文章编号:1000-0887(2012)11-1294-10

变质量 Chetaev 型非完整系统的共形不变性*

黄卫立[1,2], 蔡建乐[2]

(1. 湖南城市学院 通信与电子工程学院, 湖南 益阳 413000;
2. 杭州师范大学 理学院, 杭州 310018)

摘要: 研究变质量 Chetaev 型非完整系统的共形不变性与守恒量. 推导共形因子表达式,得到系统共形不变性同时是 Lie 对称性的充要条件,给出系统弱 Lie 对称性和强 Lie 对称性的共形不变性,导出系统相应的守恒量,并举例说明结果的应用.

关 键 词: 非完整系统; 变质量; 共形不变性; 守恒量

中图分类号: O316;O302;O313.6 **文献标志码:** A

DOI: 10.3879/j.issn.1000-0887.2012.11.005

引 言

变质量力学系统的研究,在数学、力学、航空、航天等领域中具有重要的理论意义和应用价值. 早在 19 世纪中叶,人们就提出了变质量力学的问题,但直到 1897 年,俄国的科学家 Мещерский 首先建立了变质量质点动力学的基本方程. 1929 年苏联力学家 Циолковский 提出用多级火箭实现宇宙飞行,对变质量力学作出了重要贡献. 中国力学家梅凤翔对变质量系统力学的研究做了大量工作,奠定了变质量系统分析力学研究的理论基础[1].

对称性原理是物理学中更高层次的法则,近年来,动力学系统的对称性与守恒量的研究倍受人们关注[2-5],在 Noether 对称性[6-10]、Lie 对称性[11-15] 和 Mei 对称性[16-20] 以及其它对称性[21-24]方面的研究取得重要进展. 同时,变质量系统的对称性与守恒量研究取得了一系列重要成果[25-27]. 共形不变性是寻找动力学系统守恒量的一种现代方法. Galiullin 等人研究了在特殊无限小变换下 Birkhoff 系统的共形不变性或共形对称性[28]. 最近,我们研究了 Lagrange 系统、Hamilton 系统、完整系统和非完整系统 Lie 对称性或 Mei 对称性的共形不变性及其守恒量[29-34]. 文献[35-40]分别研究了一阶微分方程、广义 Hamilton 系统、准坐标下和事件空间中以及高阶非完整系统 Lie 对称性或 Mei 对称性的共形不变性.

然而,对于变质量系统,例如喷气飞机、火箭、卫星、航天器等,由于质量的变化,系统的动力学方程变得复杂. 文献[41]研究了变质量完整系统 Mei 对称性的共形不变性. 对于非完整系统,其动力学方程也比完整系统更复杂,其对称性与守恒量或共形不变性的研究也比完整系统要困难. 本文研究变质量 Chetaev 型非完整系统的共形不变性及系统的守恒量. 当然,非完

* **收稿日期:** 2012-03-21; **修订日期:** 2012-06-21
基金项目: 国家自然科学基金资助课题(10932002);浙江省自然科学基金资助课题(LY12A02008)
作者简介: 黄卫立(1959—), 男,湖南益阳人,副教授(E-mail: amuu@163.com);
蔡建乐, 教授(联系人. E-mail: caijianle@yahoo.com.cn).

整系统的共形不变性理论自然也适合完整系统.

1 共形不变性及其确定方程

研究由 N 个质点组成的系统,第 i 个质点的质量为

$$m_i = m_i(t,\boldsymbol{q}) \qquad (i = 1,2,\cdots,N). \tag{1}$$

设力学系统的位形由 n 个广义坐标 $q_s(s = 1,2,\cdots,n)$ 来确定,其运动受有 g 个彼此相容且独立的双面理想 Chetaev 型非完整约束:

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0 \qquad (s = 1,2,\cdots,n;\ \beta = 1,2,\cdots,g). \tag{2}$$

约束(2)加在虚位移 δq_s 上的 Chetaev 条件为

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0. \tag{3}$$

采用 Euler 算子

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s},$$

变质量 Chetaev 型非完整系统的运动微分方程表为

$$E_s(L) = Q_s + P_s + \lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{4}$$

式中, $L = L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 和 $Q_s = Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 分别为系统的 Lagrange 函数和非势广义力, $\boldsymbol{\lambda}_\beta$ 为约束乘子, P_s 为广义反推力

$$P_s = \dot{m}_i(\boldsymbol{u}_i + \dot{\boldsymbol{r}}_i)\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_s} - \frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial q_s} \qquad (s = 1,2,\cdots,n;\ i = 1,2,\cdots,N), \tag{5}$$

其中, $\boldsymbol{r}_i,\dot{\boldsymbol{r}}_i$ 分别为第 i 个质点的矢径和速度, $\boldsymbol{u}_i$ 为微粒相对第 i 个质点的相对速度.

假设系统非奇异,即

$$D = \det\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}\right) \neq 0,$$

在运动微分方程积分之前可求出 $\boldsymbol{\lambda}_\beta$ 作为 $t,\boldsymbol{q},\dot{\boldsymbol{q}}$ 的函数. 于是,方程(4)可表为

$$E_s(L) = Q_s + P_s + \Lambda_s \qquad (s = 1,2,\cdots,n). \tag{6}$$

展开式(6)可得

$$F_s \equiv A_{sk}(t,\boldsymbol{q})\ddot{q}_k + B_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) - Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) - P_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) - \Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0 \qquad (s,k = 1,2,\cdots,n), \tag{7}$$

式中

$$A_{sk} = \frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k},\ B_s = \frac{\partial^2 L}{\partial \dot{q}_s\partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_s\partial t} - \frac{\partial L}{\partial q_s}. \tag{8}$$

由式(7)可求得所有广义加速度

$$\ddot{q}_s = \frac{M_{ks}}{D}\left(\frac{\partial L}{\partial q_k} - \frac{\partial^2 L}{\partial \dot{q}_k\partial t} - \frac{\partial^2 L}{\partial \dot{q}_k\partial q_j}\dot{q}_j + Q_k + P_k + \Lambda_k\right) \qquad (s,\ k,\ j = 1,2,\cdots,n), \tag{9}$$

式中, M_{ks} 为 M_{sk} 的转置, M_{sk} 表示矩阵元素 $\partial^2 L/(\partial \dot{q}_s\partial \dot{q}_k)$ 的余因子. 式(9)简记为

$$\ddot{q}_s = \alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \qquad (s = 1,2,\cdots,n). \tag{10}$$

取时间 t 和广义坐标 q_s 的无限小单参数变换群:

$$t^* = t + \varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\ q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{11}$$

其中,ε 为无限小参数,ξ_0, ξ_s 为群的无限小变换的生成元或生成函数.

定义 1 对于矩阵 Γ_s^l, 使得

$$X^{(2)}(F_s) = \Gamma_s^l(F_l) \qquad (s,l = 1,2,\cdots,n), \tag{12}$$

则方程(7)在无限小单参数变换群(11)作用下是共形不变的,式(12)是方程(7)的共形不变的确定方程,其中 Γ_s^l 为共形因子. 式中

$$\begin{cases} X^{(2)} = X^{(1)} + (\ddot{\xi}_s - 2\ddot{q}_s\dot{\xi}_0 - \dot{q}_s\ddot{\xi}_0)\dfrac{\partial}{\partial\ddot{q}_s}, \\ X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\dfrac{\partial}{\partial\dot{q}_s}, \\ X^{(0)} = \xi_0\dfrac{\partial}{\partial t} + \xi_s\dfrac{\partial}{\partial q_s}. \end{cases} \tag{13}$$

对于与变质量非完整系统(2)和(4)相应的完整系统(7),如果无限小生成元 ξ_0,ξ_s 满足确定方程

$$X^{(2)}(F_s)\big|_{F_s=0} = 0, \tag{14}$$

则称这种对称性为系统的 Lie 对称性.

2 共 形 因 子

为得到方程(7)共形不变性的共形因子表达式,计算差值

$$X^{(2)}(F_s) - X^{(2)}(F_s)\big|_{F_s=0}. \tag{15}$$

利用

$$\dot{\xi}_k = \frac{\partial\xi_k}{\partial t} + \frac{\partial\xi_k}{\partial q_r}\dot{q}_r + \frac{\partial\xi_k}{\partial\dot{q}_r}\ddot{q}_r \qquad (k = 0,1,2,\cdots,n), \tag{16}$$

$$\begin{aligned}\ddot{\xi}_k = {} & \frac{\partial^2\xi_k}{\partial t^2} + 2\frac{\partial^2\xi_k}{\partial q_r\partial t}\dot{q}_r + 2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial t}\ddot{q}_r + \frac{\partial^2\xi_k}{\partial q_r\partial q_j}\dot{q}_r\dot{q}_j + 2\frac{\partial^2\xi_k}{\partial q_r\partial\dot{q}_j}\dot{q}_r\ddot{q}_j + \\ & \frac{\partial\xi_k}{\partial q_r}\ddot{q}_r + \frac{\partial^2\xi_k}{\partial\dot{q}_r\partial\dot{q}_j}\ddot{q}_r\ddot{q}_j + \frac{\partial\xi_k}{\partial\dot{q}_r}\left(\frac{\partial\alpha_r}{\partial t} + \frac{\partial\alpha_r}{\partial q_j}\dot{q}_j + \frac{\partial\alpha_r}{\partial\dot{q}_j}\ddot{q}_j\right) \\ & (k = 0,1,2,\cdots,n;\ r,\ j = 1,2,\cdots,n),\end{aligned} \tag{17}$$

经计算得

$$\begin{aligned} X^{(2)}(F_s) = {} & A_{sk}(\ddot{\xi}_k - 2\ddot{q}_k\dot{\xi}_0 - \dot{q}_k\ddot{\xi}_0) + X^{(0)}(A_{sk})\ddot{q}_k + \\ & X^{(0)}(B_s - Q_s - P_s - \Lambda_s) + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial(B_s - Q_s - P_s - \Lambda_s)}{\partial\dot{q}_k} = \\ & A_{sk}\Big[2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial t}\ddot{q}_r + 2\frac{\partial^2\xi_k}{\partial q_j\partial\dot{q}_r}\dot{q}_j\ddot{q}_r + \frac{\partial\xi_k}{\partial q_r}\ddot{q}_r + \frac{\partial^2\xi_k}{\partial\dot{q}_r\partial\dot{q}_j}\ddot{q}_r\ddot{q}_j + \\ & \frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\ddot{q}_r - 2\ddot{q}_k\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial q_r}\dot{q}_r + \frac{\partial\xi_0}{\partial\dot{q}_r}\ddot{q}_r\right)\Big] - \\ & A_{sk}\dot{q}_k\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial t}\ddot{q}_r + 2\frac{\partial^2\xi_0}{\partial q_j\partial\dot{q}_r}\dot{q}_j\ddot{q}_r + \frac{\partial\xi_0}{\partial q_r}\ddot{q}_r + \frac{\partial^2\xi_0}{\partial\dot{q}_r\partial\dot{q}_j}\ddot{q}_r\ddot{q}_j + \frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\ddot{q}_r\right) + \end{aligned}$$

$$X^{(0)}(A_{sk})\ddot{q}_k + \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \dot{q}_k \frac{\partial \xi_0}{\partial \dot{q}_r}\right)\ddot{q}_r \frac{\partial (B_s - Q_s - P_s - \Lambda_s)}{\partial \dot{q}_k} + C(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{18}$$

式中，$C(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ 为其余不含 $\ddot{q}_s$ 项的代数和. 式(18) 中用 α 替换 $\ddot{q}$，同理可得 $X^{(2)}(F_s)\big|_{F_s=0}$ 的结果.

由于 $\alpha_s = -A^{sk}(B_k - Q_k - P_k - \Lambda_k)$，我们有

$$\begin{aligned}\ddot{q}_k - \alpha_k &= \ddot{q}_k + A^{kl}(B_l - Q_l - P_l - \Lambda_l) = \\ &A^{kl}(A_{lm}\ddot{q}_m + B_l - Q_l - P_l - \Lambda_l) = A^{kl}F_l,\end{aligned} \tag{19}$$

$$\begin{aligned}\ddot{q}_k\ddot{q}_j &= (A^{kl}F_l + \alpha_k)(A^{jm}F_m + \alpha_j) = \\ &A^{kl}F_lA^{jm}F_m + \alpha_kA^{jm}F_m + \alpha_jA^{kl}F_l + \alpha_k\alpha_j.\end{aligned} \tag{20}$$

因此

$$\begin{aligned}&X^{(2)}(F_s) - X^{(2)}(F_s)\big|_{F_s=0} = \\ &A_{sk}\left(2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_k}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_k}{\partial q_r} + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)(\ddot{q}_r - \alpha_r) + A_{sk}\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}(\ddot{q}_r\ddot{q}_j - \alpha_r\alpha_j) - \\ &2A_{sk}(\ddot{q}_k - \alpha_k)\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right) - 2A_{sk}\frac{\partial \xi_0}{\partial \dot{q}_r}(\ddot{q}_k\ddot{q}_r - \alpha_k\alpha_r) - \\ &A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_0}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_0}{\partial q_r} + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)(\ddot{q}_r - \alpha_r) - \\ &A_{sk}\dot{q}_k\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}(\ddot{q}_r\ddot{q}_j - \alpha_r\alpha_j) + X^{(0)}(A_{sk})(\ddot{q}_k - \alpha_k) + \\ &\left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \dot{q}_k\frac{\partial \xi_0}{\partial \dot{q}_r}\right)(\ddot{q}_r - \alpha_r)\frac{\partial (B_s - Q_s - P_s - \Lambda_s)}{\partial \dot{q}_k} = \\ &\boldsymbol{\beta}_s^l F_l + A_{sk}\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}A^{jm}F_mA^{rl}F_l - 2A_{sk}\frac{\partial \xi_0}{\partial \dot{q}_r}A^{rm}F_mA^{kl}F_l - \\ &A_{sk}\dot{q}_k\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}A^{jm}F_mA^{rl}F \qquad (s,\ k,\ r,\ j,\ m,\ l = 1, 2, \cdots, n),\end{aligned} \tag{21}$$

式中

$$\begin{aligned}\boldsymbol{\beta}_s^l = &A_{sk}\left(2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_k}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_k}{\partial q_r} + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)A^{rl} - \\ &A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_0}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_0}{\partial q_r} + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)A^{rl} - \\ &2\delta_s^l\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right) + X^{(0)}(A_{sk})A^{kl} + \\ &\left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \dot{q}_k\frac{\partial \xi_0}{\partial \dot{q}_r}\right)A^{rl}\frac{\partial (B_s - Q_s - P_s - \Lambda_s)}{\partial \dot{q}_k} + \\ &A_{sk}\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}(\alpha_rA^{jl} + \alpha_jA^{rl}) - 2A_{sk}\frac{\partial \xi_0}{\partial \dot{q}_r}(\alpha_kA^{rl} + \alpha_rA^{kl}) - \\ &A_{sk}\dot{q}_k\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}(\alpha_rA^{jl} + \alpha_jA^{rl}) \qquad (s,\ k,\ j,\ r,\ l = 1, 2, \cdots, n).\end{aligned} \tag{22}$$

$\boldsymbol{\beta}_s^l$ 即共形因子，忽略 F_l 的高阶项，得到

$$X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}=\beta_s^l F_l \qquad (s,l=1,2,\cdots,n).\tag{23}$$

取 $A_{sk}=\delta_{sk}$, $B_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})-Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})-P_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})-\Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})=-\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, 方程(7)可规范为标准形式

$$F_s\equiv\ddot{q}_s-\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0 \qquad (s=1,2,\cdots,n).\tag{24}$$

由式(22)容易得到共形因子

$$\begin{aligned}\beta_s^l=&2\frac{\partial^2\xi_s}{\partial\dot{q}_l\partial t}+2\frac{\partial^2\xi_s}{\partial q_j\partial\dot{q}_l}\dot{q}_j+\frac{\partial\xi_s}{\partial q_l}+\frac{\partial\xi_s}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_l}-\\&\dot{q}_s\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_l\partial t}+2\frac{\partial^2\xi_0}{\partial q_j\partial\dot{q}_l}\dot{q}_j+\frac{\partial\xi_0}{\partial q_l}+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_l}\right)-\\&2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r+\frac{\partial\xi_0}{\partial\dot{q}_r}\alpha_r\right)-\left(\frac{\partial\xi_k}{\partial\dot{q}_l}-\dot{q}_k\frac{\partial\xi_0}{\partial\dot{q}_l}\right)\frac{\partial\alpha_s}{\partial\dot{q}_k}+\\&2\frac{\partial^2\xi_s}{\partial\dot{q}_r\partial\dot{q}_l}\alpha_r-2\dot{q}_s\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial\dot{q}_l}\alpha_r-2\frac{\partial\xi_0}{\partial\dot{q}_l}\alpha_s.\end{aligned}\tag{25}$$

利用式(12)和(14)及(23),易得下述结论:

定理 1 如果方程(7)在无限小单参数变换群(11)作用下是 Lie 对称性的,且存在矩阵 (β_s^l) 满足

$$X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}=\beta_s^l F_l \qquad (s,l=1,2,\cdots,n),\tag{26}$$

则方程(7)在无限小单参数变换群(11)作用下共形不变性同时是 Lie 对称性的充分与必要条件为

$$\Gamma_s^l=\beta_s^l \qquad (s,l=1,2,\cdots,n).\tag{27}$$

3 结构方程与守恒量

利用式(6)消除 $\partial L/\partial q_s$ 可得

$$\begin{aligned}X^{(1)}(L)=&\xi_0\frac{\partial L}{\partial t}+\xi_s\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-Q_s-P_s-\Lambda_s\right)+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial L}{\partial\dot{q}_s}=\\&\frac{\mathrm{d}}{\mathrm{d}t}\left[(\xi_s-\dot{q}_s\xi_0)\frac{\partial L}{\partial\dot{q}_s}+L\xi_0\right]-L\dot{\xi}_0-(Q_s+P_s+\Lambda_s)(\xi_s-\dot{q}_s\xi_0),\end{aligned}\tag{28}$$

令

$$G=-\left[(\xi_s-\dot{q}_s\xi_0)\frac{\partial L}{\partial\dot{q}_s}+L\xi_0\right]+\mathrm{const},\tag{29}$$

显然有下述结论:

定理 2 对于满足共形因子(22)或(25)的无限小变换群(11)的生成元 ξ_0, ξ_s,如果存在规范函数 $G=G(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 满足如下 Lie 对称性的结构方程

$$L\dot{\xi}_0+X^{(1)}(L)+(Q_s+P_s+\Lambda_s)(\xi_s-\dot{q}_s\xi_0)+\dot{G}=0,\tag{30}$$

则相应于变质量 Chetaev 型非完整系统(2)和(4)的完整系统(7)的共形不变性存在守恒量

$$I=L\xi_0+\frac{\partial L}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G=\mathrm{const}.\tag{31}$$

在无限小变换(11)下,非完整约束方程(2)的不变性表为如下限制方程

$$X^{(1)}\{f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})\}=0 \qquad (\beta=1,2,\cdots,g).\tag{32}$$

若考虑到 Chetaev 条件(2)对无限小生成元 ξ_0, ξ_s 的限制,则有

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0, \tag{33}$$

称方程(33)为附加限制方程.

定义 2 对于变质量 Chetaev 型非完整系统,如果无限小生成元 ξ_0,ξ_s 满足确定方程(14)以及限制方程(32),则称这种对称性为系统的弱 Lie 对称性,相应的共形不变性是弱 Lie 对称性的共形不变性. 如果无限小生成元 ξ_0,ξ_s 满足确定方程(14)、限制方程(32)以及附加限制方程(33),则称这种对称性为系统的强 Lie 对称性,相应的共形不变性是強 Lie 对称性的共形不变性.

由命题 2 易得以下结论:

定理 3 如果 ξ_0, ξ_s 是变质量 Chetaev 型非完整系统(2)和(4)的弱(強)Lie 对称性生成元,且存在规范函数 $G = G(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 满足结构方程(30),则变质量 Chetaev 型非完整系统(2)和(4)的弱(強)Lie 对称性的共形不变性导致守恒量(31).

4 算　　例

我们讨论一个变质量 Chetaev 型非完整系统的实例,其质量、Lagrange 函数、非完整约束与非势广义力分别为

$$m = m_0 \exp(-\beta t) \qquad (m_0 \text{ 与 } \beta \text{ 均为常数}), \tag{34}$$

$$L = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2), \tag{35}$$

$$f = \dot{q}_2 - t\dot{q}_1 = 0, \tag{36}$$

$$Q_1 = -\frac{1}{t(1+t^2)} m\dot{q}_1,\ Q_2 = -\frac{1}{1+t^2} m\dot{q}_1. \tag{37}$$

设微粒分离的绝对速度为 0,由式(5)得

$$P_1 = P_2 = 0. \tag{38}$$

方程(4)表为

$$m\ddot{q}_1 = -\dot{m}\dot{q}_1 - \frac{1}{t(1+t^2)} m\dot{q}_1 - \lambda t,\ m\ddot{q}_2 = -\dot{m}\dot{q}_2 - \frac{1}{1+t^2} m\dot{q}_1 + \lambda. \tag{39}$$

由方程(34)~(39)求得

$$\lambda = \frac{1}{1+t^2} m\dot{q}_1, \tag{40}$$

$$\Lambda_1 = -\lambda t = -\frac{t}{1+t^2} m\dot{q}_1,\ \Lambda_2 = \lambda = \frac{1}{1+t^2} m\dot{q}_1. \tag{41}$$

因此

$$\ddot{q}_1 = \beta\dot{q}_1 - \dot{q}_1/t,\ \ddot{q}_2 = \beta\dot{q}_2 \tag{42}$$

或

$$F_1 = \ddot{q}_1 - \beta\dot{q}_1 + \dot{q}_1/t,\ F_2 = \ddot{q}_2 - \beta\dot{q}_2. \tag{43}$$

选取生成元函数

$$\xi_0 = 0,\ \xi_1 = q_1,\ \xi_2 = q_2 + 1, \tag{44}$$

可得共形不变性的确定方程

$$X^{(2)}(F_1) = \ddot{q}_1 - \beta\dot{q}_1 + \dot{q}_1/t = F_1,\ X^{(2)}(F_2) = \ddot{q}_2 - \beta\dot{q}_2 = F_2. \tag{45}$$

共形因子为

$$\boldsymbol{\Gamma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{46}$$

由式(25)也可得 $\beta_s^l = \Gamma_s^l$. 此时系统共形不变性同时是 Lie 对称性的.

显然,相应的对称性是系统的弱 Lie 对称性,因为生成元满足如下限制方程

$$X^{(1)}f = X^{(1)}(\dot{q}_2 - t\dot{q}_1) = 0. \tag{47}$$

但相应的对称性不是系统的强 Lie 对称性,因为生成元不满足附加限制方程(33),即

$$\frac{\partial f}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) \neq 0 \qquad (s = 1,2). \tag{48}$$

将式(35)、(37)、(38)、(41)和(44)代入结构方程(30),则有

$$\dot{G} = -m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{m}{t}q_1\dot{q}_1, \tag{49}$$

$$G = -m(q_1\dot{q}_1 + q_2\dot{q}_2). \tag{50}$$

式(31)给出守恒量

$$I = m\dot{q}_2 = \text{const}. \tag{51}$$

5 结　　论

变质量 Chetaev 型非完整系统在无限小变换下存在共形不变性. 当无限小生成元满足系统的限制方程或限制方程和附加限制方程时,我们可得到系统弱或强 Lie 对称性的共形不变性. 共形不变性满足一定条件时也可导致相应的守恒量.

参考文献(References):

[1] 杨来伍, 梅凤翔. 变质量系统力学[M]. 北京:北京理工大学出版社,1989. (YANG Lai-wu, MEI Feng-xiang. *Mechanics Variable Mass System*[M]. Beijing: Beijing Institute of Technology Press, 1989. (in Chinese))

[2] 梅凤翔. 李群和李代数对约束力学系统的应用[M]. 北京:科学出版社,1999. (MEI Feng-xiang. *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems*[M]. Beijing:Science Press, 1999. (in Chinese))

[3] 赵跃宇, 梅凤翔. 力学系统的对称性与不变量[M]. 北京: 科学出版社, 1999. (ZHAO Yue-yu, MEI Feng-xiang. *Symmetries and Invariants of Mechanical Systems*[M]. Beijing: Science Press, 1999. (in Chinese))

[4] 梅凤翔. 约束力学系统的对称性与守恒量[M]. 北京:北京理工大学出版社, 2004. (MEI Feng-xiang. *Symmetries and Conserved Quantities of Constrained Mechanical Systems*[M]. Beijing: Beijing Institute of Technology Press, 2004. (in Chinese))

[5] 罗绍凯, 张永发. 约束系统动力学研究进展[M]. 北京:科学出版社,2008. (LUO Shao-kai, ZHANG Yong-fa. *Advances in the Study of Dynamics of Constrained Systems*[M]. Beijing: Science Press, 2008. (in Chinese))

[6] Fu J L, Chen B Y, Chen L Q. Noether symmetries of discrete nonholonomic dynamical sys-

tems[J]. *Physics Letters A*, 2009, **373**(4): 409-412.

[7] Wang P, Fang J H, Wang X M. Discussion on perturbation to weak Noether symmetry and adiabatic invariants for Lagrange systems[J]. *Chinese Physics Letters*, 2009, **26**(3): 034501 doi:10.1088/0256-307X/26/3/034501.

[8] Fu J L, Chen L Q, Chen B Y. Noether-type theory for discrete mechanico-electrical dynamical systems with nonregular lattices[J]. *Science China: Physics, Mechanics & Astronomy*, 2010, **53**(9): 1687-1698.

[9] Xia L L, Shan L F. Weak Noether symmetry for a nonholonomic controllable mechanical system[J]. *Chinese Physics B*, 2010, **19**(9): 090302 doi:10.1088/1674-1056/19/9/090302.

[10] Wang P. Perturbation to Noether symmetry and Noether adiabatic invariants of discrete mechanico-electrical systems[J]. *Chinese Physics Letters*, 2011, **28**(4): 040203 doi:10.1088/0256-307X/28/4/040203.

[11] Mei F X. Lie symmetries and conserved quantities of Birkhoffian systems[J]. *Chinese Science Bulletin*, 1999, **44**: 318-320.

[12] LUO Shao-kai, JIA Li-qun. A set of Lie symmetrical conservation law for rotational relativistic Hamiltonian systems[J]. *Communication in Theoretical Physics*, 2003, **40**(3): 265-268.

[13] Fu J L, Chen B Y. Hojman conserved quantities and Lie symmetries of non-conservative systems[J]. *Modern Physics Letters B*, 2009, **23**(10):1315-1322.

[14] Zhou S, Fu H, Fu J L. Symmetry theories of Hamiltonian systems with fractional derivatives [J]. *Science China: Physics, Mechanics & Astronomy*, 2011, **54**(10): 1847-1853.

[15] Jiang W A, Li L, Li Z J, Luo S K. Lie symmetrical perturbation and a new type of non-Noether adiabatic invariants for disturbed generalized Birkhoffian systems[J]. *Nonlinear Dynamics*, 2012, **67**(2): 1075-1081.

[16] Mei F X. Form invariance of Lagrange system[J]. *Journal of Beijing Institute of Technology*, 2000, **9**(2): 120-124.

[17] Li H, Fang J H. Lie symmetry and Mei symmetry of a rotational relativistic system in phase space[J]. *Chinese Physics*, 2004, **13**(8): 1187-1190.

[18] 贾利群, 郑世旺, 张耀宇. 事件空间中非 Chetaev 型非完整系统的 Mei 对称性与 Mei 守恒量[J]. 物理学报, 2007, **56**(10): 5575-5579. (JIA Li-qun, ZHENG Shi-wang, ZHANG Yao-yu. Mei symmetry and Mei conserved quantity of nonholonomic systems of non-Chetaev's type in event space[J]. *Acta Physica Sinica*, 2007, **56**(10): 5575-5579. (in Chinese))

[19] Xia L L, Zhao X L. Generalized Mei conserved quantity of Mei symmetry for mechanico-electrical systems with nonholonomic controllable constraints[J]. *Chinese Physics Letters*, 2009, **26**(1): 010203 doi:10.1088/0256-307X/26/1/010203.

[20] 贾利群, 解银丽, 罗绍凯. 相对运动动力学系统 Appell 方程 Mei 对称性导致的 Mei 守恒量[J]. 物理学报, 2011, **60**(4): 040201. (JIA Li-qun, XIE Yin-li, LUO Shao-kai. Mei conserved quantity deduced from Mei symmetry of Appell equation in a dynamical system of relative motion [J]. *Acta Physica Sinica*, 2011, **60**(4): 040201. (in Chinese))

[21] Xu X J, Qin M C, Mei F X. Unified symmetry of holonomic mechanical systems[J]. *Chinese Physics*, 2005, **14**(7): 1287-1289.

[22] Wu H B, Mei F X. Symmetry of Lagrangians of holonomic systems in terms of quasi-coordinates[J]. *Chinese Physics B*, 2009, **18**(8): 3145-3149.

[23] Jiang W A, Luo S K. A new type of non-Noether exact invariants and adiabatic invariants of generalized Hamiltonian systems[J]. *Nonlinear Dynamics*, 2012, **67**(1): 475-482.

[24] Xia L L, Li Y C, Hou Q B, Wang J. Unified symmetry of nonholonomic mechanical systems with variable mass[J]. *Chinese Physics B*, 2004, **15**(5): 903-906.

[25] Cui J C, Zhang Y Y, Yang X F, Jia L Q. Mei symmetry and Mei conserved quantity of Appell equations for avariable mass holonomic system[J]. *Chinese Physics B*, 2010, **19**(3): 31-35.

[26] Zhang M L, Sun X T, Wang X X, Xie Y L, Jia L Q. Lie symmetry and the generalized Hojman conserved quantity of Nielsen equations for a variable mass holonomic system of relative motion[J]. *Chinese Physics B*, 2011, **20**(11): 19-22.

[27] Wu H B, Mei F X. Symmetry of Lagrangians of holonomic variable mass system[J]. *Chinese Physics B*, 2012, **21**(6): 064501 doi: 10.1088/1674-1056/21/6/064501.

[28] Galiullin A S, Gafarov G G, Malaishka R P, Khwan A M. *Analytical Dynamics of Helmholtz, Birkhoff and Nambu Systems*[M]. Moscow: UFN, 1997.

[29] 蔡建乐, 梅凤翔. Lagrange 系统 Lie 点变换下的共形不变性与守恒量[J]. 物理学报, 2008, **57**(9): 5369-5373. (CAI Jian-le, MEI Feng-xiang. Conformal invariance and conserved quantity of Lagrange systems under Lie point transformation[J]. *Acta Physica Sinica*, 2008, **57**(9): 5369-5373. (in Chinese))

[30] CAI Jian-le. Conformal invariance and conserved quantities of general holonomic systems[J]. *Chinese Physics Letters*, 2008, **25**(5): 1523-1526.

[31] Cai J L, Luo S K, Mei F X. Conformal invariance and conserved quantity of Hamilton systems[J]. *Chinese Physics B*, 2008, **17**(9): 3170-3174.

[32] Cai J L. Conformal invariance and conserved quantities of Mei symmetry for Lagrange systems[J]. *Acta Physica Polonica A*, 2009, **115**(5): 854-856.

[33] 蔡建乐. 一般完整系统 Mei 对称性的共形不变性与守恒量[J]. 物理学报, 2009, **58**(1): 22-27. (CAI Jian-le. Conformal invariance and conserved quantities of Mei symmetry for general holonomic systems[J]. *Acta Physica Sinica*, 2009, **58**(1): 22-27. (in Chinese))

[34] Cai J L. Conformal invariance and conserved quantity for the nonholonomic system of Chetaev's type[J]. *International Journal of Theoretical Physics*, 2010, **49**(1): 201-211.

[35] He G, Mei F X. Conformal invariance and integration of first-order differential equations[J]. *Chinese Physics B*, 2008, **17**(8): 2764-2766.

[36] Luo Y P. Generalized conformal symmetries and its application of Hamilton systems[J]. *International Journal of Theoretical Physics*, 2009, **48**: 2665-2671.

[37] Luo Y P, Fu J L. Conformal invariance and Hojman conserved quantities for holonomic systems with quasi-coordinates[J]. *Chinese Physics B*, 2010, **19**(9): 94-99.

[38] Zhang Y. Conformal invariance and Noether symmetry, Lie symmetry of holonomic mechanical systems in event space[J]. *Chinese Physics B*, 2009, **18**(11): 4636-4642.

[39] Zhang M J, Fang J H, Lu K. Conformal invariance and conserved quantity of third-order Lagrange equations for non-conserved mechanical systems[J]. *Chinese Physics B*, 2009, **18**(11): 4650-4656.

[40] Huang W L, Cai J L. Conformal invariance and conserved quantity of Mei symmetry for higher-order nonholonomic system[J]. *Acta Mechanica*, 2012, **223**(2): 433-440.

[41] Cai J L. Conformal invariance of Mei symmetry for the holonomic system with variable mass

[J]. *Chinese Journal of Physics*, 2010, **48**: 728-735.

Conformal Invariance for the Nonholonomic System of Chetaev's Type With Variable Mass

HUANG Wei-li[1,2], CAI Jian-le[2]

(1. *Department of Physics and Telecom Engineering*, *Hunan City University*, *Yiyang*, *Hunan* 413000, *P. R. China*;
2. *College of Science*, *Hangzhou Normal University*, *Hangzhou* 310018, *P. R. China*)

Abstract: Conformal invariance and conserved quantities for the nonholonomic system of Chetaev's type with variable mass were studied. The conformal factor expressions were deduced. The necessary and sufficient conditions that the system's conformal invariance would be Lie symmetry were obtained. The conformal invariance of weak and strong Lie symmetry for the system was given. And the system's corresponding conserved quantities were derived. Lastly, an example was taken to illustrate the application of the result.

Key words: nonholonomic system; variable mass; conformal invariance; conserved quantity