

基于广义函数空间的不连续梁振动分析*

陈小超, 毛崎波, 薛晓理

(南昌航空大学 飞行器工程学院, 南昌 330063)

摘要: 首先运用广义函数建立了轴向力作用下含任意不连续点的弹性基础 Euler(欧拉)梁的自由振动的统一微分方程.不连续点的影响由广义函数(Dirac delta 函数)引入梁的振动方程.微分方程运用 Laplace 变换方法求解;与传统方法不同的是,该文方法求得的模态函数为整个不连续梁的一般解.由于模态函数的统一化以及连续条件的退化,特征值的求解得到了极大地简化.最后,以梁-质量块模型和轴向力作用下弹性基础裂纹梁模型为例验证了该文方法的正确性与有效性.

关键词: 自由振动; 广义函数; 轴向力; 弹性基础; 不连续梁

中图分类号: O32; TB123; O29 **文献标志码:** A

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引言

随着现代土木和机械工程结构的大型化与复杂化,不连续梁式结构得到越来越多的应用;同时,结构发生破坏也会使结构产生不连续的工况.对这些不连续梁进行振动分析,对保证结构完整性和安全性具有越来越重要的意义.

常见的不连续梁求解的数值方法有有限元法^[1-2]、微分求积法^[3]、Adomian 分解法^[4-7]等.含不连续点的梁振动分析解析方法,较早期也是最常用的振动模型分析方法称为离散子结构法^[8].这种分析方法将梁从不连续点处离散,每一段子梁均为连续梁并满足连续梁的运动微分方程;求解每段连续子梁后利用连续条件和边界条件组装成超越方程组,然后求解特征方程.对于一个含 n 个不连续点的梁,每段子梁的位移函数均包括 4 个未知常数,特征值的求解依赖于解一个 $4(n+1)$ 阶的特征矩阵. Shifrin 等^[9]提出光滑函数方法将特征值矩阵的阶数降为 $(n+1)$,然而该方法仅适用于裂纹这一种不连续点,且计算难度仍然受裂纹数量的影响.由 Khiem 和 Lien^[10]提出的基于传递矩阵的方法,建立每段子梁左右两端状态变量之间的关系,然后运用子梁序列的传递矩阵将整个梁左右边界进行联系,进而将特征矩阵的阶数降为 4 阶. Li^[11-12]提出一种新颖的、利用梁的线性无关特解逐段构造每一段子梁的基本解,每一段子梁的模态形状表达可以通过前一段的表达式用递归方法得出. Binici^[13]随后将这一方法扩展到受轴向力作用下的梁.

上述方法或计算较为复杂(如离散子空间法),或仅仅适用一种特定的不连续点类型(如光滑函数法);并且当含有多个不连续点时,模态解析表达式相当复杂^[14].因此,一种求解模态

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作者简介: 陈小超(1988—),男,重庆人,硕士生(通讯作者. E-mail: keithisxc@gmail.com).

简单并能包含任意不连续点的分析方法显得尤为重要。近来, Yavari 等^[15-16]运用分布理论研究了含多个不连续点的梁的静响应, 结论显示, 广义函数可以为不连续函数提供一种有效的处理方法。整个不连续梁的静位移响应可以用一个统一的位移函数表述, 并对包括外力、变抗弯刚度、横向位移、弹性铰接等多种类型不连续点进行了研究。Wang 等^[17-18]将分布理论运用到振动分析中, 通过在每个不连续点引入一个基本的模态位移函数, 使整个梁的位移模态函数可以通过递归方法得到一个统一的表达式。本文在文献[17]的基础上, 首先利用 Heaviside 函数和 Dirac 函数引入不连续点的影响; 进一步运用分布理论在广义函数空间内建立 Winkler 弹性基础上轴向力作用下含任意不连续点均匀 Euler 梁的微分方程; 然后运用 Laplace 变换方法求解微分方程。由于广义函数的引入, 梁的模态函数可以通过一个一般解的形式表示。最后通过数值模拟和对比, 验证了方法的有效性。

1 理论模型

1.1 轴向力作用下含多个不连续点的弹性基础梁

考虑一任意边界条件下支撑在刚度为 k 的 Winkler 基础上的等截面梁, 含 n 个不连续点, 其两端受轴向力 P 作用, 如图 1 所示。

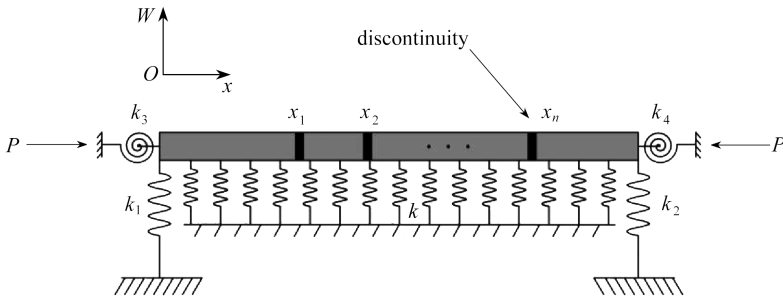


图 1 含不连续点的轴向力作用下的弹性基础 Euler 梁

Fig.1 An elastic foundation Euler beam with multiple discontinuities subject to axial load

由文献[3]可知图 1 中的连续梁微分方程可写为

$$W^{(4)}(x, t) + \frac{P}{EI} W^{(2)}(x, t) + \frac{\rho A}{EI} \ddot{W}(x, t) + \frac{k}{EI} W(x, t) = 0, \quad (1)$$

式中, $W(x, t)$ 为梁横向位移, P 为轴向力, E 为弹性模量, I 为梁截面惯性矩, A 为梁横截面面积, ρ 为密度, k 为 Winkler 弹性基础系数。对于第 i 个不连续点, 该不连续位置两侧梁的微分方程可表示为

$$W_m^{(4)}(x, t) + \frac{P}{EI} W_m^{(2)}(x, t) + \frac{\rho A}{EI} \ddot{W}_m(x, t) + \frac{k}{EI} W_m(x, t) = 0, \quad (2)$$

式中, 当 $m = i$ 时为不连续点左侧梁, 当 $m = i + 1$ 时为不连续点右侧梁。

根据分布理论^[15], 梁横向位移可表示为

$$W(x, t) = W_1(x, t) + \sum_{i=1}^n \Delta W(x_i, t) H(x - x_i), \quad (3)$$

式中, $H(x - x_i)$ 为 Heaviside 函数, 且有

$$\Delta W(x_i, t) = W_{i+1}(x_i, t) - W_i(x_i, t). \quad (4)$$

对式(3)两端分别以位移 x 为变量求 4 次偏导, 可得

$$W^{(4)}(x, t) = W_1^{(4)}(x, t) + \sum_{i=1}^n \left(\Delta W^{(4)}(x_i, t) H(x - x_i) + \sum_{s=0}^3 \Delta W^{(s)}(x_i, t) \delta^{(3-s)}(x - x_i) \right), \quad (5)$$

式中, δ 为 Dirac 函数.

把式(2)代入式(3)化简后可得

$$W_1^{(4)}(x, t) + \sum_{i=1}^n \Delta W^{(4)}(x_i, t) H(x - x_i) = -\frac{P}{EI} W^{(2)}(x, t) - \frac{\rho A}{EI} \ddot{W}(x, t) - \frac{k}{EI} W(x, t). \quad (6)$$

进一步把式(6)代入式(5)右端, 则式(5)可重新表示为

$$W^{(4)}(x, t) + \frac{P}{EI} W^{(2)}(x, t) + \frac{\rho A}{EI} \ddot{W}(x, t) + \frac{k}{EI} W(x, t) = \sum_{i=1}^n \sum_{s=0}^3 \Delta W^{(s)}(x_i, t) \delta^{(3-s)}(x - x_i). \quad (7)$$

式(7)即为轴向力作用下含 n 个不连续点的弹性基础 Euler 梁的自由振动方程.

根据振动分析理论^[19]可知, $W(x, t)$ 可写为如下分离变量形式:

$$W(x, t) = W(x) e^{j\omega t}, \quad (8)$$

式中, $j = \sqrt{-1}$, ω 为圆频率, $W(x)$ 为梁的模态函数.

把式(8)代入式(7), 进行变量分离, 并表示为无量纲形式:

$$W^{(4)}(x) + \gamma W^{(2)}(x) - \beta^4 W(x) = \sum_{i=1}^n \sum_{s=0}^3 \Delta W^{(s)}(x_i, t) \delta^{(3-s)}(x - x_i), \quad (9)$$

式中, $\gamma = P/EI$ 为无量纲轴向力; $\beta^4 = (\rho A \omega^2 - k)/EI$.

注意到式(9)为一个 4 阶常微分方程, 且包含有广义函数 δ , 直接求解有一定困难. 文献 [7] 指出, 通过 Laplace 变换可将求解高阶微分方程问题转化为求解变量为 s 的代数方程, 并能很好地处理广义函数.

首先对式(9)进行 Laplace 变换, 得

$$W(s) = \frac{s^3 + \gamma s}{s^4 + \gamma s^2 - \beta^4} W(0) + \frac{s^2 + \gamma}{s^4 + \gamma s^2 - \beta^4} W^{(1)}(0) + \frac{s}{s^4 + \gamma s^2 - \beta^4} W^{(2)}(0) + \frac{1}{s^4 + \gamma s^2 - \beta^4} W^{(3)}(0) + \sum_{i=1}^n \frac{s^3 e^{-sx_i}}{s^4 + \gamma s^2 - \beta^4} \Delta W(x_i) + \frac{s^2 e^{-sx_i}}{s^4 + \gamma s^2 - \beta^4} \Delta W^{(1)}(x_i) + \frac{s e^{-sx_i}}{s^4 + \gamma s^2 - \beta^4} \Delta W^{(2)}(x_i) + \frac{e^{-sx_i}}{s^4 + \gamma s^2 - \beta^4} \Delta W^{(3)}(x_i). \quad (10)$$

然后再对式(10)进行 Laplace 逆变换, 即可得到式(9)的解 $W(x)$:

$$W(x) = (S_0(x) + \gamma S_2(x)) W(0) + (S_1(x) + \gamma S_3(x)) W'(0) + S_2(x) W''(0) + S_3(x) W'''(0) + \sum_{i=1}^n \sum_{s=0}^3 (S_s(x - x_i) \Delta W^{(s)}(x_i)) H(x - x_i), \quad (11)$$

式中

$$S_3(x) = L^{-1} \left(\frac{1}{s^4 + \gamma s^2 - \beta^4} \right) = \sum_{i=1}^4 \frac{\exp(z_i x)}{2z_i(2z_i^2 + \gamma)},$$

$$S_2(x) = L^{-1} \left(\frac{s}{s^4 + \gamma s^2 - \beta^4} \right) = \sum_{i=1}^4 \frac{\exp(z_i x)}{2(2z_i^2 + \gamma)},$$

$$S_1(x) = L^{-1} \left(\frac{s^2}{s^4 + \gamma s^2 - \beta^4} \right) = \sum_{i=1}^4 \frac{z_i \exp(z_i x)}{2(2z_i^2 + \gamma)},$$

$$S_0(x) = L^{-1} \left(\frac{s^3}{s^4 + \gamma s^2 - \beta^4} \right) = \sum_{i=1}^4 \frac{z_i^2 \exp(z_i x)}{2(2z_i^2 + \gamma)},$$

$$z_1 = -z_2 = \sqrt{\frac{\sqrt{4\beta^4 + \gamma^2}}{2} - \frac{\gamma}{2}},$$

$$z_3 = -z_4 = \sqrt{-\frac{\sqrt{4\beta^4 + \gamma^2}}{2} - \frac{\gamma}{2}}.$$

1.2 不连续点与边界条件分析

由式(11)可知,梁模态函数表达式 $W(x)$ 中包含 $4n + 5$ 个未知数,其中 $W^{(j)}(0)$ 由边界条件决定,而 $\Delta W^{(j)}(x_i)$ 由梁所包含的不连续点决定.根据振动分析理论可知, $\Delta W(x_i)$, $\Delta W^{(1)}(x_i)$, $\Delta W^{(2)}(x_i)$, $\Delta W^{(3)}(x_i)$ 分别表示梁在 x_i 处的位移、转角、弯矩、剪力.因此,式(11)适用于任意边界条件下包含任意类型的不连续点(或不同类型不连续点的组合)的梁的自由振动分析.

1.2.1 不连续点

一般来说,结构包含的不连续点可分为两类:1)附加不连续点,如附加质量等;2)内部不连续点,如裂纹或内部弹性链接等.上述两类不连续点均可通过梁协调条件表示.为简便起见,本文在以下的计算中只给出附加质量块和裂纹两种不连续情况(如表1所示).但是需要指出的是,本文方法适用于任意类型不连续点,如裂纹、附加质量块、多跨梁弹性(或刚性)支撑点等.

表1中, x_i 为不连续点位置; $\alpha = M/\rho AL$ 表示无量纲附加质量; $\lambda^4 = \rho A \omega^2/EI$ 为无量纲固有频率; J 为质量块转动惯量, $J = Mr^2$, r 为质量块重心到梁中性层距离; θ 为裂纹简化成的弹簧的无量纲柔度, $\theta = 5.034 6hG(\zeta)$, 其中 $\zeta = h_c/h$ 表示裂纹的无量纲相对深度, h 为梁的厚度, h_c 为裂纹深度,

$$G(\zeta) = 1.862 4\zeta^2 - 3.95\zeta^3 + 16.37\zeta^4 - 34.226\zeta^5 + 76.81\zeta^6 - 126.9\zeta^7 + 172\zeta^8 - 143.97\zeta^9 + 66.56\zeta^{10}.$$

表1 质量块和裂纹连续条件

Table 1 The compatibility conditions of mass and crack

	the classical condition	the presented condition
point mass (moment of inertia due to eccentricity)	$W_i(x_i) = W_{i+1}(x_i)$	$\Delta W(x_i) = 0$
	$W'_i(x_i) = W'_{i+1}(x_i)$	$\Delta W'(x_i) = 0$
	$EIW''_i(x_i, t) + J\dot{W}'_i(x_i, t) = EIW''_{i+1}(x_i, t)$	$\Delta W''(x_i) = -\frac{J\omega^2 W'(x_i)}{EI}$
	$W''_i(x_i) + \alpha\lambda^4 W_i(x_i) = W''_{i+1}(x_i)$	$\Delta W'''(x_i) = \frac{M\omega^2}{EI} W(x_i)$
crack	$W_i(x_i) = W_{i+1}(x_i)$	$\Delta W(x_i) = 0$
	$W'_i(x_i) + \theta W''_i(x_i) = W'_{i+1}(x_i)$	$\Delta W'(x_i) = \theta W''(x_i)$
	$W''_i(x_i) = W''_{i+1}(x_i)$	$\Delta W''(x_i) = 0$
	$W'''_i(x_i) = W'''_{i+1}(x_i)$	$\Delta W'''(x_i) = 0$

1.2.2 边界条件

图1所示任意边界条件可写为如下形式:

$$\begin{cases} W'''(0) = -\frac{k_1}{EI}W(0) + \gamma W'(0), W''(0) = \frac{k_3}{EI}W'(0), \\ W'''(L) = \frac{k_2}{EI}W(L) - \gamma W'(L), W''(L) = -\frac{k_4}{EI}W'(L). \end{cases} \quad (12)$$

1.3 特征值与模态求解

如前文所述,含 n 个不连续点的梁位移函数表达式(11)中包含 $4n + 5$ 个未知数.将边界条件和连续条件代入式(11),即可得到 $4n + 4$ 个方程组成的线性方程组:

$$\begin{bmatrix} a_{11}(\omega) & \cdots & a_{1\xi}(\omega) \\ \vdots & \ddots & \vdots \\ a_{\xi 1}(\omega) & \cdots & a_{\xi\xi}(\omega) \end{bmatrix} [W^{(s)}(x_1) \cdots W^{(s)}(x_i) \cdots W^{(s)}(x_\xi)]^T = \mathbf{0}, \quad (13)$$

式中 $\xi = 4n + 4$; x_i 为边界和不连续点位置.

式(13)是一个齐次代数方程组,且系数中包含有梁的固有频率 ω .式(13)存在解的条件为系数矩阵行列式为 0,由此可解出梁的固有频率 ω ,将频率 ω 代回式(13)即可解得 $W^{(s)}(x_i)$.将解得的 ω 和 $W^{(s)}(x_i)$ 代回式(11)即可得不连续梁模态函数 $W(x)$.

需要指出的是,对于某些不连续点类型,方程组系数矩阵阶数远小于 $4n + 4$.以含 n 个不连续点为例,若其中含 p 个质量块, q 个裂纹, $p + q = n$;由于连续条件中 $\Delta W^{(j)}(x_i)$ 相当一部分为 0,其系数矩阵阶数则缩减为 $2p + q + 4$;若边界条件为经典边界,则系数矩阵阶数进一步缩减为 $2p + q + 2$.

2 数值模拟

为了验证本文方法的有效性和准确性,本节选取经典的不连续梁模型进行数值模拟.

2.1 算例 1

含有附加质量块的梁的振动分析已经被广泛的研究,Maiz 等^[20]所建立的振动分析模型中考虑了质量块转动惯量,使分析结果更加精确.图 2 为含两个附加质量块的固支梁模型.图中梁长度为 L ,两质量块大小与位置分别为 M_1, M_2 及 x_1, x_2 .

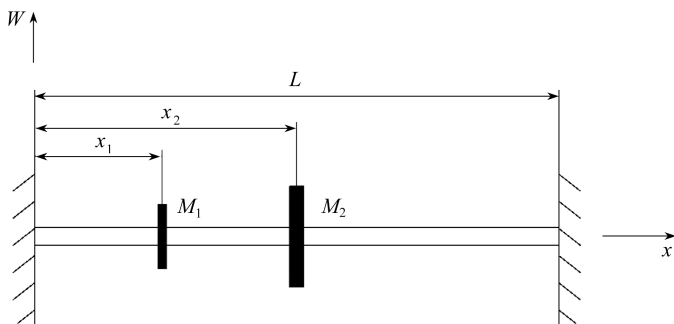


图 2 含两质量块固支梁

Fig.2 A clamped-clamped beam with 2 masses

由式(11)及表 1 中质量块连续条件可知,图 2 中梁的横向位移函数可写为

$$W(x) = S_2(x)W'''(0) + S_3(x)W''(0) + \sum_{i=1}^2 \left(S_3(x - x_i) \frac{M\omega^2}{EI} W(x_i) - S_2(x - x_i) \frac{J\omega^2 W'(x_i)}{EI} \right) H(x - x_i). \quad (14)$$

由式(14)及边界条件即可得出梁振动的特征值矩阵(推导过程见附录):

$$\bar{\mathbf{A}} = \begin{bmatrix} a_{11}(\omega) & \cdots & a_{16}(\omega) \\ \vdots & \ddots & \vdots \\ a_{61}(\omega) & \cdots & a_{66}(\omega) \end{bmatrix}. \quad (15)$$

求解 $|\bar{\mathbf{A}}| = 0$ 即可解得固有频率,进一步可求得模态形状.为便于比较,将固有频率无量纲化并设置与文献[19]相同的参数,即质量块偏心距: $C_i = r_i/L$; 质量块位置: $\eta_i = x_i/L$; 质量块比: $\alpha_i = M_i/\rho AL$.

表 2 表示在不同附加质量和偏心距下,两端固支梁的前 5 阶固有频率.图 3 与图 4 表示不同质量块和偏心距时的第 2、3 阶模态形状.从表 2 可以发现,本文的计算结果与文献[19]完全吻合.这就验证了本文方法的正确性.

表 2 含两质量块固支梁前 5 阶无量纲固有频率

Table 2 First 5 non-dimensional eigen frequencies of the clamped-clamped beam with 2 masses of different uniform magnitudes and different uniform eccentricities

		$\eta_1 = 0.25; \eta_2 = 0.5$				
		$C_1 = C_2 = 0$	$C_1 = C_2 = 0.025$	$C_1 = C_2 = 0.05$	$C_1 = C_2 = 0.075$	$C_1 = C_2 = 0.1$
$\alpha_1 = \alpha_2 = 0$		4.730 0				
		7.853 2				
		10.995 6				
		14.137 1				
		17.278 8				
$\alpha_1 = \alpha_2 = 0.25$		4.068 1	4.066 0	4.059 7	4.049 3	4.034 7
		7.039 9	7.019 7	6.957 4	6.849 1	6.691 8
		9.659 8	9.639 5	9.567 2	9.402 0	9.088 8
		14.008 1	13.580 2	12.449 7	11.319 2	10.601 3
		16.317 8	15.949 8	14.614 8	12.775 5	11.418 8
$\alpha_1 = \alpha_2 = 0.5$		3.702 7	3.700 0	3.692 2	3.679 1	3.660 6(3.660 6)
		6.481 4	6.458 3	6.385 5	6.253 9	6.057 5(6.057 5)
		9.268 3	9.217 5	9.021 8	8.587 1	8.026 9(8.026 9)
		13.969 3	13.112 2	11.390 1	10.252 5	9.441 0(9.441 0)
		16.087 6	15.347 9	12.970 3	11.112 5	10.298 2(10.298 2)
$\alpha_1 = \alpha_2 = 0.75$		3.458 0	3.455 2	3.446 8	3.432 8	3.413 1
		6.077 2	6.054 3	5.981 0	5.845 1	5.638 0
		9.095 6	9.010 6	8.665 3	8.005 5	7.368 9
		13.950 2	12.695 4	10.726 8	9.561 8	8.614 9
		15.983 6	14.847 1	11.971 1	10.445 6	10.013 9
$\alpha_1 = \alpha_2 = 1$		3.277 2	3.274 4	3.265 8	3.251 5	3.231 4(3.231 4)
		5.769 3	5.747 2	5.675 5	5.540 3	5.331 2(5.331 2)
		9.000 3	8.879 1	8.375 0	7.566 0	6.911 1(6.911 1)
		13.938 8	12.328 9	10.265 2	9.004 9	8.047 5(8.047 5)
		15.924 3	14.377 3	11.319 5	10.155 6	9.878 4(9.878 4)

注 括号内为文献[19]结果.

Note The bracketed values are given in reference [19].

由表 2 可以进一步发现,随着质量块转动惯量增大,前 5 阶固有频率均减小且高阶频率的变化率更大.由图 3 和图 4 可以看出,由于转动惯量的影响,第 3 阶模态形状变化比第 2 阶模

态形状更大.由此可以得出与文献[19]一致的结论:质量块转动惯量对梁高阶模态影响大于对低阶模态的影响.

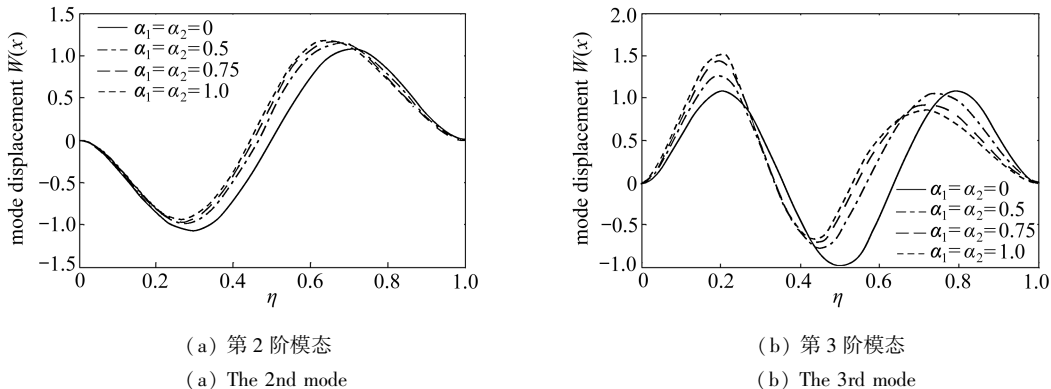


图3 含两质量块的固支梁第2、3阶模态形状 ($C_1 = C_2 = 0.1$)

Fig.3 The 2nd and 3rd mode shapes of the clamped-clamped beam with 2 masses of different uniform eccentricities ($C_1 = C_2 = 0.1$)

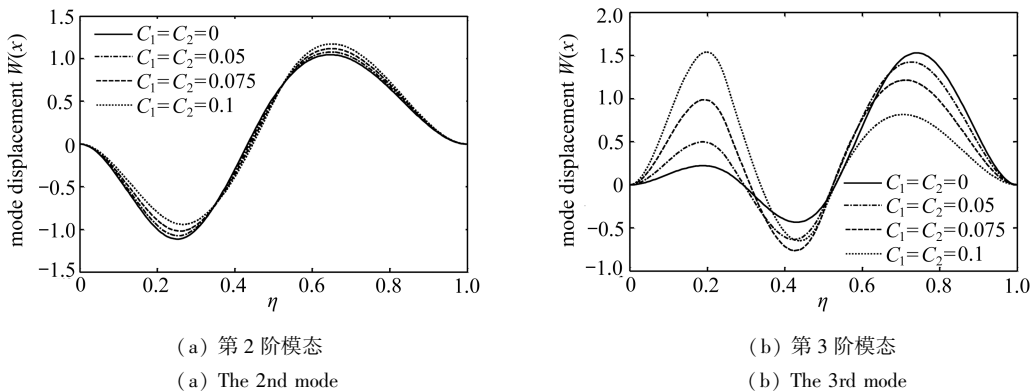


图4 含两质量块的固支梁第2、3阶模态形状 ($\alpha_1 = \alpha_2 = 1$)

Fig.4 The 2nd and 3rd mode shapes of the clamped-clamped beam with 2 masses of different uniform magnitudes ($\alpha_1 = \alpha_2 = 1$)

2.2 算例2

由于固有频率有良好的抗噪性且易于实验测得,因而基于固有频率的损伤检测方法长期受到广泛地关注.然而,需要考虑所有对频率的影响因素以得出更精确的损伤信息,例如桥墩所受轴向力,铁轨的弹性支撑等.因此,近年来较为复杂的损伤模型受到越来越多地关注^[3,13].本文讨论在轴向力作用下弹性基础多裂纹悬臂梁模型,如图4所示.图中 $x_i (i = 1, 2, 3)$ 分别为各个裂纹的位置, P 为轴向力, k 为 Winkler 弹性基础刚度.

由于轴向力的存在,裂纹处除表1中所示的转角不连续外还有轴向力引起的剪力不连续:

$$\Delta W'''(x_i) = -\gamma \Delta W'(x_i), \tag{16}$$

因此图5中所示裂纹梁位移函数可表示为

$$W(x) = S_2(x) W''(0) + S_3(x) W'''(0) + \sum_{i=1}^2 (S_1(x - x_i) \Delta W'(x_i) + S_3(x - x_i) \Delta W'''(x_i)) H(x - x_i). \tag{17}$$

由式(17)及表1中裂纹连续条件可知,图5中所示裂纹梁位移函数可写为

$$W(x) = S_2(x)W''(0) + S_3(x)W'''(0) + \sum_{i=1}^2 [(S_1(x-x_i) - \gamma S_3(x-x_i))\theta_i W''(x_i)]H(x-x_i). \tag{18}$$

通过类似于算例 1 的计算过程,可以得到梁的频率矩阵,进一步可解得固有频率以及相应的模态形状.

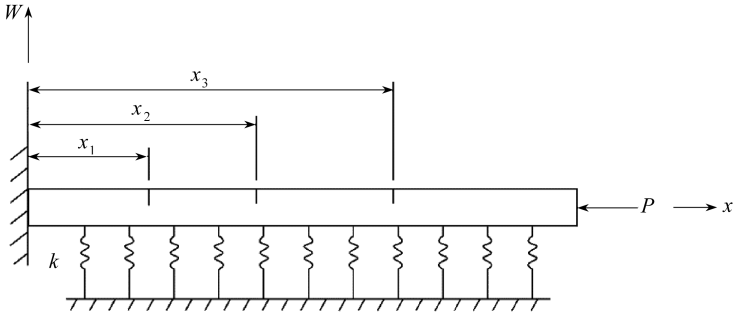
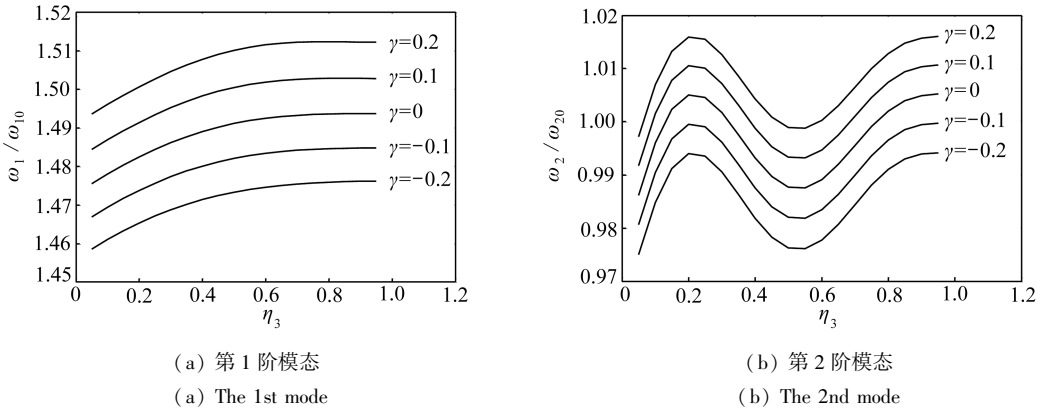


图 5 轴向力作用下弹性基础三裂纹悬臂梁

Fig.5 An elastic foundation cantilever beam with 3 cracks subject to axial load



(a) 第 1 阶模态
(a) The 1st mode

(b) 第 2 阶模态
(b) The 2nd mode

图 6 轴向力和裂纹位置对第 1、2 阶固有频率的影响

$$(\eta_1 = 0.15, \eta_2 = 0.35, \zeta_1 = \zeta_2 = \zeta_3 = 0.3, \kappa = 1)$$

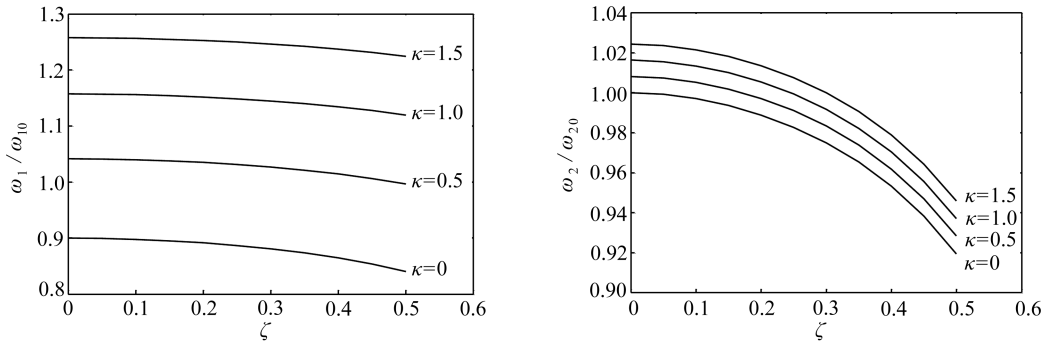
Fig.6 The 1st and 2nd eigen frequencies of the elastic foundation cantilever beam with 3 cracks vs. axial crack location η_3 under different axial loads

$$(\eta_1 = 0.15, \eta_2 = 0.35, \zeta_1 = \zeta_2 = \zeta_3 = 0.3, \kappa = 1)$$

图 6 表示轴向力和裂纹位置对损伤梁第 1、2 阶固有频率的影响.图中 ω_1 和 ω_2 分别为损伤梁第 1、2 阶固有频率, ω_{10} 和 ω_{20} 为无损梁第 1、2 阶固有频率(其中 $\omega_{10} = (1.875/L)^2 \sqrt{EI/\rho A}$, $\omega_{20} = (4.694/L)^2 \sqrt{EI/\rho A}$; κ 为无量纲弹性刚度, $\kappa = k/EI$).从图 6 可以发现,当无量纲轴向力 γ 从 -0.2 变化至 0.2 时,第 1 和第 2 阶固有频率的变化率分别为 4% 和 2%.由此可得出结论:梁受压力作用时,固有频率随着压力增大而减小;梁受拉力作用时,固有频率随着拉力增大而增大;且轴向力对较低阶频率影响大于对较高阶频率的影响.

图 7 给出了弹性基础和裂纹深度对梁前两阶固有频率的影响.图 7 中,当无量纲弹性刚度 κ 从 0 增加到 1.5 时,梁的第 1 阶固有频率约增大 72%,而第 2 阶固有频率只增大 2.5%.此外,第 1、2 阶固有频率随着裂纹深度增加,均呈减小趋势.由此可得出结论:弹性基础的存在会使梁固有频率增大,且频率变化率随着弹性基础刚度增大而增大;弹性基础对低阶频率影响大于

对高阶频率影响。



(a) 第 1 阶模态 (b) 第 2 阶模态
(a) The 1st mode (b) The 2nd mode

图 7 弹性基础刚度和裂纹深度对第 1,2 阶固有频率影响

$$(\eta_1 = 0.15, \eta_2 = 0.35, \eta_3 = 0.5, \zeta_1 = \zeta_2 = \zeta_3 = \zeta, P = 0)$$

Fig.7 The 1st and 2nd eigen frequencies of the elastic foundation cantilever beam with 3 cracks vs. crack depth ζ at different foundation moduli

$$(\eta_1 = 0.15, \eta_2 = 0.35, \eta_3 = 0.5, \zeta_1 = \zeta_2 = \zeta_3 = \zeta, P = 0)$$

3 结 论

本文建立了广义函数空间内的轴向力作用下弹性基础不连续 Euler-Bernoulli (欧拉伯努利) 梁微分方程; 由于使用 Heaviside 函数和 Dirac 函数以引入不连续点的影响, 梁的模态位移函数得以用一个统一的表达式表示. 与传统方法相比较, 新的连续条件表达方式可以使特征值矩阵的求解得以大大简化. 算例 1 研究了附加质量块的转动惯量对梁-质量块系统的影响, 计算结果与已有文献的一致即证明了本文方法的准确性. 算例 2 模拟了轴向力弹性基础裂纹梁的参数变化, 验证了本文方法具有广泛的适用性.

附录 含两质量块固支梁特征矩阵

将式(14)重写为

$$W(x) = S_2(x)W''(0) + S_3(x)W'''(0) + \sum_{i=1}^2 \left(S_3(x-x_i) \frac{M\omega^2}{EI} W(x_i) - S_2(x-x_i) \frac{J\omega^2 W'(x_i)}{EI} \right) H(x-x_i), \tag{A1}$$

固支梁右端边界条件为

$$W(L) = 0, W'(L) = 0, \tag{A2}$$

把式(A1)代入式(A2)可得

$$W(L) = S_2(L)W''(0) + S_3(L)W'''(0) + \sum_{i=1}^2 \left(S_3(L-x_i) \frac{M\omega^2}{EI} W(x_i) - S_2(L-x_i) \frac{J\omega^2 W'(x_i)}{EI} \right), \tag{A3}$$

$$W'(L) = S_2'(L)W''(0) + S_3'(L)W'''(0) + \sum_{i=1}^2 \left(S_3'(L-x_i) \frac{M\omega^2}{EI} W(x_i) - S_2'(L-x_i) \frac{J\omega^2 W'(x_i)}{EI} \right). \tag{A4}$$

式(A1)中未知变量 $W(x_i)$ 和 $W'(x_i)$ 可分别表示为

$$W(x_1) = S_2(x_1)W''(0) + S_3(x_1)W'''(0), \tag{A5}$$

$$W(x_2) = S_2(x_2)W''(0) + S_3(x_2)W'''(0) +$$

$$S_3(x_2 - x_1) \frac{M\omega^2}{EI} W(x_2) - S_2(x_2 - x_1) \frac{J\omega^2 W'(x_2)}{EI}, \quad (A6)$$

$$W'(x_1) = S_2'(x_1) W''(0) + S_3'(x_1) W'''(0), \quad (A7)$$

$$W'(x_2) = S_2(x_2)' W''(0) + S_3'(x_2) W'''(0) + S_3'(x_2 - x_1) \frac{M\omega^2}{EI} W(x_2) - S_2'(x_2 - x_1) \frac{J\omega^2 W'(x_2)}{EI}. \quad (A8)$$

把式(A3)~(A8)表示为矩阵形式,即可得出梁的特征方程:

$$\bar{\mathbf{A}} \bar{\mathbf{W}} = \mathbf{0}, \quad (A9)$$

式中

$$\bar{\mathbf{A}} = \begin{bmatrix} a_{11}(\omega) & \cdots & a_{16}(\omega) \\ \vdots & \ddots & \vdots \\ a_{61}(\omega) & \cdots & a_{66}(\omega) \end{bmatrix},$$

$$\bar{\mathbf{W}} = [W''(0) \quad W'''(0) \quad W(x_1) \quad W(x_2) \quad W'(x_1) \quad W'(x_2)]^T.$$

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Free Vibration Analysis of Elastic Foundation Euler Beams With Different Discontinuities Based on Generalized Functions

CHEN Xiao-chao, MAO Qi-bo, XUE Xiao-li

(School of Aircraft Engineering, Nanchang Hangkong University,
Nanchang 330063, P.R.China)

Abstract: The general governing differential equations for the vibration of elastic foundation Euler-Bernoulli beams with different discontinuities subject to axial forces were established based on generalized functions. For each discontinuity at a given location, a basic modal displacement function (Dirac delta function) starting at that location was introduced. The differential equations were then solved by means of Laplace transformation. Unlike the classical vibration solutions to problems of beams with discontinuities, the generalized solution was in a single unified expression for the whole beam. Due to unification of the modal function and degeneration of the compatibility conditions, solution of the eigenvalues was greatly simplified. Finally, the free vibration problems of (a) an elastic foundation beam with multiple masses and corresponding rotary inertias, and (b) an elastic foundation beam with multiple cracks under axial force, were solved with the proposed method. Results show that the present method is accurate and efficient for free vibration analysis of beams with different discontinuities.

Key words: free vibration; generalized function; axial force; elastic foundation; beam with discontinuity

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