

# 微极性流体在上下正交移动的 渗透平行圆盘间的流动\*

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**摘要:** 分析了上下正交运动的两平行圆盘间的非稳态的不可压缩的二维微极性流体的流动. 应用 von Kármán 类型的一个相似变换, 偏微分方程组 (PDEs) 被转化成一组耦合的非线性常微分方程 (ODEs). 应用同伦分析方法, 得到方程的解析解, 并且详细讨论了不同的物理参数, 像膨胀率, 渗透 Reynolds 数等, 对流体的速度场的影响.

**关键词:** 同伦分析方法; 膨胀率; 正交移动的渗透圆盘

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## 引 言

两圆盘间的流体流动在旋转机械、计算机存储设备、黏度测定、润滑、晶体生长过程、热质交换器、生物力学和海洋学等方面有广泛的科技应用. 在止推轴承中, 两圆盘间通过喷注, 而达到润滑分离的目的. 而且, 在现代润滑科技上, 具有聚合物添加剂的流体被应用作为润滑油<sup>[1]</sup>. Elerat<sup>[2]</sup>证明了具有渗透的两静止圆盘间的流动解的存在性和唯一性. Rasmussen<sup>[3]</sup>推广了 Berman<sup>[4]</sup>的相似变换, 把控制方程变成了耦合的非线性微分方程, 分析了两渗透圆盘间的稳态流动.

Eringen<sup>[5]</sup>和 Aero 等<sup>[6]</sup>首先提出的微极性流体理论, 对这样的一类流体进行了描述: 随机分布的小颗粒悬浮在粘性的介质中, 小颗粒具有旋转或者平移的状态, 是 Navier-Stokes 控制方程的推广. 这样的流体在血液流、润滑、多孔介质、湍流剪切流和毛细血管和微尺寸管道都有应用<sup>[7-8]</sup>. 文献[9-11]对微极性流体及其应用有详细的论述. Guram 和 Anwar<sup>[12-13]</sup>数值分析了稳态的不可压缩的一旋转一静止的两平行圆盘间的流动. Kamal 等<sup>[7]</sup>比较了 Newton 流和微极性流体, 并且数值分析了两渗透的平行圆盘间的对称流. Ashraf 等<sup>[8]</sup>应用 SOR 方法, 分析了两渗透圆盘间 Reynolds 数影响和微极性结构.

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应用相似变换, Nazir 和 Mahnood<sup>[14]</sup> 首先考虑了两正交运动的平行旋转圆盘间的流动和热传导特征, 然而, 在他们的工作中, 认为圆盘是不可渗透的. 而上下正交运动的这样的特殊的边界条件, 是由 Uchida 等<sup>[15]</sup> 为了分析胀缩的血管中的生物流体的流动, 隔膜间的振动以及呼吸系统空气中的空气循环而首先提出的一个流动模型. Ohki<sup>[16]</sup> 研究了半无限长多孔管道中的非稳态流动, 管道沿轴向可以延伸然而其半径不发生改变. Barron 等<sup>[17]</sup> 试验得到了壁面随时间运动的具有多孔的矩形管道流动, 他们用二氧化碳的升华过程去模拟壁面的喷注过程, 而且, 在升华的过程中, 管道壁面膨胀或者收缩. Majdalani 等<sup>[18-19]</sup>, Dauenhauer 等<sup>[20]</sup> 得到了不同的渗透 Reynolds 数下膨胀收缩的矩形管道数值和渐近解. 近年来 Asghar 等<sup>[21]</sup> 应用 Adomian 分解方法, 讨论了弱渗透 Reynolds 数影响下, 微膨胀的管道内的流动. Dinarvand 等<sup>[22]</sup> 得到两平行胀缩渗透平板间的解析近似解. Si<sup>[23-24]</sup> 把这一问题推广到微极性流体和粘弹性流体. 然而, 以上所有的工作都认为壁面的膨胀率为一常数. Xu 等<sup>[25]</sup> 对这一工作做了推广, 考虑壁面的膨胀率为时间的函数, 由  $\alpha_0$  变到  $\alpha_1$ . 但是结果发现这一和时间相关的解很快接近于稳定的状态.

受以上的工作的启发, 本文对两正交移动的渗透圆盘间的非稳态的不可压缩的微极性流体进行了分析, 忽略了体积力和力偶的影响. 圆盘间的流动是由对称的两渗透圆盘的喷注或者吸附所驱动, 本文应用由 Liao<sup>[26-27]</sup> 提出的同伦分析方法 (HAM) 求解速度分布情况, HAM 已经成功地被应用于解决了很多的非线性问题<sup>[28-34]</sup>. 应用同伦分析方法, 得到壁面的移动、圆盘的渗透对流体速度和微旋转速度的影响.

## 1 问题描述和控制方程

对两正交移动的渗透圆盘间的非稳态的不可压缩的微极性流体进行了分析, 忽略了体积力和力偶的影响. 两圆盘间的距离是  $2a(t)$ , 并且两圆盘有相同的渗透速度, 以速率  $\dot{a}(t)$  接近或者远离中心. 如图 1 所示, 选择柱面坐标, 中心原点位于两圆盘的中间位置. 速度  $u, w$  为  $r, z$  方向的速度分量,  $\phi$  为微极性分量.

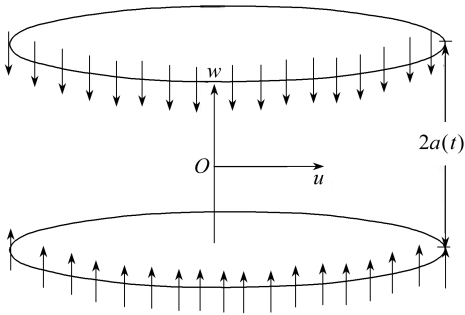


图 1 两正交移动的渗透圆盘间的流动模型

Fig. 1 Model for orthogonally moving porous disks

根据以上假设, 控制方程可以表示为以下形式<sup>[7-8]</sup>:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \quad (1)$$

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = \\ - \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\mu + \kappa}{\rho} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \right. \\ \left. \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\kappa}{\rho} \frac{\partial \phi}{\partial z}, \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = \\ - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu + \kappa}{\rho} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{\kappa}{\rho} \left( \frac{\partial \phi}{\partial z} + \frac{\phi}{r} \right), \end{aligned} \quad (3)$$

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial r} + w \frac{\partial \phi}{\partial z} = \frac{\gamma}{\rho j} \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{\phi}{r^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + \frac{\kappa}{\rho j} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) - 2 \frac{\kappa}{\rho j} \phi, \quad (4)$$

这里  $\rho, \nu$  分别为密度、粘性系数,  $j, \gamma, \kappa$  分别表示单位质量的微惯性、旋转梯度粘度、涡流粘度系数.  $\gamma$  表示为如下形式<sup>[35]</sup>:

$$\gamma = \left( \mu + \frac{\kappa}{2} \right) j, \quad (5)$$

这里  $\mu$  为动力粘性系数. 由于微元素在壁面附近的强聚集性, 我们假设靠近壁面的微元素处于静止状态, 没有旋转<sup>[36]</sup>. 以  $j = a^2$  作为参考尺寸长度, 则相应的边界条件为

$$u = 0, \quad w = 2v_w = A\dot{a}, \quad \phi = 0, \quad z = a(t), \quad (6)$$

$$u = 0, \quad w = -2v_w = -A\dot{a}, \quad \phi = 0, \quad z = -a(t), \quad (7)$$

这里  $A = 2v_w / \dot{a}$  是壁面渗透强度的表征值<sup>[15,19]</sup>.

考虑 von Kármán<sup>[37]</sup> 类型的相似变换, 受流函数定义的启发<sup>[7-8,20]</sup>, 在本文中, 我们定义

$$\begin{cases} u = -\frac{1}{r} \frac{\partial \psi}{\partial z} = -\frac{\nu r}{a^2} F_\eta(\eta, t), \\ w = \frac{1}{r} \frac{\partial \psi}{\partial r} = \frac{2\nu}{a} F(\eta, t), \\ \phi = -\frac{\nu r}{a^3} G(\eta, t), \quad \eta = \frac{z}{a}. \end{cases} \quad (8)$$

把  $u, w, \phi$  代入到方程(1)~(4), 消去方程中的压力项, 得到如下的非线性方程组:

$$(1 + K)F_{\eta\eta\eta\eta} + \alpha(3F_{\eta\eta} + \eta F_{\eta\eta\eta}) - 2FF_{\eta\eta\eta} - KG_{\eta\eta} - a^2\nu^{-1}F_{\eta\eta t} = 0, \quad (9)$$

$$(1 + K/2)G_{\eta\eta} + \alpha(3G + \eta G_\eta) + KF_{\eta\eta} - 2KG + F_\eta G - 2FG_\eta - a^2\nu^{-1}G_t = 0, \quad (10)$$

边界条件(6)、(7)变成

$$F'(1) = 0, \quad F(1) = Re, \quad G(1) = 0, \quad (11)$$

$$F'(-1) = 0, \quad F(-1) = -Re, \quad G(-1) = 0, \quad (12)$$

这里  $K = \kappa/\mu, \alpha = a\dot{a}/\nu$  为壁面膨胀率<sup>[20]</sup>, 当其值为负数时, 表示两圆盘间的距离接近,  $Re = av_w/\nu$  为渗透 Reynolds 数, 为正数时表示喷注.

令

$$f = \frac{F}{Re}, \quad g = \frac{G}{Re}. \quad (13)$$

把式(13)代入方程组(9)、(10), 偏微分方程组(9)、(10)变成

$$(1 + K)f_{\eta\eta\eta\eta} + \alpha(3f_{\eta\eta} + \eta f_{\eta\eta\eta}) - 2Reff_{\eta\eta\eta} - Kg_{\eta\eta} - a^2\nu^{-1}f_{\eta\eta t} = 0, \quad (14)$$

$$(1 + K/2)g_{\eta\eta} + \alpha(3g + \eta g_\eta) + Kf_{\eta\eta} - 2Kg + Re(f_\eta g - 2fg_\eta) - a^2\nu^{-1}g_t = 0. \quad (15)$$

根据 Uchida 等<sup>[15]</sup>, Majdalani 等<sup>[18]</sup> 以及 Dauenhauer 等<sup>[20]</sup> 所描述的变换, 可以得到关于时间和空间的相似解: 假设  $\alpha$  是常数并且  $f = f(\eta)$ , 这样  $f_{\eta\eta t} = 0$ . 相似地<sup>[23]</sup>, 在本文中我们同样假设  $g = g(\eta)$ . 从物理的角度来看<sup>[15,19-20]</sup>, 该模型基于下述假设, 认为壁面运动速度随着两盘间距离增大而减小,

$$\alpha = \frac{\dot{a}a}{\nu} = \frac{\dot{a}_0 a_0}{\nu} = \text{const}, \quad (16)$$

这里  $a_0$  和  $\dot{a}_0$  表示初始的圆盘高度和膨胀速度. 对式(16)进行积分, 即可以得到相似解, 其结果为

$$\frac{a}{a_0} = \frac{v_w(0)}{v_w(t)} = \sqrt{1 + 2\nu\alpha t a_0^{-2}}. \quad (17)$$

因为  $2v_w = A\dot{a}, A = \text{const}$ <sup>[15,19-20]</sup>, 则可以得到壁面的喷注速度的表达式.

根据如上假设, 微分方程(14)、(15)变成

$$(1 + K) f'''' + \alpha(3 f'' + \eta f''') - K g'' - 2Re f'''' = 0, \quad (18)$$

$$(1 + K/2) g'' + \alpha(3g + \eta g') + K f'' - 2Kg + Re(f'g - 2fg') = 0, \quad (19)$$

边界条件为

$$f'(1) = 0, f(1) = 1, g(1) = 0, \quad (20)$$

$$f'(-1) = 0, f(-1) = -1, g(-1) = 0. \quad (21)$$

方程(18)、(19)也可以表示为

$$f'''' + \alpha/(1 + K)(3f'' + \eta f''') - c_1 g'' - 2Rff'''' = 0, \quad (22)$$

$$g'' + \alpha/(1 + K/2)(3g + \eta g') + c_2(f'' - 2g) + c_3(f'g - 2fg') = 0, \quad (23)$$

这里

$$R = \rho av_w / (\mu + \kappa), c_1 = \mu / (\mu + \kappa), c_2 = \kappa a^2 / \gamma, c_3 = av_w j / (\nu \gamma).$$

边界条件同式(20)、(21)一样. 当  $\alpha = 0$  时, 则是 Kamal 等<sup>[7]</sup>所讨论的情况.

## 2 速度的同伦分析解

选择初始猜测函数

$$f_0 = -\frac{1}{2} \eta^3 + \frac{3}{2} \eta, g_0 = 0 \quad (24)$$

和辅助线性算子为

$$\mathcal{L}_1(f) = f'''' , \mathcal{L}_2(f) = f'' , \quad (25)$$

辅助算子满足

$$\mathcal{L}_1(C_1 + C_2 \eta + C_3 \eta^2 + C_4 \eta^3) = 0, \mathcal{L}_2(C_5 + C_6 \eta) = 0, \quad (26)$$

这里  $C_i, i = 1, \dots, 6$  为常数.

由以上定义, 可得零阶形变方程

$$(1 - p) \mathcal{L}_1(\hat{f} - f_0) = ph \mathfrak{N}_1(\hat{f}, \hat{g}), \quad (27)$$

$$\hat{f}'(-1, p) = 0, \hat{f}(-1, p) = -1, \hat{f}'(1, p) = 0, \hat{f}(1, p) = 1, \quad (28)$$

$$(1 - p) \mathcal{L}_2(\hat{g} - g_0) = ph \mathfrak{N}_2(\hat{f}, \hat{g}), \quad (29)$$

$$\hat{g}(-1, p) = 0, \hat{g}(1, p) = 0, \quad (30)$$

$$\begin{aligned} \mathfrak{N}_1(\hat{f}, \hat{g}) = & (1 + K) \frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} - 2Re \hat{f}(\eta, p) \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + \\ & 3\alpha \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} + \alpha \eta \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} - K \frac{\partial^2 \hat{g}(\eta, p)}{\partial \eta^2}, \end{aligned} \quad (31)$$

$$\begin{aligned} \mathfrak{N}_2(\hat{f}, \hat{g}) = & \left(1 + \frac{K}{2}\right) \frac{\partial^2 \hat{g}(\eta, p)}{\partial \eta^2} + \\ & Re\left(\frac{\partial \hat{f}(\eta, p)}{\partial \eta} \hat{g}(\eta, p) - 2\hat{f}(\eta, p) \frac{\partial \hat{g}(\eta, p)}{\partial \eta}\right) + 3\alpha \hat{g}(\eta, p) + \\ & \alpha \eta \frac{\partial \hat{g}(\eta, p)}{\partial \eta} - 2K \hat{g}(\eta, p) + K \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2}, \end{aligned} \quad (32)$$

这里,  $p \in [0, 1]$  是植入参数,  $h$  为非零的辅助参数. 当  $p$  由 0 连续变到 1, 则函数  $\hat{f}(\eta, p)$ ,  $\hat{g}(\eta, p)$  分别从  $f_0(\eta)$ ,  $g_0(\eta)$  变到  $f(\eta)$ ,  $g(\eta)$ . 应用 Taylor 定理, 可以得到

$$\hat{f}(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \quad f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \hat{f}(\eta, p)}{\partial p^m} \right|_{p=0}, \quad (33)$$

$$\hat{g}(\eta, p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta) p^m, \quad g_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \hat{g}(\eta, p)}{\partial p^m} \right|_{p=0}. \quad (34)$$

级数(33)、(34)的收敛性强烈地依赖于  $h$  的取值. 一旦  $h$  的取值选定, 则级数(33)、(34) 在  $p = 1$  收敛. 根据级数(33)、(34), 可得

$$f = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad (35)$$

$$g = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta). \quad (36)$$

对式(27)、(29)关于  $p$  求  $m$  阶导数, 令  $p = 0$ , 两边同时除以  $m!$ , 则可以得到  $m$  阶形变公式

$$\mathcal{L}_1(f_m(\eta) - \chi_m f_{m-1}(\eta)) = h \mathfrak{N}_m^f(\eta), \quad (37)$$

$$f_m'(-1) = 0, f_m(-1) = 0, f_m'(1) = 0, f_m(1) = 0, \quad (38)$$

$$\mathfrak{N}_m^f(\eta) = (1 + K)f_{m-1}^{(4)} + \alpha(3f_{m-1}'' + \eta f_{m-1}''') - Kg_{m-1}'' - 2Re \sum_{k=0}^{m-1} f_{m-k-1}''' f_k, \quad (39)$$

$$\mathcal{L}_2(g_m(\eta) - \chi_m g_{m-1}(\eta)) = h \mathfrak{N}_m^g(\eta), \quad (40)$$

$$g_m(-1) = 0, g_m(1) = 0, \quad (41)$$

$$\mathfrak{N}_m^g(\eta) = \left(1 + \frac{K}{2}\right) g_{m-1}'' + \alpha(3g_{m-1} + \eta g_{m-1}') + K(f_{m-1}'' - 2g_{m-1}) + Re \sum_{k=0}^{m-1} (f_{m-k-1}' g_k - 2f_{m-k-1} g_k'), \quad (42)$$

这里

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (43)$$

方程(37)、(40)的通解是

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 \eta + C_3 \eta^2 + C_4 \eta^3, \quad (44)$$

$$g_m(\eta) = g_m^*(\eta) + C_5 + C_6 \eta, \quad (45)$$

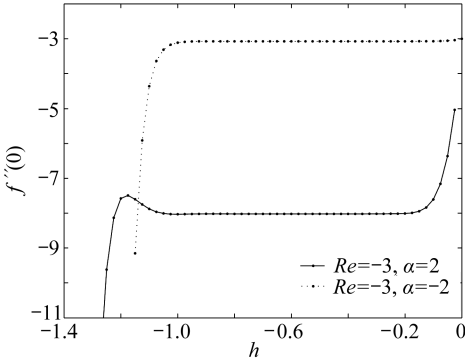
这里,  $f_m^*(\eta)$ ,  $g_m^*(\eta)$  表示方程(37)、(40)的特解, 常数  $C_i$  ( $i = 1, \dots, 6$ ) 可以根据边界条件(38)、(41)确定.

应用 Maple, 我们可以逐项求解出线性方程(37)、(40) ( $m = 1, 2, 3, \dots$ ).

### 3 结果和讨论

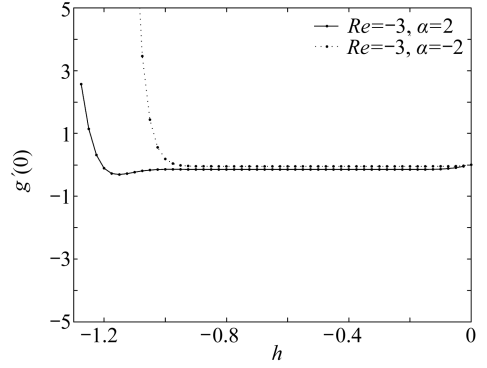
如 Liao<sup>[26-27]</sup> 所指出, 级数(33)、(34)的收敛性依赖于  $h$ , 并且  $h$  的值确定了 HAM 的收敛区间和收敛速度. 为了达到这样的目的, 在图 2 中, 我们画出所对应的  $h$  曲线. 一旦确定了合适的  $h$  值, 则我们可以得到收敛的 HAM 级数解. 从理论上讲<sup>[38]</sup>, 如图 3, 我们也可以通过定义  $k$  阶近似的残差的平方得到最优的  $h$  值, 用以提高收敛速度.

$$\Delta_k = \int_0^1 \left( \mathfrak{N}_1 \left( \sum_{m=0}^k f_m(\eta), \sum_{m=0}^k g_m(\eta) \right) \right)^2 + \left( \mathfrak{N}_2 \left( \sum_{m=0}^k f_m(\eta), \sum_{m=0}^k g_m(\eta) \right) \right)^2 d\eta. \quad (46)$$



(a)  $f''(0)$ - $h$  曲线

(a)  $f''(0)$ - $h$  curve



(b)  $g'(0)$ - $h$  曲线

(b)  $g'(0)$ - $h$  curve

图2 当  $K = 0.2$  时,  $f''(0)$ ,  $g'(0)$  的 20 阶近似所对应的  $h$  曲线

Fig.2  $h$ -curve on the 20th-order approximation for  $f''(0)$ ,  $g'(0)$  as  $K = 0.2$

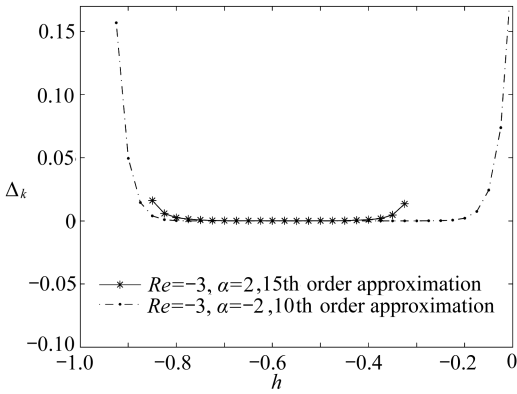
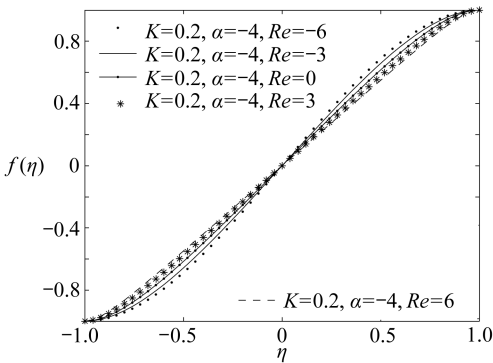


图3 当  $K = 0.2$  时, 残差的平方  $\Delta_k$  在不同的  $h$  处的取值

Fig.3  $\Delta_k$  for different  $h$  as  $K = 0.2$

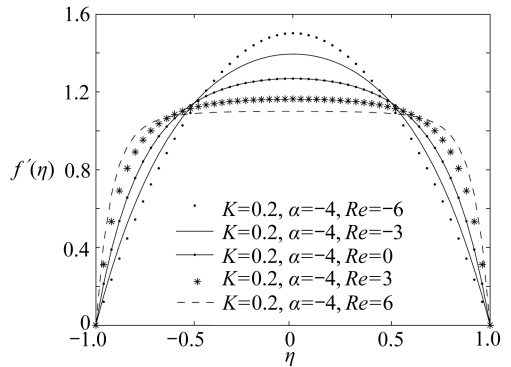
下面我们给出两圆盘间的速度和微旋转速度的图像,分析一些物理参数的影响.图4、图5给出渗透 Reynolds 数对速度的影响.图像表明,对于不同的  $Re$ ,其速度关于中心  $\eta = 0$  对称,而且,速度的最大值随着  $Re$  的增加而减少.图6给出在不同的渗透 Reynolds 数  $Re$  对微旋转的影响.从其变化趋势上来看,在两圆盘之间,存在两个极值点和一个零点.当  $\alpha = -4$  时,随着 Reynolds 数  $Re$  的增加,极值点的绝对值向圆盘位置移动,然而,当  $\alpha = 2$  时,随着 Reynolds 数  $Re$  的增加,极值点向圆盘中心移动.

图7~图9给出当  $K, Re$  不变的时候,负的  $\alpha$  对速度和微旋转的影响. $f(\eta)$  在两圆盘之间改变



(a)  $f(\eta)$  的曲线

(a) Characteristic of  $f(\eta)$

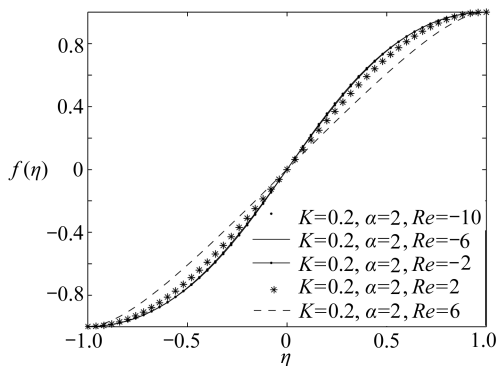


(b)  $f'(\eta)$  的曲线

(b) Characteristic of  $f'(\eta)$

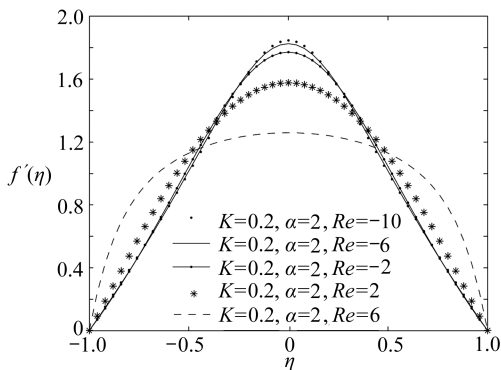
图4 当  $\alpha = -4$ ,  $K = 0.2$  时,  $f, f'$  随  $Re$  的变化曲线

Fig.4 Characteristic of  $f, f'$  for different  $Re$  as  $\alpha = -4$ ,  $K = 0.2$



(a)  $f(\eta)$  的曲线

(a) Characteristic of  $f(\eta)$

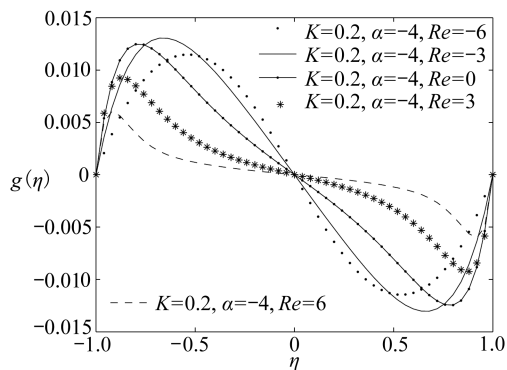


(b)  $f'(\eta)$  的曲线

(b) Characteristic of  $f'(\eta)$

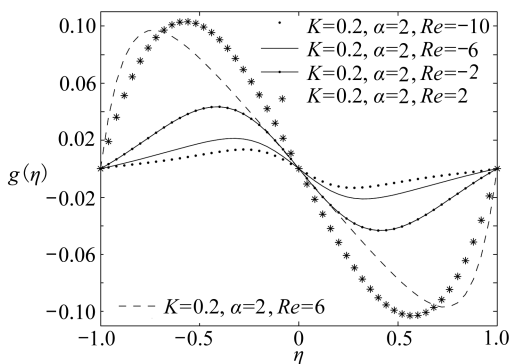
图 5 当  $\alpha = 2, K = 0.2$  时,  $f, f'$  随  $Re$  的变化曲线

Fig. 5 Characteristic of  $f, f'$  for different  $Re$  as  $\alpha = 2, K = 0.2$



(a)  $g(\eta)$  的曲线 ( $\alpha = -4$ )

(a) Characteristic of  $g(\eta)$  ( $\alpha = -4$ )

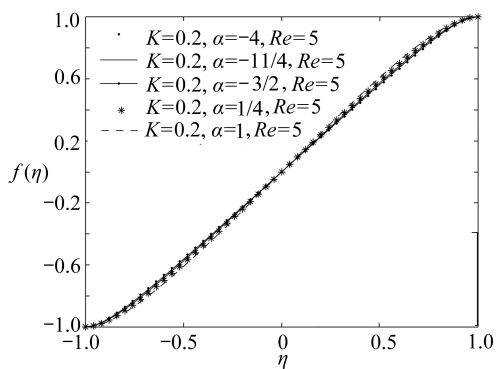


(b)  $g(\eta)$  的曲线 ( $\alpha = 2$ )

(b) Characteristic of  $g(\eta)$  ( $\alpha = 2$ )

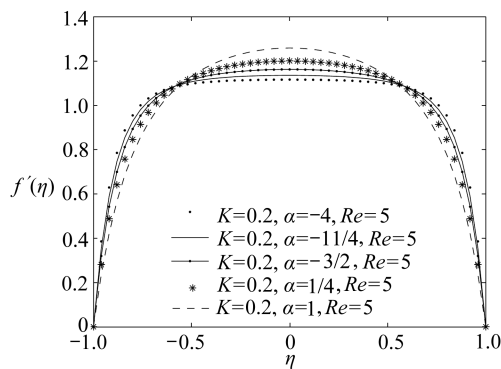
图 6 当  $\alpha = -4, K = 0.2$  和  $\alpha = 2, K = 0.2$  时,  $g$  随  $Re$  的变化曲线

Fig. 6 Characteristic of  $g$  for different  $Re$  as  $\alpha = -4, K = 0.2$  and  $\alpha = 2, K = 0.2$ , respectively



(a)  $f(\eta)$  的曲线

(a) Characteristic of  $f(\eta)$



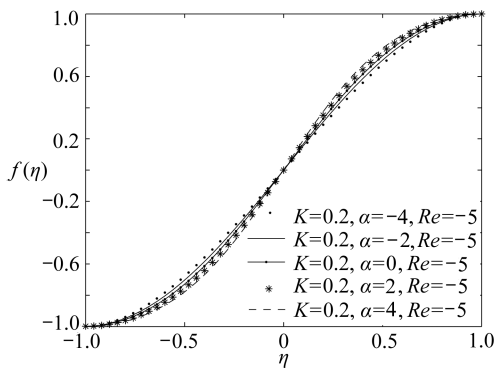
(b)  $f'(\eta)$  的曲线

(b) Characteristic of  $f'(\eta)$

图 7 当  $Re = 5, K = 0.2$  时,  $f, f'$  随  $\alpha$  的变化曲线

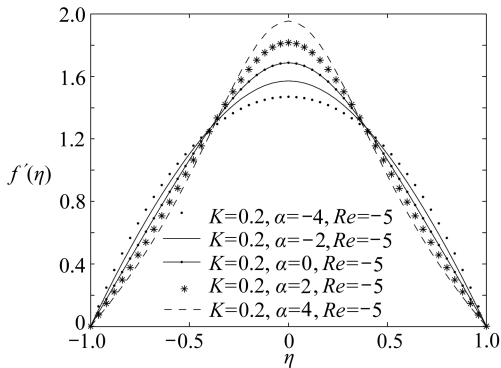
Fig. 7 Characteristic of  $f, f'$  for different  $\alpha$  as  $Re = 5, K = 0.2$





(a)  $f(\eta)$  的曲线

(a) Characteristic of  $f(\eta)$

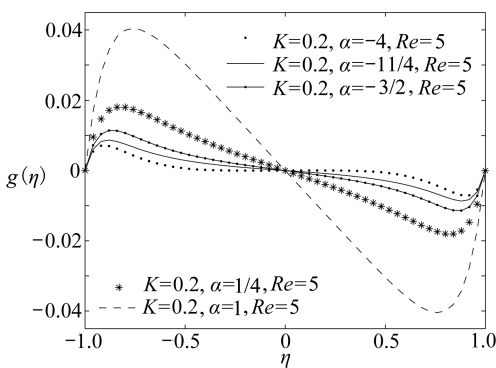


(b)  $f'(\eta)$  的曲线

(b) Characteristic of  $f'(\eta)$

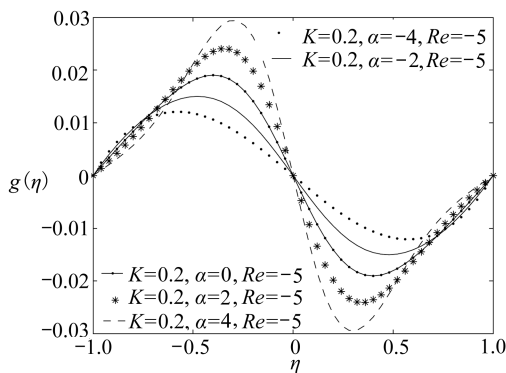
图 8 当  $Re = -5, K = 0.2$  时,  $f, f'$  随  $\alpha$  的变化曲线

Fig. 8 Characteristic of  $f, f'$  for different  $\alpha$  as  $Re = -5, K = 0.2$



(a)  $g(\eta)$  的曲线 ( $Re = 5$ )

(a) Characteristic of  $g(\eta)$  ( $Re = 5$ )

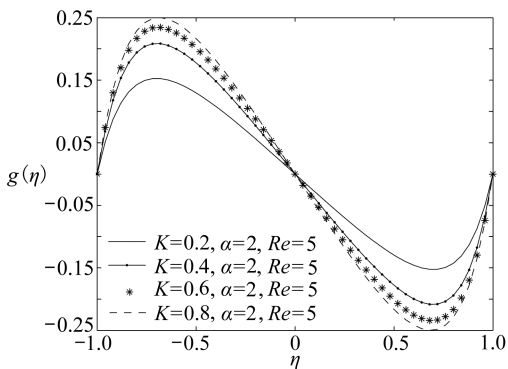


(b)  $g(\eta)$  的曲线 ( $Re = -5$ )

(b) Characteristic of  $g(\eta)$  ( $Re = -5$ )

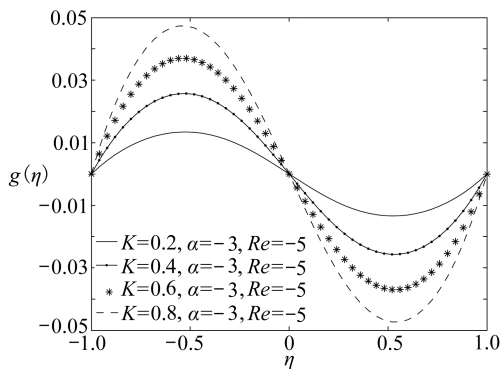
图 9 当  $Re = 5, K = 0.2$  和  $Re = -5, K = 0.2$  时,  $g(\eta)$  随  $\alpha$  的变化曲线

Fig. 9 Characteristic of  $g(\eta)$  for different  $\alpha$  as  $Re = 5, K = 0.2$  and  $Re = -5, K = 0.2$ , respectively



(a)  $g(\eta)$  的曲线 ( $Re = 5, \alpha = 2$ )

(a) Characteristic of  $g(\eta)$  ( $Re = 5, \alpha = 2$ )



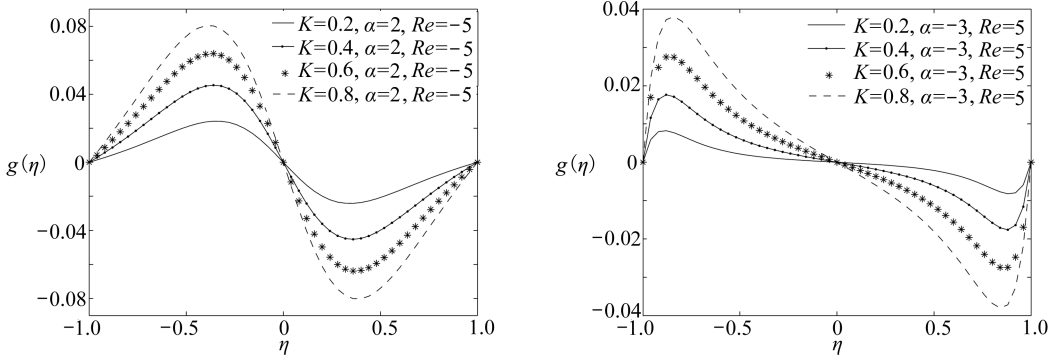
(b)  $g(\eta)$  的曲线 ( $Re = -5, \alpha = -3$ )

(b) Characteristic of  $g(\eta)$  ( $Re = -5, \alpha = -3$ )

图 10 当  $Re = 5, \alpha = 2$  和  $Re = -5, \alpha = -3$  时,  $g(\eta)$  随  $K$  的变化曲线

Fig. 10 Characteristic of  $g(\eta)$  for different  $K$  as  $Re = 5, \alpha = 2$  and  $Re = -5, \alpha = -3$ , respectively



(a)  $g(\eta)$  的曲线 ( $Re = -5, \alpha = 2$ )(a) Characteristic of  $g(\eta)$  ( $Re = -5, \alpha = 2$ )(b)  $g(\eta)$  的曲线 ( $Re = 5, \alpha = -3$ )(b) Characteristic of  $g(\eta)$  ( $Re = 5, \alpha = -3$ )图 11 当  $Re = -5, \alpha = 2$  和  $Re = 5, \alpha = -3$  时,  $g(\eta)$  随  $K$  的变化曲线Fig. 11 Characteristic of  $g(\eta)$  for different  $K$  as  $Re = -5, \alpha = 2$  and  $Re = 5, \alpha = -3$ , respectively

其凹凸性。同样速度关于原点对称,最大值位于两圆盘间的中心位置,而且为变量  $\alpha$  的增函数。然而,对于变量  $f(\eta)$ ,当  $Re = 5$  时,随着  $\alpha$  的变化,其没有明显的变化。对于微旋转速度  $g(\eta)$ ,在靠近壁面的位置,符号相反,两圆盘之间存在一个零点,两个极值点。符号相反是基于这样的一个事实,在靠近壁面的位置其剪切应力使得流体以相反的方向旋转,在中心位置,其相互抵消,达到平衡状态。

图 10、图 11 给出在不同的  $K$  值影响下微旋转的变化图像。可以看出,最大值点的绝对值为变量  $K$  的增函数,且对于  $Re = 5, \alpha = 2$  的情况,  $K$  的影响比其他的更大。

## 4 结 论

本文分析了两正交移动的圆盘间的微极性流体的发展特征。静止壁面状态下的两对称的圆盘间的流动为本文研究的一个特例,对应参数  $\alpha = 0$ 。详细讨论了不同的参数,如渗透 Reynolds 数  $Re$ , 壁面的膨胀率  $\alpha$  以及微极性参数  $K$  等对流体的影响。

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## Flow of a Micropolar Fluid Between Two Orthogonally Moving Porous Disks

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**Abstract:** The unsteady, laminar, incompressible and two dimensional flow of a micropolar fluid between two orthogonally moving porous coaxial disks was considered. An extension of von Kármán's similarity transformations was applied to reduce the governing partial differential equations (PDEs) to a set of non-linear coupled ordinary differential equations (ODEs) in dimensionless form. The analytical solutions were obtained by employing the homotopy analysis method. The effects of various physical parameters like the expansion ratio, the permeability Reynolds number on the velocity fields were discussed in detail.

**Key words:** homotopy analysis method; expansion ratio; orthogonally moving porous disks