow\varepsilon^2 f$,

$\eta\leftrightarrow\varepsilon\eta,\alpha\leftrightarrow\varepsilon\alpha$，则系统围绕曲线平衡位形的控制方程为

$$\eta_{,\tau\tau}+2M_{\mathrm{r}}u\eta_{,\xi\tau}+(2k\pi)^2(u^2-u_k^2)\sin(k\pi\xi)\int_0^1\eta\sin(k\pi\xi)\mathrm{d}\xi+ (k\pi)^2\eta_{,\xi\xi}+\eta_{,\xi\xi\xi\xi}=\varepsilon N(\eta,\tau), \tag{4}$$

其中

$$\begin{aligned}N(\eta,\tau)={}&f\omega^2\sin(\omega\tau)-\alpha\eta_{,\tau\xi\xi\xi\xi}\pm\\&k\pi\kappa\sqrt{u^2-u_k^2}\left[2\eta_{,\xi\xi}\int_0^1\eta\sin(k\pi\xi)\mathrm{d}\xi-\sin(k\pi\xi)\int_0^1\eta_{,\xi}^2\mathrm{d}\xi\right]\mp\\&4\alpha k\pi(u^2-u_k^2)\sin(k\pi\xi)\int_0^1\eta_{,\xi\tau}\cos(k\pi\xi)\mathrm{d}\xi+\\&\frac{\varepsilon\kappa^2}{2}\eta_{,\xi\xi}\left(\int_0^1\eta_{,\xi}^2\mathrm{d}\xi+2\varepsilon\alpha\int_0^1\eta_{,\xi}\eta_{,\xi\tau}\mathrm{d}\xi\right)\pm\\&\varepsilon\alpha\kappa\sqrt{u^2-u_k^2}\left[2\eta_{,\xi\xi}\int_0^1\eta_{,\xi\tau}\cos(k\pi\xi)\mathrm{d}\xi-2k\pi\sin(k\pi\xi)\int_0^1\eta_{,\xi}\eta_{,\xi\tau}\mathrm{d}\xi\right].\end{aligned} \tag{5}$$

1.2 离散方程

研究关注流速在 $u_{(1)}<u<u_{(2)}$ 的超临界范围内的情形，采用满足边界条件的 4 阶位移模式

$$\eta(\xi,\tau)=\sum_{r=1}^{4}\phi_r(\xi)q_r(\tau), \tag{6}$$

其中 $\phi_r(\xi)$ 为两端简支梁的振型，以它近似代替相同边界条件下输液管道的振型；$q_r(\tau)$ 为广义坐标。对扰动方程(4)使用 Galerkin 截断技巧，得到离散的常微分方程：

$$\begin{aligned}\ddot q_1+g_{12}\dot q_2+g_{14}\dot q_4+k_{11}q_1={}&\varepsilon h_{11}\sin(\omega\tau)+\varepsilon\alpha_{11}\dot q_1+\varepsilon\alpha_{12}q_1^2+\varepsilon\alpha_{13}q_2^2+\varepsilon\alpha_{14}q_3^2+\varepsilon\alpha_{15}q_4^2+\\&\varepsilon^2\alpha_{16}q_1^3+\varepsilon^2\alpha_{17}q_1q_2^2+\varepsilon^2\alpha_{18}q_1q_3^2+\varepsilon^2\alpha_{19}q_1q_4^2+\varepsilon^2\alpha_{110}q_1\dot q_1+\\&\varepsilon^2\alpha_{111}q_2\dot q_2+\varepsilon^2\alpha_{112}q_3\dot q_3+\varepsilon^2\alpha_{113}q_4\dot q_4+\varepsilon^3\alpha_{114}q_1^2\dot q_1+\\&\varepsilon^3\alpha_{115}q_1q_2\dot q_2+\varepsilon^3\alpha_{116}q_1q_3\dot q_3+\varepsilon^3\alpha_{117}q_1q_4\dot q_4,\end{aligned} \tag{7a}$$

$$\begin{aligned}\ddot q_2+g_{21}\dot q_1+g_{23}\dot q_3+k_{21}q_2={}&\varepsilon\alpha_{21}\dot q_2+\varepsilon\alpha_{22}q_1q_2+\varepsilon^2\alpha_{23}q_1^2q_2+\varepsilon^2\alpha_{24}q_2^3+\varepsilon^2\alpha_{25}q_2q_3^2+\\&\varepsilon^2\alpha_{26}q_2q_4^2+\varepsilon^2\alpha_{27}q_2\dot q_1+\varepsilon^3\alpha_{28}q_1q_2\dot q_1+\varepsilon^3\alpha_{29}q_2^2\dot q_2+\\&\varepsilon^3\alpha_{210}q_2q_3\dot q_3+\varepsilon^3\alpha_{211}q_2q_4\dot q_4,\end{aligned} \tag{7b}$$

$$\begin{aligned}\ddot q_3+g_{32}\dot q_2+g_{34}\dot q_4+k_{31}q_3={}&\varepsilon h_{31}\sin(\omega\tau)+\varepsilon\alpha_{31}\dot q_3+\varepsilon\alpha_{32}q_1q_3+\varepsilon^2\alpha_{33}q_1^2q_3+\varepsilon^2\alpha_{34}q_2^2q_3+\\&\varepsilon^2\alpha_{35}q_3^3+\varepsilon^2\alpha_{36}q_3q_4^2+\varepsilon^2\alpha_{37}q_3\dot q_1+\varepsilon^3\alpha_{38}q_1q_3\dot q_1+\\&\varepsilon^3\alpha_{39}q_2q_3\dot q_2+\varepsilon^3\alpha_{310}q_3^2\dot q_3+\varepsilon^3\alpha_{311}q_3q_4\dot q_4,\end{aligned} \tag{7c}$$

$$\begin{aligned}\ddot q_4+g_{41}\dot q_1+g_{43}\dot q_3+k_{41}q_4={}&\varepsilon\alpha_{41}\dot q_4+\varepsilon\alpha_{42}q_1q_4+\varepsilon^2\alpha_{43}q_1^2q_4+\varepsilon^2\alpha_{44}q_2^2q_4+\varepsilon^2\alpha_{45}q_3^2q_4+\\&\varepsilon^2\alpha_{46}q_4^3+\varepsilon^2\alpha_{47}q_4\dot q_1+\varepsilon^3\alpha_{48}q_1q_4\dot q_1+\varepsilon^3\alpha_{49}q_2q_4\dot q_2+\\&\varepsilon^3\alpha_{410}q_3q_4\dot q_3+\varepsilon^3\alpha_{411}q_4^2\dot q_4,\end{aligned} \tag{7d}$$

式中的相关系数，可以由 Galerkin 方法确定。

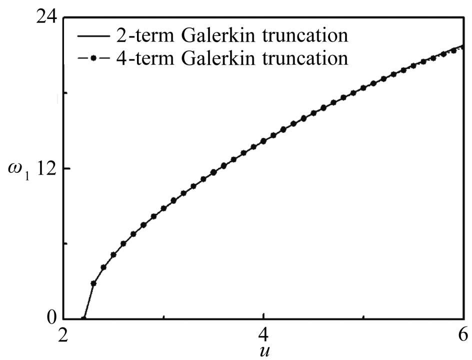
2 数值算例

文献[16-18]的研究表明,在超临界流速范围内,输液管振动系统中的固有频率出现可公度的频率关系.给定具体的数值算例 $M_r = 0.447, P = -5, \alpha = 0.001, f = 0.1, \kappa = 4$, 求解方程(7)的齐次部分,可以得到系统的前4阶固有频率,具体得到的频率见表1.观察发现当流速为 $u \approx 4.835\ 36$ 时,系统的第2阶固有频率是第1阶固有频率的两倍,此时发生2∶1内共振.此外,内共振的调谐参数可通过 $\omega_2 - 2\omega_1$ 确定,同时也就确定了调谐参数相对于第1阶固有频率所占的比重情况.

表1　前4阶固有频率随流速变化

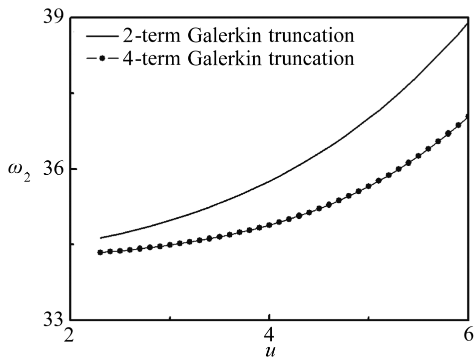
Table 1　Variation of natural frequencies of the first 4 modes with steady flow velocity u

u	ω_1	ω_2	ω_3	ω_4	$\omega_2 - 2\omega_1$	$(\omega_2 - 2\omega_1)/\omega_1$
4.820	17.686 9	35.479 5	84.601 0	156.997	0.105 7	0.597 6%
4.825	17.706 4	35.484 2	84.602 9	157.006	0.071 4	0.403 2%
4.830	17.726 0	35.488 8	84.604 7	157.014	0.036 8	0.207 6%
4.835 36	17.746 9	35.493 8	84.606 7	157.023	0	0
4.840	17.765 0	35.498 1	84.608 4	157.031	−0.031 9	−0.179 6%
4.845	17.784 5	35.502 8	84.610 2	157.039	−0.066 2	−0.372 2%
4.850	17.803 9	35.507 5	84.612 1	157.048	−0.100 3	−0.563 3%



(a) 一阶固有频率

(a) The 1st-order frequency



(b) 二阶固有频率

(b) The 2nd-order frequency

图2　比较2阶与4阶Galerkin截断的固有频率

Fig.2　Comparison of frequencies between the 2-term Galerkin and 4-term Galerkin truncation methods

研究表明[18],Galerkin截断方法可以通过更多的截断阶数,使离散系统能更好地近似一个连续系统.图2对比了2阶与4阶Galerkin截断的前两阶固有频率,对比结果发现计算的一阶固有频率与截断的阶数关系不明显,频率几乎一致.然而,2阶固有频率的结果显示,4阶截断的固有频率要比2阶截断的固有频率明显偏小,表明随着Galerkin截断阶数的增加,可以获得更为精确的固有频率.

为了研究超临界输液管的振动,考虑内共振条件下,系统受到主共振时前两阶模态响应的情况,分别引入两个调谐参数, σ_1 表示内共振调谐参数, σ_2 表示一阶主共振的调谐参数,因此,可以描述发生各种共振时的接近程度

$$\omega_2 = 2\omega_1 + \varepsilon\sigma_1,\ \omega = \omega_1 + \varepsilon\sigma_2. \tag{8}$$

通过Runge-Kutta数值方法对方程(7)进行数值求解,获得不同流速下的稳态响应后,再利用

式(6)对管道横截面不同位置坐标($\xi = 1/2, 1/3, 1/4$)分别进行计算,可以系统地得到输液管在内共振前后的幅频响应特性.图3显示了当流速 $u = 4.820$ 时,系统呈现明显的软特性即响应曲线向左弯曲;图4显示了当流速 $u = 4.835\ 36$ 时,系统呈现双跳跃现象,这一现象仅发生在流速完全接近2∶1内共振附近;图5显示了当流速 $u = 4.850$ 时,这时流速已经远离内共振点,系统呈现向右弯曲的硬特性.观察图3,4,5,发现管道在中点($\xi = 1/2$)的横截面位置时,系统的振幅响应最小.

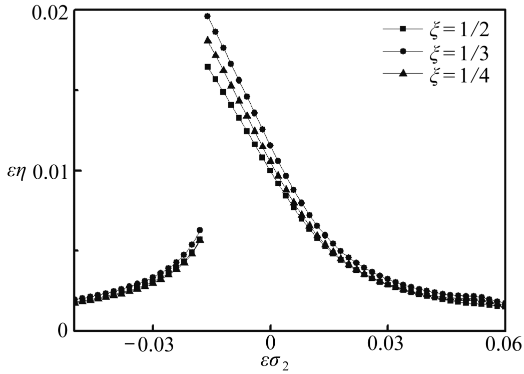


图3　软特性($u = 4.820$, $\sigma_1 = 10.57$)

Fig.3　The softening type for $u = 4.820$ and $\sigma_1 = 10.57$

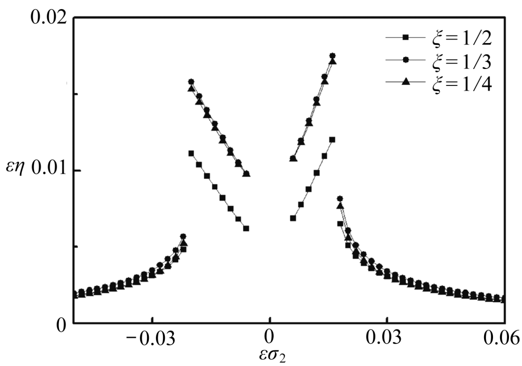


图4　双跳跃($u = 4.835\ 36$, $\sigma_1 = 0$)

Fig.4　The double jumps for $u = 4.835\ 36$ and $\sigma_1 = 0$

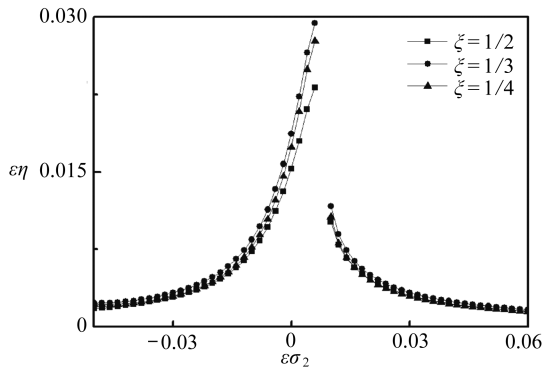


图5　硬特性($u = 4.850$, $\sigma_1 = -10.02$)

Fig.5　The hardening type for $u = 4.850$ and $\sigma_1 = -10.02$

3　结　　论

利用 Galerkin 截断的技巧,通过数值计算处理了超临界简单支承的受迫输液管系统,并求解固有频率,发现存在内共振情况.数值结果表明,在远低于内共振流速时,系统的响应曲线存在软特性;在完全调谐的内共振附近时,响应出现了双跳跃现象;在远高于内共振流速时,系统的响应呈现硬特性.通过对数值计算结果的比较,发现管中点受到的响应振幅值最小,数值结果验证了高阶截断的理论,说明了超临界系统的复杂性.

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A Galerkin Numerical Method for the Pipe Conveying Supercritical Fluid Under Forced Vibration

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Abstract: For the pipe conveying fluid, as the flow rate increased over a critical value, the equilibrium configuration was found to get unstable and bifurcate into curved equilibrium patterns. The nonlinear dynamic model for the simply-supported pipe was built and converted to variable-coefficient partial differential control equations through coordinate transformation. The 4-term Galerkin truncation procedure was then applied and the control equations of motion were transformed to 2nd-order ordinary differential equations to be solved with numerical techniques. The natural frequencies of the simply-supported pipe conveying fluid were calculated, and the result comparison was made between the 2-term and 4-term Galerkin truncation methods to give that the latter had higher accuracy. For specific system parameters, the 2nd-order natural frequency was approximately two times of the 1st-order one within a certain range of flow velocity, and the 2-to-1 internal resonance occurred. Massive computation of the amplitude-frequency responses of the pipe conveying fluid before and after internal resonance was conducted with the Runge-Kutta numerical technique. The results show that, as the flow rate and tuning parameter vary, the softening, hardening and double jumping phenomena will be respectively identified by the amplitude-frequency responses of the pipe.

Key words: supercritical fluid; Galerkin procedure; internal resonance; softening and hardening; double jumping

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