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流变计算的高性能有限元收敛性分析*

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摘要: 文中研究非 Newton(牛顿)流体流变问题的混合型双曲抛物一阶偏微分方程的收敛性,采用耦合的偏微分方程组(Cauchy 流体方程、P-T/T 应力方程),模拟自由表面元或由过度拉伸元素产生的流域.使用半离散有限元方法进行求解,对于含有时间变量的耦合方程,在空间上用有限元法,利用三线性泛函来解决偏微分方程组的非线性;在时间上用 Euler(欧拉)格式,得出方程组的收敛精度可达到 $O(h^2 + \Delta t)$.通过高性能计算的预估计和后估计得到方程的数值结果,并显示网格变形的大小.

关 键 词: 非 Newton 流体; 半离散有限元; 耦合方程; 收敛
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引 言

非 Newton 流体力学是新兴的边缘科学,它是近四、五十年发展起来的,起源于高聚物加工的需要,涉及广泛的工业领域,是力学、现代数学、化学和各工程科学的交叉与综合,特别是与材料科学有着十分密切的联系,它是现代流体力学的重要分支,同时也是现代流变学的重要组成部分.当今世界,从事化工、石油、水利、生物工程、轻工、食品材料科学的专家们对非 Newton 流体的研究越来越深入,应用也越来越广泛.非 Newton 流体力学以其强大的生命力迅速发展.蜂窝多孔结构材料的流变问题、橡胶流变问题在文献[1-2]中有详细论述.在关于非 Newton 材料的研究中,科学家们发现 Maxwell 方程可用于软固体材料(如尼龙、橡胶制品)[3-5]制作流程的非 Newton 流体应力场计算.

在现代科学技术中非线性问题变得越来越重要,因为现在面临的问题常常是非线性介质与材料(材料非线性),它们都导致非线性微分方程组.非线性问题可能出现一些线性问题所没有的奇特性质和结构,如奇点、分岔、多解、孤立波等等,是一般线性方法所不能解决的.而有限元方法是数值求解偏微分方程的一种有效方法.它是变分法、逼近论和分片多项式插值理论相结合的产物.目前有限元方法已广泛应用于结构力学、流体力学、电磁学以及其它科学研究领域,并且在各个工业部门,如飞机、轮船、油田开发等大型问题的设计计算,模拟仿真与优化方面都已获得巨大的成功.

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有限元的收敛性是有限元方法研究中的一个热点,其中的积分恒等式和积分展开技术是获得收敛的一种方法.对问题广泛的适用性与其它算法耦合的方便性是其特点,积分恒等式和积分展开式在有限元以及混合有限元中得到很好的应用.有限元的收敛性很好地解决了非线性方程在迭代计算过程中的收敛性问题.

1　流固耦合方程及其变分

1.1　流固耦合方程

在汽车短暂的碰撞过程中,汽车表面材料会出现非常大的变形[6],接触表面会呈现出近似流体的性质.Cauchy 方程[7-8]用于计算应力流场中速度 $\boldsymbol{u}$ 的变化,进而描述形变

$$\frac{\partial \boldsymbol{u}}{\partial t} = \frac{1}{\rho}\nabla\cdot\boldsymbol{\tau} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \tag{1}$$

其中 $\boldsymbol{u} = (u_x, u_y)^{\mathrm{T}}$, $\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{pmatrix}$ 且 $\tau_{xy} = \tau_{yx}$, ρ 是流体密度.

PTT 本构方程(增加了非线性指数冲击项的 Maxwell 方程[9-10],该指数冲击项是用来描述非 Newton 流体的特征)是用于计算非 Newton 流体的应力场 $\boldsymbol{\tau}$ 的分布和变化规律[11]:

$$\lambda\frac{\partial\boldsymbol{\tau}}{\partial t} = 2\eta\boldsymbol{D} - \exp\left[\frac{\varepsilon\lambda}{\eta_0}(\tau_{xx}+\tau_{yy})\right]\boldsymbol{\tau} - \lambda[\boldsymbol{u}\cdot\nabla\boldsymbol{\tau} - \nabla\boldsymbol{u}\cdot\boldsymbol{\tau} - (\nabla\boldsymbol{u}\cdot\boldsymbol{\tau})^{\mathrm{T}} + \xi[\boldsymbol{D}\cdot\boldsymbol{\tau} + (\boldsymbol{D}\cdot\boldsymbol{\tau})^{\mathrm{T}}]],$$

其中 $\boldsymbol{D} = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}})/2$,$\boldsymbol{D}$ 是应变, ε 是伸长率, ξ 是切变因子, λ 是松弛因子, η 是粘度.将 PTT 方程变形为

$$\frac{\partial\boldsymbol{\tau}}{\partial t} = \frac{2\eta}{\lambda}\boldsymbol{D} - \frac{1}{\lambda}\boldsymbol{\tau} - \frac{\varepsilon}{\eta_0}(\tau_{xx}+\tau_{yy})\boldsymbol{\tau} - \boldsymbol{u}\cdot\nabla\boldsymbol{\tau} + \left(1-\frac{\xi}{2}\right)[\nabla\boldsymbol{u}\cdot\boldsymbol{\tau} + (\nabla\boldsymbol{u}\cdot\boldsymbol{\tau})^{\mathrm{T}}] - \frac{\xi}{2}[\boldsymbol{\tau}\cdot\nabla\boldsymbol{u} + (\boldsymbol{\tau}\cdot\nabla\boldsymbol{u})^{\mathrm{T}}]. \tag{2}$$

1.2　流固耦合方程的变分

设 Ω 是 R^d 上的光滑有界区域, $\Gamma = \Gamma_1 \cup \Gamma_2$ 是其边界.定义 $S = \Gamma\times(0,T)$,设应变和应力在滑动边界上为 0, 即 $\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_1} = 0$,$\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_2} = 0$.当 $t = 0$ 时, $\boldsymbol{u}^0 = \boldsymbol{u}(x,0)$,$\boldsymbol{\tau}^0 = \boldsymbol{\tau}(x,0)$.

引入下面的空间:

$$X = H_0^1(\Omega)^2 = \{\boldsymbol{u}\in H^1(\Omega)^2,\ \boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_1} = 0,\ \boldsymbol{u}|_{\Gamma_2} = 0\},$$

$$Y = H_0^1(\Omega)^{2\times 2} = \{\boldsymbol{\tau}\in H^1(\Omega)^{2\times 2},\ \boldsymbol{\tau}|_{\Gamma} = 0\}.$$

定义

$$B_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{v}:\boldsymbol{w}\,\mathrm{d}x, \qquad \forall\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in X,$$

$$B_2(\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{\sigma}) = \int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{\tau}:\boldsymbol{\sigma}\,\mathrm{d}x, \qquad \forall\,\boldsymbol{u}\in X;\ \boldsymbol{\tau},\boldsymbol{\sigma}\in Y,$$

$$b(\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{\sigma}) = \int_\Omega\left\{\left(1-\frac{\xi}{2}\right)[\nabla\boldsymbol{u}\cdot\boldsymbol{\tau} + (\nabla\boldsymbol{u}\cdot\boldsymbol{\tau})^{\mathrm{T}}] - \frac{\xi}{2}[\boldsymbol{\tau}\cdot\nabla\boldsymbol{u} + (\boldsymbol{\tau}\cdot\nabla\boldsymbol{u})^{\mathrm{T}}]:\boldsymbol{\sigma}\right\}\mathrm{d}x, \qquad \forall\,\boldsymbol{u}\in X;\ \boldsymbol{\tau},\boldsymbol{\sigma}\in Y.$$

显然 $B_1(\cdot,\cdot,\cdot)$,$B_2(\cdot,\cdot,\cdot)$,$b(\cdot,\cdot,\cdot)$ 分别是 $X\times X\times X$,$X\times Y\times Y$,$X\times Y\times Y$ 上的三线性

泛函,并且有

$$|B_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant c\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|\ \|\boldsymbol{w}\| \leqslant c(\|\nabla\boldsymbol{u}\|\ \|\boldsymbol{v}\| + \|\nabla\boldsymbol{v}\|\ \|\boldsymbol{u}\|)\|\boldsymbol{w}\|,$$

$$|B_2(\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{\sigma})| \leqslant c\|\boldsymbol{u}\|\ \|\boldsymbol{\tau}\|\ \|\boldsymbol{\sigma}\| \leqslant c(\|\nabla\boldsymbol{u}\|\ \|\boldsymbol{\tau}\| + \|\nabla\boldsymbol{\tau}\|\ \|\boldsymbol{u}\|)\|\boldsymbol{\sigma}\|,$$

$$|b(\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{\sigma})| \leqslant c\|\nabla\boldsymbol{u}\|\ \|\boldsymbol{\tau}\|\ \|\boldsymbol{\sigma}\|.$$

对于耦合方程组(1)和(2), 求解 $(\boldsymbol{u},\boldsymbol{\tau}) \in (X,Y)$, 使得

$$\begin{cases}(\boldsymbol{u}_t,\boldsymbol{w}) = -\dfrac{1}{\rho}(\nabla\boldsymbol{w},\boldsymbol{\tau}) + B_1(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) + Q(\boldsymbol{w},\boldsymbol{\tau}),\\ (\boldsymbol{\tau}_t,\boldsymbol{v}) = \dfrac{2\eta}{\lambda}(\boldsymbol{D},\boldsymbol{v}) - \dfrac{1}{\lambda}(\boldsymbol{\tau},\boldsymbol{v}) - \dfrac{\varepsilon}{\eta_0}((\tau_{xx}+\tau_{yy})\boldsymbol{\tau},\boldsymbol{v}) - \\ \qquad B_2(\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{v}).\end{cases} \tag{3}$$

这里

$$Q(\boldsymbol{w},\boldsymbol{\tau}) = \int_\Gamma [w_x\tau_{xx}\cos(\boldsymbol{n},x) + w_y\tau_{yy}\cos(\boldsymbol{n},y)]\mathrm{d}s,$$

其中 $(\boldsymbol{n},x)$ 及 $(\boldsymbol{n},y)$ 分别是边界 Γ 上的法向量 $\boldsymbol{n}$ 与两正向坐标轴的夹角.

2 对于耦合方程的有限元分析

对求解区域进行四边形剖分, 记 h_K 为单元 K 的直径, $h = \max\{h_K\}$,得到有限元空间 (X_h, Y_h).在这里 $X_h = P_m(\Omega)^2 \cap X$, $Y_h = P_m(\Omega)^{2\times2} \cap Y$,其中 $P_m(\Omega)$ 表示定义在 Ω 上的所有次数小于等于 m 的多项式集合, 显然 $(X_h,Y_h) \subset (X,Y)$.

假设 1 存在一个映射 $X:\boldsymbol{u} \to \bar{\boldsymbol{u}} \in X_h$,使得 $(\nabla(\boldsymbol{u}-\bar{\boldsymbol{u}}),\boldsymbol{v}) = 0$, 并且满足

$$\|\boldsymbol{u}-\bar{\boldsymbol{u}}\|_l \leqslant ch^{m+1-l}\|\boldsymbol{u}\|_{m+1}, \qquad l = 0,1;$$

假设 2 存在逆不等式 $\|\nabla\boldsymbol{u}_h\| \leqslant ch^{-1}\|\boldsymbol{u}_h\|$;

假设 3 存在有限元投影 $\bar{\boldsymbol{u}}$,满足 $(\nabla(\boldsymbol{u}-\bar{\boldsymbol{u}}),\boldsymbol{v}_h) = 0$.

耦合问题的弱解形式是寻求 $(\boldsymbol{u}_h,\boldsymbol{\tau}_h) \in X_h \times Y_h$,对一切 $t \in (0,T)$ 满足

$$\begin{cases}(\boldsymbol{u}_{h,t},\boldsymbol{w}) = -\dfrac{1}{\rho}(\nabla\boldsymbol{w},\boldsymbol{\tau}_h) + B_1(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{w}) + Q(\boldsymbol{w},\boldsymbol{\tau}_h),\\ (\boldsymbol{\tau}_{h,t},\boldsymbol{v}) = \dfrac{2\eta}{\lambda}(\boldsymbol{D}_h,\boldsymbol{v}) - \dfrac{1}{\lambda}(\boldsymbol{\tau}_h,\boldsymbol{v}) - \dfrac{\varepsilon}{\eta_0}((\tau_{xx}+\tau_{yy})\boldsymbol{\tau}_h,\boldsymbol{v}) - \\ \qquad B_2(\boldsymbol{u}_h,\boldsymbol{\tau}_h,\boldsymbol{v}) + b(\boldsymbol{u}_h,\boldsymbol{\tau}_h,\boldsymbol{v}).\end{cases} \tag{4}$$

定理 1 设耦合问题的解为 $(\boldsymbol{u},\boldsymbol{\tau})$,$(\boldsymbol{u}_h,\boldsymbol{\tau}_h)$ 是该问题的 m 次有限元解, 存在正常数 c_2, 对于任意的 t 满足 $1/\lambda + \varepsilon c_2/\eta_0 + c_2h^{-1} < T^{-3/2}$, 则有收敛性预估计:

$$\|\boldsymbol{u}-\boldsymbol{u}_h\| \leqslant \frac{c_1T^{3/2}h^m + c_3T^{3/2}h^{m-1}}{1 - \dfrac{1}{B}\left(\dfrac{c\eta h^{-2}}{\rho\lambda} + \dfrac{c\eta h^{-3}}{\lambda} + \dfrac{ch^{-1}}{\rho} + ch^{-2}\right)T^3} + ch^m,$$

$$\|\boldsymbol{\tau}-\boldsymbol{\tau}_h\| \leqslant \left(1+\frac{\eta}{\lambda}\right)\frac{cT^3h^{m-1} + cT^3h^{m-2}}{B - \left(\dfrac{c\eta h^{-2}}{\rho\lambda} + \dfrac{c\eta h^{-3}}{\lambda} + \dfrac{ch^{-1}}{\rho} + ch^{-2}\right)T^3} + \frac{cT^{3/2}h^{m-1}}{B},$$

其中 $B = 1 - \left(\dfrac{1}{\lambda} - \dfrac{\varepsilon c_2}{\eta_0} - c_2h^{-1}\right)T^{3/2}$,$c,c_1,c_3$ 为正常数.

证明　记 $\boldsymbol{e}_1=\boldsymbol{u}-\boldsymbol{u}_h=\boldsymbol{u}-\bar{\boldsymbol{u}}+\bar{\boldsymbol{u}}-\boldsymbol{u}_h=\boldsymbol{e}_1^*+\boldsymbol{\theta}_1,\boldsymbol{e}_2=\boldsymbol{\tau}-\boldsymbol{\tau}_h$.将式(3)、(4)相减得

$$\begin{cases}(\boldsymbol{\theta}_{1,t},\boldsymbol{w})=-\dfrac{1}{\rho}(\nabla\boldsymbol{w},\boldsymbol{\tau}-\boldsymbol{\tau}_h)+B_1(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{w})+\\ \qquad B_1(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w})-(\boldsymbol{e}_{1,t}^*,\boldsymbol{w})+Q(\boldsymbol{w},\boldsymbol{\tau}-\boldsymbol{\tau}_h),\\ (\boldsymbol{e}_{2,t},\boldsymbol{v})=\dfrac{2\eta}{\lambda}(\boldsymbol{D}-\boldsymbol{D}_h,\boldsymbol{v})-\dfrac{1}{\lambda}(\boldsymbol{\tau}-\boldsymbol{\tau}_h,\boldsymbol{v})-\\ \qquad \dfrac{\varepsilon}{\eta_0}[((\tau_{xx}+\tau_{yy})(\boldsymbol{\tau}-\boldsymbol{\tau}_h)+(\tau_{xx}+\tau_{yy}-\tau_{xx,h}-\tau_{yy,h})\boldsymbol{\tau}_h,\boldsymbol{v})]-\\ \qquad B_2(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\tau},\boldsymbol{v})-B_2(\boldsymbol{u}_h,\boldsymbol{\tau}-\boldsymbol{\tau}_h,\boldsymbol{v})+b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\tau},\boldsymbol{v})+b(\boldsymbol{u}_h,\boldsymbol{\tau}-\boldsymbol{\tau}_h,\boldsymbol{v}),\end{cases}$$

其中

$$Q(\boldsymbol{w},\boldsymbol{\tau}-\boldsymbol{\tau}_h)=\int_\Gamma[w_x(\tau_{xx}-\tau_{xx,h})\cos(\boldsymbol{n},x)+w_y(\tau_{yy}-\tau_{yy,h})\cos(\boldsymbol{n},y)]\mathrm{d}s\leqslant$$
$$\int_\Gamma w_y(\tau_{yy}-\tau_{xx}+\tau_{xx,h}-\tau_{yy,h})\cos(\boldsymbol{n},y)\mathrm{d}s.$$

由 Hölder 不等式:

$$Q(\boldsymbol{w},\boldsymbol{\tau}-\boldsymbol{\tau}_h)\leqslant\left[\int_\Gamma|w_y|^2\mathrm{d}s\right]^{1/2}\left[\int_\Gamma|\boldsymbol{\tau}-\boldsymbol{\tau}_h|^2\mathrm{d}s\right]^{1/2}\leqslant c\|\boldsymbol{w}\|_1\|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_1,$$

故有

$$\begin{cases}(\boldsymbol{\theta}_{1,t},\boldsymbol{w})\leqslant-\dfrac{1}{\rho}(\nabla\boldsymbol{w},\boldsymbol{\tau}-\boldsymbol{\tau}_h)+B_1(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{w})+B_1(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w})-\\ \qquad (\boldsymbol{e}_{1,t}^*,\boldsymbol{w})+c\|\boldsymbol{w}\|_1\|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_1,\\ (\boldsymbol{e}_{2,t},\boldsymbol{v})=\dfrac{2\eta}{\lambda}(\boldsymbol{D}-\boldsymbol{D}_h,\boldsymbol{v})-\dfrac{1}{\lambda}(\boldsymbol{\tau}-\boldsymbol{\tau}_h,\boldsymbol{v})-\\ \qquad \dfrac{\varepsilon}{\eta_0}[((\tau_{xx}+\tau_{yy})(\boldsymbol{\tau}-\boldsymbol{\tau}_h)+(\tau_{xx}+\tau_{yy}-\tau_{xx,h}-\tau_{yy,h})\boldsymbol{\tau}_h,\boldsymbol{v})]-\\ \qquad B_2(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\tau},\boldsymbol{v})-B_2(\boldsymbol{u}_h,\boldsymbol{\tau}-\boldsymbol{\tau}_h,\boldsymbol{v})+b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\tau},\boldsymbol{v})+b(\boldsymbol{u}_h,\boldsymbol{\tau}-\boldsymbol{\tau}_h,\boldsymbol{v}).\end{cases}$$

令 $\boldsymbol{w}=\boldsymbol{\theta}_{1,t},\boldsymbol{v}=\boldsymbol{e}_{2,t}$, 则

$$\begin{cases}(\boldsymbol{\theta}_{1,t},\boldsymbol{\theta}_{1,t})=-\dfrac{1}{\rho}(\nabla\boldsymbol{\theta}_{1,t},\boldsymbol{e}_2)+B_1(\boldsymbol{e}_1,\boldsymbol{u}_h,\boldsymbol{\theta}_{1,t})+\\ \qquad B_1(\boldsymbol{u},\boldsymbol{\theta}_1,\boldsymbol{\theta}_{1,t})-(\boldsymbol{e}_{1,t}^*,\boldsymbol{\theta}_{1,t})+c\|\boldsymbol{\theta}_{1,t}\|_1\|\boldsymbol{e}_2\|_1,\\ (\boldsymbol{e}_{2,t},\boldsymbol{\theta}_{2,t})=\dfrac{2\eta}{\lambda}(\boldsymbol{D}(\boldsymbol{\theta}_1),\boldsymbol{e}_{2,t})-\dfrac{1}{\lambda}(\boldsymbol{e}_2,\boldsymbol{e}_{2,t})-\\ \qquad \dfrac{\varepsilon}{\eta_0}[((\tau_{xx}+\tau_{yy})\boldsymbol{e}_2+(\tau_{xx}+\tau_{yy}-\tau_{xx,h}-\tau_{yy,h})\boldsymbol{\tau}_h,\boldsymbol{e}_{2,t})]-\\ \qquad B_2(\boldsymbol{e}_1,\boldsymbol{\tau},\boldsymbol{e}_{2,t})-B_2(\boldsymbol{u}_h,\boldsymbol{e}_2,\boldsymbol{e}_{2,t})+b(\boldsymbol{e}_1,\boldsymbol{\tau},\boldsymbol{e}_{2,t})+b(\boldsymbol{u}_h,\boldsymbol{e}_2,\boldsymbol{e}_{2,t}).\end{cases}$$

利用假设 1~3,可以得到

$$\|\boldsymbol{\theta}_{1,t}\|^2\leqslant\frac{ch^{-1}}{\rho}\|\boldsymbol{\theta}_{1,t}\|\cdot\|\boldsymbol{e}_2\|+$$
$$ch^{-1}(\|\boldsymbol{u}_h\|+\|\boldsymbol{u}\|)\|\boldsymbol{\theta}_{1,t}\|\cdot\|\boldsymbol{e}_1\|+\|\boldsymbol{e}_{1,t}^*\|\cdot\|\boldsymbol{\theta}_{1,t}\|+ch^{-2}\|\boldsymbol{e}_2\|,$$

$$\|\boldsymbol{e}_{2,t}\|^2\leqslant\frac{ch^{-1}\eta}{\lambda}\|\boldsymbol{\theta}_1\|\cdot\|\boldsymbol{e}_{2,t}\|+\frac{1}{\lambda}\|\boldsymbol{e}_2\|\cdot\|\boldsymbol{e}_{2,t}\|+$$

$$\frac{c\varepsilon}{\eta_0}(\|\boldsymbol{\tau}\|+\|\boldsymbol{\tau}_h\|)\|\boldsymbol{e}_2\|\cdot\|\boldsymbol{e}_{2,t}\|+ch^{-1}[\|\boldsymbol{\tau}\|\cdot\|\boldsymbol{e}_1\|\cdot\|\boldsymbol{e}_{2,t}\|+\|\boldsymbol{u}_h\|\cdot\|\boldsymbol{e}_2\|\cdot\|\boldsymbol{e}_{2,t}\|+\|\boldsymbol{\theta}_1\|\cdot\|\boldsymbol{\tau}\|\cdot\|\boldsymbol{e}_{2,t}\|].$$

化简后得到

$$\|\boldsymbol{\theta}_{1,t}\|\leqslant\frac{ch^{-1}}{\rho}\|\boldsymbol{e}_2\|+ch^{-1}(\|\boldsymbol{u}_h\|+\|\boldsymbol{u}\|)\|\boldsymbol{e}_1\|+\|\boldsymbol{e}_{1,t}^{*}\|+ch^{-2}\|\boldsymbol{e}_2\|\leqslant\frac{ch^{-1}}{\rho}\|\boldsymbol{e}_2\|+ch^{-1}\|\boldsymbol{e}_1\|+\|\boldsymbol{e}_{1,t}^{*}\|+ch^{-2}\|\boldsymbol{e}_2\|,$$

$$\|\boldsymbol{e}_{2,t}\|\leqslant\frac{ch^{-1}\eta}{\lambda}\|\boldsymbol{\theta}_1\|+\frac{1}{\lambda}\|\boldsymbol{e}_2\|+\frac{c\varepsilon}{\eta_0}(\|\boldsymbol{\tau}\|+\|\boldsymbol{\tau}_h\|)\|\boldsymbol{e}_2\|+ch^{-1}[\|\boldsymbol{\tau}\|\cdot\|\boldsymbol{e}_1\|+\|\boldsymbol{u}_h\|\cdot\|\boldsymbol{e}_2\|+\|\boldsymbol{\theta}_1\|\cdot\|\boldsymbol{\tau}\|],$$

$$\|\boldsymbol{e}_{2,t}\|\leqslant\frac{ch^{-1}\eta}{\lambda}\|\boldsymbol{\theta}_1\|+\frac{1}{\lambda}\|\boldsymbol{e}_2\|+\frac{c_2\varepsilon}{\eta_0}\|\boldsymbol{e}_2\|+c_2h^{-1}[\|\boldsymbol{e}_1\|+\|\boldsymbol{\theta}_1\|]+c_2h^{-1}\|\boldsymbol{e}_2\|.$$

利用 $\|\boldsymbol{\theta}(t)\|^2\leqslant T\int_0^t\|\boldsymbol{\theta}_t\|^2\mathrm{d}t$, 可以得到

$$\|\boldsymbol{\theta}_1(t)\|\leqslant\left(\frac{ch^{-1}T^{3/2}}{\rho}+ch^{-2}T^{3/2}\right)\|\boldsymbol{e}_2\|+ch^{-1}T^{3/2}\|\boldsymbol{e}_1\|+cT^{3/2}h^{m},\tag{5}$$

$$\|\boldsymbol{e}_2(t)\|\leqslant\frac{ch^{-1}\eta T^{3/2}}{\lambda}\|\boldsymbol{\theta}_1\|+\left(\frac{1}{\lambda}+\frac{\varepsilon c_2}{\eta_0}+c_2h^{-1}\right)T^{3/2}\|\boldsymbol{e}_2\|+c_2h^{-1}T^{3/2}[\|\boldsymbol{e}_1\|+\|\boldsymbol{\theta}_1\|],\tag{6}$$

$$\left[1-\left(\frac{1}{\lambda}-\frac{\varepsilon c_2}{\eta_0}-c_2h^{-1}\right)T^{3/2}\right]\|\boldsymbol{e}_2(t)\|\leqslant\frac{c\eta h^{-1}T^{3/2}}{\lambda}\|\boldsymbol{\theta}_1\|+c_2h^{-1}T^{3/2}[\|\boldsymbol{e}_1\|+\|\boldsymbol{\theta}_1\|].\tag{7}$$

令 $$1-\left(\frac{1}{\lambda}-\frac{\varepsilon c_2}{\eta_0}-c_2h^{-1}\right)T^{3/2}=B.$$

将式(7)代入式(5), 并利用 $\|\boldsymbol{e}_1\|\leqslant\|\boldsymbol{\theta}_1\|+ch^{m}$, 得到

$$\|\boldsymbol{\theta}_1(t)\|\leqslant\frac{1}{B}\left(\frac{ch^{-1}T^{3/2}}{\rho}+ch^{-2}T^{3/2}\right)\left\{\frac{c\eta h^{-1}T^{3/2}}{\lambda}\|\boldsymbol{\theta}_1\|+c_2h^{-1}T^{3/2}[\|\boldsymbol{e}_1\|+\|\boldsymbol{\theta}_1\|]\right\}+ch^{-1}T^{3/2}\|\boldsymbol{e}_1\|+cT^{3/2}h^{m}\leqslant\frac{1}{B}\left(\frac{ch^{-1}T^{3/2}}{\rho}+ch^{-2}T^{3/2}\right)\left[\frac{c\eta h^{-1}T^{3/2}}{\lambda}\|\boldsymbol{\theta}_1\|+cT^{3/2}\|\boldsymbol{\theta}_1\|\right]+cT^{3/2}h^{m}+cT^{3/2}h^{m-1},$$

所以

$$\|\boldsymbol{\theta}_1(t)\|\leqslant\frac{cT^{3/2}h^{m}+cT^{3/2}h^{m-1}}{1-\frac{1}{B}\left(\frac{c\eta h^{-2}}{\rho\lambda}+\frac{c\eta h^{-3}}{\lambda}+\frac{ch^{-1}}{\rho}+ch^{-2}\right)T^{3}},$$

$$\|\boldsymbol{e}_1(t)\|\leqslant\|\boldsymbol{\theta}_1(t)\|+ch^{m}\leqslant$$

$$\frac{c_1T^{3/2}h^m + c_3T^{3/2}h^{m-1}}{1 - \frac{1}{B}\left(\frac{c\eta h^{-2}}{\rho\lambda} + \frac{c\eta h^{-3}}{\lambda} + \frac{ch^{-1}}{\rho} + ch^{-2}\right)T^3} + ch^m. \tag{8}$$

将式(8)代入式(7)可得

$$\|\boldsymbol{e}_2(t)\| \leqslant \left(1 + \frac{\eta}{\lambda}\right)\frac{cT^3h^{m-1} + cT^3h^{m-2}}{B - \left(\frac{c\eta h^{-2}}{\rho\lambda} + \frac{c\eta h^{-3}}{\lambda} + \frac{ch^{-1}}{\rho} + ch^{-2}\right)T^3} + \frac{cT^{3/2}h^{m-1}}{B}.$$

证毕.

3　对时间的 Euler 格式

对于应力、应变在时间 t 上的变化,采用差分格式,设 $0 = t_0 < t_1 < \cdots < t_N = T$,并记 $I_n = (t_{n-1}, t_n)$,$k_n = t_n - t_{n-1}$ 为时间步长, $k = \max k_n$.记 $\boldsymbol{U}^n$,$\boldsymbol{\tau}^n$ 为初边值问题真解 $\boldsymbol{u}(x,t)$,$\boldsymbol{\tau}(x,t)$ 在节点 $t_n(n = 1,2,\cdots,N)$ 的近似值, 并且对于初值

$$\|\boldsymbol{U}_h^0 - \boldsymbol{U}^0\| \leqslant ch^{m+1-l}\|\boldsymbol{U}^0\|_{m+1}, \qquad l = 0,1,$$
$$\|\boldsymbol{\tau}_h^0 - \boldsymbol{\tau}^0\| \leqslant ch^{m+1-l}\|\boldsymbol{\tau}^0\|_{m+1}, \qquad l = 0,1.$$

在时间上采用 Euler 向后差分格式, 求 $(\boldsymbol{U}^n, \boldsymbol{\tau}^n) \in (X_h, Y_h)$,$(n = 1,2,\cdots,N)$ 满足

$$\begin{cases}\left(\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k_n}, \boldsymbol{w}\right) = -\frac{1}{\rho}(\nabla\boldsymbol{w}, \boldsymbol{\tau}^n) + B_1(\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{w}) + Q(\boldsymbol{w}, \boldsymbol{\tau}^n), \\ \left(\frac{\boldsymbol{\tau}^n - \boldsymbol{\tau}^{n-1}}{k_n}, \boldsymbol{v}\right) = \frac{2\eta}{\lambda}(\boldsymbol{D}(\boldsymbol{U}^n), \boldsymbol{v}) - \frac{1}{\lambda}(\boldsymbol{\tau}^n, \boldsymbol{v}) - \\ \quad \frac{\varepsilon}{\eta_0}((\tau_{xx}^n + \tau_{yy}^n)\boldsymbol{\tau}^n, \boldsymbol{v}) - B_2(\boldsymbol{U}^n, \boldsymbol{\tau}^n, \boldsymbol{v}) + b(\boldsymbol{U}^n, \boldsymbol{\tau}^n, \boldsymbol{v}).\end{cases} \tag{9}$$

为了简化记法,下面的 $\boldsymbol{u} = \boldsymbol{u}(t_n)$,$\boldsymbol{\tau} = \boldsymbol{\tau}(t_n)$, 由式(3)和(9)可得

$$\begin{cases}\left(\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k_n} - \boldsymbol{u}_t, \boldsymbol{w}\right) = -\frac{1}{\rho}(\nabla\boldsymbol{w}, \boldsymbol{\tau}^n - \boldsymbol{\tau}) + \\ \quad B_1(\boldsymbol{U}^n - \boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) + B_1(\boldsymbol{U}^n, \boldsymbol{U}^n - \boldsymbol{u}, \boldsymbol{w}) + Q(\boldsymbol{w}, \boldsymbol{\tau}^n - \boldsymbol{\tau}), \\ \left(\frac{\boldsymbol{\tau}^n - \boldsymbol{\tau}^{n-1}}{k_n} - \boldsymbol{\tau}_t, \boldsymbol{v}\right) = \frac{2\eta}{\lambda}(\boldsymbol{D}(\boldsymbol{U}^n - \boldsymbol{u}), \boldsymbol{v}) - \frac{1}{\lambda}(\boldsymbol{\tau}^n - \boldsymbol{\tau}, \boldsymbol{v}) - \\ \quad \frac{\varepsilon}{\eta_0}((\tau_{xx}^n + \tau_{yy}^n)\boldsymbol{\tau}^n - (\tau_{xx} + \tau_{yy})\boldsymbol{\tau}, \boldsymbol{v}) + B_2(\boldsymbol{u} - \boldsymbol{U}^n, \boldsymbol{\tau}, \boldsymbol{v}) + \\ \quad B_2(\boldsymbol{U}^n, \boldsymbol{\tau} - \boldsymbol{\tau}^n, \boldsymbol{v}) + b(\boldsymbol{U}^n - \boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{v}) + b(\boldsymbol{U}^n, \boldsymbol{\tau}^n - \boldsymbol{\tau}, \boldsymbol{v}).\end{cases}$$

定理 2　设 $\boldsymbol{u}$,$\boldsymbol{\tau}$ 为方程的精确解, 而 $\{\boldsymbol{U}^n, \boldsymbol{\tau}^n\}$ 是式(4)定义的近似解, 则有后估计:

$$\left(q^n - \frac{c(h^{-2}k^2 + \rho kh^{-3})(\lambda + \eta)}{(1 - c_1h^{-1}k)\rho\lambda}\right)\|\boldsymbol{\tau}^n - \boldsymbol{\tau}(t_n)\| \leqslant$$
$$ck\int_0^{t_n}\|\boldsymbol{\tau}_{tt}(s)\|\mathrm{d}s + \frac{ch^{-1}k(\lambda + \eta)}{(1 - c_1h^{-1}k)\lambda}\left(k\int_0^{t_n}\|\boldsymbol{u}_{tt}(s)\|\mathrm{d}s + ch^{m+1}\|\boldsymbol{u}^0\|_{m+1}\right) +$$
$$ch^{m+1}\|\boldsymbol{\tau}^0\|_{m+1},$$
$$\left((1 - c_1h^{-1}k)^n - \frac{ch^{-2}k^2(\lambda + \eta)}{\rho q\lambda}\right)\|\boldsymbol{U}^n - \boldsymbol{u}(t_n)\| \leqslant$$

$$ck\int_0^{t_n}\|\boldsymbol{u}_{tt}(s)\|\mathrm{d}s+\frac{c(h^{-1}k+\rho h^{-2})}{\rho q}\left(k\int_0^{t_n}\|\boldsymbol{\tau}_{tt}(s)\|\mathrm{d}s+ch^{m+1}\|\boldsymbol{\tau}^0\|_{m+1}\right)+ch^{m+1}\|\boldsymbol{u}^0\|_{m+1},$$

其中 c_1 满足 $c_1h^{-1}k<1$.

证明 令 $\boldsymbol{e}^n=\boldsymbol{U}^n-\boldsymbol{u}(t_n)$, $\boldsymbol{\theta}^n=\boldsymbol{\tau}^n-\boldsymbol{\tau}(t_n)$, 引入差分格式:

$$\bar{\partial}_t\boldsymbol{e}^n=\frac{\boldsymbol{e}^n-\boldsymbol{e}^{n-1}}{k_n},\ \bar{\partial}_t\boldsymbol{\theta}^n=\frac{\boldsymbol{\theta}^n-\boldsymbol{\theta}^{n-1}}{k_n}.$$

首先考虑 Cauchy 方程

$$(\bar{\partial}_t\boldsymbol{e}^n,\boldsymbol{w})=-\frac{1}{\rho}(\nabla\boldsymbol{w},\boldsymbol{\tau}^n-\boldsymbol{\tau})+B_1(\boldsymbol{U}^n-\boldsymbol{u},\boldsymbol{u},\boldsymbol{w})+B_1(\boldsymbol{U}^n,\boldsymbol{U}^n-\boldsymbol{u},\boldsymbol{w})+Q(\boldsymbol{w},\boldsymbol{\tau}^n-\boldsymbol{\tau})-(\boldsymbol{R}^n,\boldsymbol{w}),$$

其中 $$\boldsymbol{R}^n=\frac{\boldsymbol{u}(t_n)-\boldsymbol{u}(t_{n-1})}{k_n}-\boldsymbol{u}_t(t_n)=\bar{\partial}_t\boldsymbol{u}(t_n)-\boldsymbol{u}_t(t_n).$$

令 $\boldsymbol{w}=\boldsymbol{e}^n$, 由于 $(\bar{\partial}_t\boldsymbol{e}^n,\boldsymbol{e}^n)=(\|\boldsymbol{e}^n\|^2-(\boldsymbol{e}^{n-1},\boldsymbol{e}^n))/k_n$, 利用定理 1 类似的过程, 可以得到

$$\|\boldsymbol{e}^n\|^2\leqslant\|\boldsymbol{e}^{n-1}\|\|\boldsymbol{e}^n\|+k_n\|\boldsymbol{R}^n\|\|\boldsymbol{e}^n\|+\frac{c_1h^{-1}k_n}{\rho}\|\boldsymbol{e}^n\|\|\boldsymbol{\theta}^n\|+c_1h^{-1}k_n\|\boldsymbol{e}^n\|^2+ch^{-2}\|\boldsymbol{\theta}^n\|\|\boldsymbol{e}^n\|,$$

这里的常数 c_1 满足 $c_1h^{-1}k<1$, 即

$$\|\boldsymbol{e}^n\|\leqslant\|\boldsymbol{e}^{n-1}\|+k_n\|\boldsymbol{R}^n\|+\frac{c_1h^{-1}k_n}{\rho}\|\boldsymbol{\theta}^n\|+c_1h^{-1}k_n\|\boldsymbol{e}^n\|+ch^{-2}\|\boldsymbol{\theta}^n\|,$$

$$(1-c_1h^{-1}k)\|\boldsymbol{e}^n\|-\|\boldsymbol{e}^{n-1}\|\leqslant k_n\|\boldsymbol{R}^n\|+\frac{c_1h^{-1}k_n+c\rho h^{-2}}{\rho}\|\boldsymbol{\theta}^n\|.$$

逐步递推可得

$$(1-c_1h^{-1}k)^n\|\boldsymbol{e}^n\|-\|\boldsymbol{e}^0\|\leqslant\sum_{j=1}^n(1-c_1h^{-1}k)^{j-1}k_j\|\boldsymbol{R}^j\|+\sum_{j=1}^n(1-c_1h^{-1}k)^{j-1}\frac{c_1h^{-1}k_n+c\rho h^{-2}}{\rho}\|\boldsymbol{\theta}^j\|\leqslant\sum_{j=1}^n k_j\|\boldsymbol{R}^j\|+\frac{c_1h^{-1}k_n+c\rho h^{-2}}{\rho}\|\boldsymbol{\theta}^n\|.$$

由于

$$\boldsymbol{R}^j=\frac{\boldsymbol{u}(t_j)-\boldsymbol{u}(t_{j-1})}{k_j}-\boldsymbol{u}_t(t_j)=-k_j^{-1}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\boldsymbol{u}_{tt}(s)\mathrm{d}s,$$

$$\sum_{j=1}^n k_j\|\boldsymbol{R}^j\|\leqslant\sum_{j=1}^n\left\|\int_{t_{j-1}}^{t_j}(s-t_{j-1})\boldsymbol{u}_{tt}(s)\mathrm{d}s\right\|\leqslant ck\int_0^{t_n}\|\boldsymbol{u}_{tt}(s)\|\mathrm{d}s,$$

所以

$$(1-c_1h^{-1}k)^n\|\boldsymbol{e}^n\|-\|\boldsymbol{e}^0\|\leqslant ck\int_0^{t_n}\|\boldsymbol{u}_{tt}(s)\|\mathrm{d}s+\frac{c_1h^{-1}k_n+c\rho h^{-2}}{\rho}\|\boldsymbol{\theta}^n\|. \tag{10}$$

最后研究 PTT 方程的后估计:

$$(\bar{\partial}_t\boldsymbol{\theta}^n,\boldsymbol{v})=\frac{2\eta}{\lambda}(\boldsymbol{D}(\boldsymbol{e}^n),\boldsymbol{v})-\frac{1}{\lambda}(\boldsymbol{\theta}^n,\boldsymbol{v})-\frac{\varepsilon}{\eta_0}((\tau_{xx}^n+\tau_{yy}^n)\boldsymbol{\tau}^n-(\tau_{xx}+\tau_{yy})\boldsymbol{\tau},\boldsymbol{v})+$$

$$B_2(\boldsymbol{u}-\boldsymbol{U}^n,\boldsymbol{\tau},\boldsymbol{v})+B_2(\boldsymbol{U}^n,\boldsymbol{\tau}-\boldsymbol{\tau}^n,\boldsymbol{v})+b(\boldsymbol{U}^n-\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{v})+$$
$$b(\boldsymbol{U}^n,\boldsymbol{\tau}^n-\boldsymbol{\tau},\boldsymbol{v})-(\boldsymbol{R}^n,\boldsymbol{w}),$$

其中
$$\boldsymbol{R}^n=\frac{\boldsymbol{\tau}(t_n)-\boldsymbol{\tau}(t_{n-1})}{k_n}-\boldsymbol{\tau}_t(t_n)=\bar{\partial}_t\boldsymbol{\tau}(t_n)-\boldsymbol{\tau}_t(t_n).$$

令 $\boldsymbol{v}=\boldsymbol{\theta}^n$ 由于$(\bar{\partial}_t\boldsymbol{\theta}^n,\boldsymbol{\theta}^n)=(\|\boldsymbol{\theta}^n\|^2-(\boldsymbol{\theta}^{n-1},\boldsymbol{\theta}^n))/k_n$，利用定理1类似的过程，可以得到

$$\frac{1}{k_n}\|\boldsymbol{\theta}^n\|^2\leqslant\frac{c_2\eta h^{-1}}{\lambda}\|\boldsymbol{e}^n\|\,\|\boldsymbol{\theta}^n\|+\left(\frac{c_2\varepsilon}{\eta_0}-\frac{1}{\lambda}\right)\|\boldsymbol{\theta}^n\|^2+$$
$$c_2h^{-1}(\|\boldsymbol{e}^n\|+\|\boldsymbol{\theta}^n\|)\|\boldsymbol{\theta}^n\|+\frac{1}{k_n}\|\boldsymbol{\theta}^n\|\,\|\boldsymbol{\theta}^{n-1}\|+\|\boldsymbol{\theta}^n\|\,\|\boldsymbol{R}^n\|,$$

$$\|\boldsymbol{\theta}^n\|\leqslant k_n\left(\frac{c_2\eta h^{-1}}{\lambda}\|\boldsymbol{e}^n\|+\left(\frac{c_2\varepsilon}{\eta_0}-\frac{1}{\lambda}\right)\|\boldsymbol{\theta}^n\|+\right.$$
$$\left.c_2h^{-1}(\|\boldsymbol{e}^n\|+\|\boldsymbol{\theta}^n\|)+\|\boldsymbol{R}^n\|\right)+\|\boldsymbol{\theta}^{n-1}\|.$$

整理得到

$$\left(1-k_n\left(\frac{c_2\varepsilon}{\eta_0}-\frac{1}{\lambda}+c_2h^{-1}\right)\right)\|\boldsymbol{\theta}^n\|-\|\boldsymbol{\theta}^{n-1}\|\leqslant$$
$$k_n\|\boldsymbol{R}^n\|+\left(c_2h^{-1}+\frac{c_2\eta h^{-1}}{\lambda}\right)k_n\|\boldsymbol{e}^n\|,$$

这里
$$k\left(\frac{c_2\varepsilon}{\eta_0}-\frac{1}{\lambda}+c_2h^{-1}\right)<1.$$

令
$$q=1-k_n\left(\frac{c_2\varepsilon}{\eta_0}-\frac{1}{\lambda}+c_2h^{-1}\right),$$

逐步递推可得

$$q^n\|\boldsymbol{\theta}^n\|-\|\boldsymbol{\theta}^0\|\leqslant$$
$$\sum_{j=1}^n q^{j-1}k_j\|\boldsymbol{R}^j\|+\sum_{j=1}^n q^{j-1}\left(c_2h^{-1}+\frac{c_2\eta h^{-1}}{\lambda}\right)k_n\|\boldsymbol{e}^j\|\leqslant$$
$$\sum_{j=1}^n k_j\|\boldsymbol{R}^j\|+ch^{-1}\left(1+\frac{\eta}{\lambda}\right)k\|\boldsymbol{e}^n\|.$$

由于

$$\boldsymbol{R}^j=\frac{\boldsymbol{\tau}(t_j)-\boldsymbol{\tau}(t_{j-1})}{k_j}-\boldsymbol{\tau}_t(t_j)=-k_j^{-1}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\boldsymbol{\tau}_{tt}(s)\,\mathrm{d}s,$$

$$\sum_{j=1}^n k_j\|\boldsymbol{R}^j\|\leqslant\sum_{j=1}^n\left\|\int_{t_{j-1}}^{t_j}(s-t_{j-1})\boldsymbol{\tau}_{tt}(s)\,\mathrm{d}s\right\|\leqslant ck\int_0^{t_n}\|\boldsymbol{\tau}_{tt}(s)\|\,\mathrm{d}s,$$

所以

$$q^n\|\boldsymbol{\theta}^n\|-\|\boldsymbol{\theta}^0\|\leqslant ck\int_0^{t_n}\|\boldsymbol{\tau}_{tt}(s)\|\,\mathrm{d}s+ch^{-1}\left(1+\frac{\eta}{\lambda}\right)k\|\boldsymbol{e}^n\|.\tag{11}$$

由式(10)和(11)可得后估计：

$$\left(q^n-\frac{c(h^{-2}k^2+\rho kh^{-3})(\lambda+\eta)}{(1-c_1h^{-1}k)\rho\lambda}\right)\|\boldsymbol{\theta}^n\|-\|\boldsymbol{\theta}^0\|\leqslant$$

$$ck\int_0^{t_n} \| \boldsymbol{\tau}_{tt}(s) \| \mathrm{d}s + \frac{ch^{-1}k(\lambda + \eta)}{(1 - c_1 h^{-1}k)\lambda}\left(k\int_0^{t_n} \| \boldsymbol{u}_{tt}(s) \| \mathrm{d}s + \| \boldsymbol{e}^0 \| \right),$$

$$\left((1 - c_1 h^{-1}k)^n - \frac{ch^{-2}k^2(\lambda + \eta)}{\rho q \lambda}\right) \| \boldsymbol{e}^n \| - \| \boldsymbol{e}^0 \| \leqslant$$

$$ck\int_0^{t_n} \| \boldsymbol{u}_{tt}(s) \| \mathrm{d}s + \frac{c(h^{-1}k + \rho h^{-2})}{\rho q}\left(k\int_0^{t_n} \| \boldsymbol{\tau}_{tt}(s) \| \mathrm{d}s + \| \boldsymbol{\theta}^0 \| \right).$$

证毕.

4　数值试验

采用九点双二次型 Lagrange 形函数作为有限元基函数来求解耦合方程组,在时间上使用 Euler 差分格式进行数值计算.由于该耦合方程组的复杂性,我们不能得到它的精确解.利用本文的算法给出 $\| \boldsymbol{U}^n - \boldsymbol{U}^{n-1} \|$ 的后估计数值结果,见下表 1 所示.

表 1　误差分析

Table 1　Error analysis

sample	number of iterations n				
	6	7	8	10	11
1	0.000 337 08	0.000 229 98	0.000 267 6	0.000 574 29	0.000 375 69
2	8.820 8E-005	4.559 5E-005	4.618 1E-005	8.098 5E-005	4.876 E-005
3	-5.069 3E-005	-2.056E-005	-1.841 2E-005	-3.078 2E-005	-1.367 6E-005
4	8.745 6E-006	6.674E-005	0.000 113 58	0.000 233 98	0.000 310 4
5	5.997 8E-005	6.943 7E-005	7.442 8E-005	9.143 7E-005	9.169 1E-005
6	-3.548 6E-005	-3.389 5E-005	-3.011 4E-005	-2.855 5E-005	-2.046E-005
7	-1.852 4E-005	4.107 8E-006	3.417 5E-005	3.982 3E-005	0.000 134 88
8	3.515 8E-005	3.739 4E-005	4.082 2E-005	3.155 2E-005	5.529 5E-005
9	-2.148E-005	-2.372 9E-005	-2.422 6E-005	-1.539 7E-005	-2.88E-005

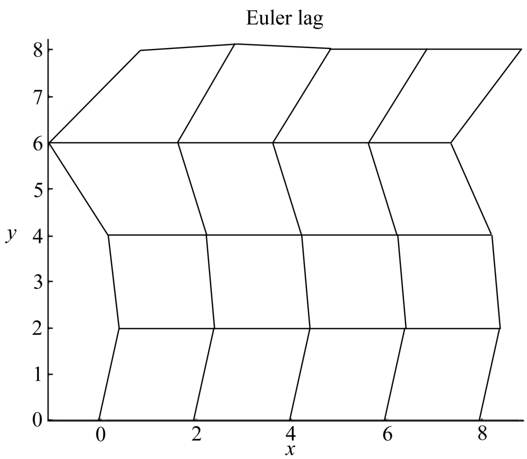

图 1　网格变形结果

Fig.1　Mesh deformation result

由上表可以看出,误差都小于 0.001.根据上面的理论,误差到达 3 阶精度,理论结果与数据实验结果相符合.增加网格数量,能更加直观地看出网格的变形.图 1 是时间步长 $\Delta t = 1$,速度值从 $U = 0.04$ 变到 $U = 0$ 时,网格的最终形态.见图 1.

5 总　　结

本文使用宏观尺度方法研究非 Newton 流变问题.使用非线性偏微分耦合方程组描述非 Newton 流域的形变和应力变化,该方程组使用半离散有限元方法求解.从本文可以看到,解的整体光滑性不仅依赖于初值和边值的光滑性,以及它们在连接面上的高阶协调性,而且这些数据的不光滑性能传播到区域内部,解没有磨光性质.由于这些光滑和协调条件一般不容易满足,因此解的光滑性一般不高.在一定光滑性条件下,可以获得它的高性能计算收敛性估计.

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Convergence of Finite Element Method in Rheology

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Abstract: Convergence of the first-order mixed-type hyperbolic parabola partial differential equations in non-Newtonian fluid problems was studied. The coupling partial differential equations (Cauchy fluid equation, P-T/T stress equation) were used to simulate the flow zone generated by the free surface elements or excessively tensile elements. The semi-discrete finite element method was applied to solve these equations coupling with time. The finite element method was used in space. The trilinear functional was employed to solve the nonlinear problems of partial differential equations. In the time domain the Euler scheme was adopted. The convergence order of the equation set reached $O(h^2 + \Delta t)$. Numerical results of the equations were obtained through priori and posteriori error estimation of high performance computation. And the deformed sizes of the grids were presented.

Key words: non-Newtonian fluid; semi-discrete finite element; coupling equations; convergence

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