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广义 Oldroyd-B 流体作 MHD 旋转流动时的精确解*

T·哈亚特[1,2], S·翟贝[1], S·阿斯哈[3], A·A·亨迪[2]

(1. 真纳大学 数学系 45320,伊斯兰堡 44000,巴基斯坦;
2. 沙特国王大学 科学学院 物理系,邮政信箱 1846,
利雅得 11321,沙特阿拉伯;
3. COMSATS 信息职业技术学院 数学系,伊斯兰堡 44000,巴基斯坦)

摘要: 对充满多孔空间的广义 Oldroyd-B 流体,研究有滑移时磁流体动力学(MHD)的旋转流动. 数学建模时采用了分式积分法. 给出平板振荡和周期性压力梯度诱导的 3 个示例,并得到了每种情况下的精确解. 比较了有滑移和没有滑移条件下两种情况的结果. 曲线图表表明,滑移对速度分布的影响是巨大的.

关 键 词: 滑移条件; 精确解; 分式积分
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引 言

在过去的数十年间,非 Newton 流体的流动动力学受到了广泛关注,这是由于非 Newton 流体的流动与许多工业应用有关. 例如,食品加工、制药、泡沫材料、化学和生物化学的流体介质,一般都呈现出非 Newton 流体特性. 这种流体具有非线性剪切应力-应变比特点. 通常,表示非 Newton 流体本构关系的方程高度非线性,阶次也比 Navier-Stokes 方程高得多. 因此,研究人员求解这类问题的唯一解时,需要补充额外的边界条件. 文献[1-3]给出了出色的研究报告.

在过去,大量的研究致力于在非旋转坐标系中不同比率类型流体的流动. 最近文献[4-15]提及了旋转坐标系中流动的研究. 文献[16-17]还讨论了非 Newton 流体稳定的和不稳定的旋转流动. 文献[18]对该方向的研究做出非常有益的评论. 文献[19]作者注意到滑移条件对三阶流体旋转流动稳定性的影响.

就我们所知道,滑移对不同类型流体在旋转坐标系中流动的影响很少受到关注. 然而,迄今为止没有讨论过滑移对比率类型流体流动的影响. 本文的目的是,研究滑移对多孔材料中 Oldroyd-B 流体旋转流动的影响. 流体受到横向磁场作用. 控制方程包含了 Oldroyd-B 流体修正

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作者简介: T. Hayat, 教授(联系人. Tel:+92-51-90642127; Fax:+92-51-2601171; E-mail: pensy_t@yahoo.com).
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的 Darcy 定律. 事实上本文的目的是,对 3 种振荡流动估计和量化滑移参数的取值范围. 数学分析中采用了分式积分方法[20-30]. 作为结果,给出了速度分布;结果表明,得到的粘性二阶流体和 Maxwell 流体看作存在闭式解的极限情况.

1　基 本 方 程

不可压缩流体在旋转坐标系中流动时的控制方程为

$$\rho\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}+2\Omega\times\boldsymbol{V}+\Omega\times(\Omega\times r)\right)=-\nabla p+\mathrm{div}\boldsymbol{S}+\boldsymbol{J}\times\boldsymbol{B}+\boldsymbol{R},\tag{1}$$

$$\mathrm{div}\boldsymbol{V}=0,\tag{2}$$

其中, $\boldsymbol{V}=(u,v,w)$ 为速度场,ρ 为流体密度,p 为压力,Ω 为旋转坐标系的角速度,$r^2=x^2+y^2$, $\boldsymbol{S}$ 为广义 Oldroyd-B 流体中的附加应力,即[4]

$$\left(1+\lambda^{\alpha}\frac{\mathrm{D}^{\alpha}}{\mathrm{D}t^{\alpha}}\right)\boldsymbol{S}=\mu\left(1+\theta^{\beta}\frac{\mathrm{D}^{\beta}}{\mathrm{D}t^{\beta}}\right)\boldsymbol{A}_1,\tag{3}$$

其中,μ 为动力学粘度,λ 为松弛时间,θ 为延迟时间,α 和 β 为分式积分参数,且 $0\leqslant\alpha\leqslant\beta\leqslant 1$, $\alpha\leqslant\beta$. $\boldsymbol{A}_1$ 为 Rivlin-Erickson 第一张量,由下式给出:

$$\boldsymbol{A}_1=\nabla\boldsymbol{V}+(\nabla\boldsymbol{V})^{\mathrm{T}},\tag{4}$$

其中,∇为梯度算子,T 表示矩阵的转置,且

$$\frac{\mathrm{D}^{\alpha}\boldsymbol{S}}{\mathrm{D}t^{\alpha}}=\frac{\tilde{\mathrm{D}}^{\alpha}\boldsymbol{S}}{\tilde{\mathrm{D}}t^{\alpha}}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{S}-(\nabla\boldsymbol{V})\boldsymbol{S}-\boldsymbol{S}(\nabla\boldsymbol{V})^{\mathrm{T}},\tag{5}$$

这里,对 t 的 α 阶分式导数定义如下:

$$\frac{\tilde{\mathrm{D}}^{\alpha}f(t)}{\tilde{\mathrm{D}}t^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\varepsilon)^{-\alpha}f(t)\,\mathrm{d}\varepsilon,\qquad 0<\alpha<1,\tag{6}$$

其中 $\Gamma(\cdot)$ 为 Gamma 函数. 当 $\alpha=\beta=1$ 时,方程(3)简化为 Oldroyd-B 流体方程. 当 $\lambda=0$ 且 $\mu\theta=\alpha_1$ (流体的材料参数)时,得到广义的二阶流体方程;当 $\theta=0$ 时,即得到广义的 Maxwell 流体方程. 当 $\lambda=\theta=0$ 和 $\alpha=\beta=1$ 时,推演出 Navier-Stokes 流体方程.

Maxwell 方程可写为

$$\mathrm{div}\,\boldsymbol{B}=0,\ \mathrm{curl}\,\boldsymbol{B}=\mu_{\mathrm{m}}\boldsymbol{J},\ \mathrm{curl}\,\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t},\tag{7}$$

其中,$\boldsymbol{J}$ 为电流密度,$\boldsymbol{B}$ 为总磁场,且 $\boldsymbol{B}=\boldsymbol{B}_0+\boldsymbol{b}$,$\boldsymbol{B}_0$ 和 $\boldsymbol{b}$ 分别为外加作用磁场和感应磁场,μ_{m} 为磁导率,$\boldsymbol{E}$ 为电场.

本分析假定磁 Reynolds 数很小, 因此感应磁场可以忽略不计. 磁场 $\boldsymbol{B}_0$ 作用于 z 方向, 且 $\boldsymbol{E}=0$. 鉴于这些假设,Lorentz 力变为

$$\boldsymbol{J}\times\boldsymbol{B}=-\sigma B_0^2\boldsymbol{V}.\tag{8}$$

在广义 Oldroyd-B 流体中,Darcy 阻力 $\boldsymbol{R}$ 的表达式为[4]

$$\left(1+\lambda^{\alpha}\frac{\mathrm{D}^{\alpha}}{\mathrm{D}t^{\alpha}}\right)\boldsymbol{R}=-\frac{\mu\phi}{K}\left(1+\theta^{\beta}\frac{\mathrm{D}^{\beta}}{\mathrm{D}t^{\beta}}\right)\boldsymbol{V},\tag{9}$$

其中,ϕ 为介质的孔隙率,K 为介质的渗透率.

2　控 制 方 程

考虑不可压缩的广义 Oldroyd-B 流体,在非导电刚性平板 ($z=0$) 上,作磁流体(MHT)旋转流动. 流体和平板以匀角速度平行于 z 轴旋转(垂直于平板). 对一块无限大的平板,除了压

力外,所有的物理量均与 z 和 t 有关. 本文中,附加应力张量 $\boldsymbol{S}$ 和速度 $\boldsymbol{V}$ 分布为

$$\boldsymbol{S}(z,t)=\begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}, \tag{10}$$

$$\boldsymbol{V}(z,t)=(u(z,t),v(z,t),0). \tag{11}$$

由方程(11),连续方程自动满足,并由方程(1)和(3)导得

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)S_{xz}=\mu\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial u}{\partial z}, \tag{12}$$

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)S_{yz}=\mu\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial v}{\partial z}, \tag{13}$$

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left(\frac{\partial u}{\partial t}-2\Omega v\right)=-\frac{1}{\rho}\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\frac{\partial\hat{p}}{\partial x}+\nu\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial^{2}u}{\partial z^{2}}-\frac{\sigma B_0^2}{\rho}\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)u-\frac{\nu\phi}{k}\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)u, \tag{14}$$

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left(\frac{\partial v}{\partial t}+2\Omega u\right)=-\frac{1}{\rho}\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\frac{\partial\hat{p}}{\partial y}+\nu\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial^{2}v}{\partial z^{2}}-\frac{\sigma B_0^2}{\rho}\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)v-\frac{\nu\phi}{k}\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)v, \tag{15}$$

$$\frac{\partial\hat{p}}{\partial z}=0,$$

其中,ν 为运动学粘度,修正压力 $p=p-(\rho/2)\Omega^2r^2$,且上述方程意味着 $p\neq p(z)$.

记

$$F=u+\mathrm{i}v, \tag{16}$$

方程(14)和(15)可以组合成下列微分方程:

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left[\frac{\partial F}{\partial t}+2\mathrm{i}\Omega F\right]=-\frac{1}{\rho}\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left(\frac{\partial\hat{p}}{\partial x}+\mathrm{i}\frac{\partial\hat{p}}{\partial y}\right)+\nu\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial^{2}F}{\partial z^{2}}-\frac{\sigma B_0^2}{\rho}\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)F-\frac{\nu\phi}{k}\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)F. \tag{17}$$

下节将讨论 3 种流动实例.

3 一般周期振荡引起的流动

本节讨论在滑移条件下,平板作一般周期振荡时产生的流动. 根据剪应力定义的滑移条件可用如下的数学表达式表示:

$$u(0,t)-\frac{\gamma}{\mu}S_{xz}=U_0\sum_{k=-\infty}^{\infty}a_k\mathrm{e}^{\mathrm{i}k\omega_0 t},\ v(0,t)-\frac{\gamma}{\mu}S_{yz}=0, \tag{18}$$

其中 γ 为滑移参数,且

$$u,v\to 0,\qquad \text{当 } z\to\infty. \tag{19}$$

式(18)中 Fourier 级数的系数为

$$a_k=\frac{1}{T_0}\int_{T_0}f(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\mathrm{d}t. \tag{20}$$

根据 F 边界条件简化为

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)F(z,t)-\gamma\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial F}{\partial z}=$$

$$U_0\sum_{k=-\infty}^{\infty}a_k\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\mathrm{e}^{\mathrm{i}k\omega_0 t},\qquad \text{当 } z=0, \tag{21}$$

$$F(\infty,t)=0. \tag{22}$$

考虑到式(17),并没有修正的压力梯度,又,利用

$$\begin{cases} z^*=\dfrac{zU_0}{\nu},\ F^*=\dfrac{F}{U_0},\ t^*=\dfrac{tU_0^2}{\nu},\ \omega_0^*=\dfrac{\omega_0\nu}{U_0^2},\\ \lambda_1^*=\dfrac{\lambda U_0^2}{\nu},\ \theta^*=\dfrac{\theta U_0^2}{\nu},\ \Omega^*=\dfrac{\Omega\nu}{U_0^2},\\ M^{*2}=\dfrac{\sigma B_0^2\nu}{\rho U_0^2},\ \dfrac{1}{K}=\dfrac{\phi\nu^2}{kU_0^2},\ \gamma^*=\dfrac{U_0\gamma}{\nu}, \end{cases} \tag{23}$$

有

$$\begin{cases} \left(1+\lambda^{\alpha}\dfrac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left(\dfrac{\partial F}{\partial t}+2\mathrm{i}\Omega F\right)=\left(1+\theta^{\beta}\dfrac{\partial^{\beta}}{\partial t^{\beta}}\right)\dfrac{\partial^2 F}{\partial z^2}-\\ \qquad M^2\left(1+\lambda^{\alpha}\dfrac{\partial^{\alpha}}{\partial t^{\alpha}}\right)F-\dfrac{1}{K}\left(1+\theta^{\beta}\dfrac{\partial^{\beta}}{\partial t^{\beta}}\right)F,\\ \left(1+\lambda^{\alpha}\dfrac{\partial^{\alpha}}{\partial t^{\alpha}}\right)F(z,t)-\gamma\left(1+\theta^{\beta}\dfrac{\partial^{\beta}}{\partial t^{\beta}}\right)\dfrac{\partial F}{\partial z}=\\ \qquad \displaystyle\sum_{k=-\infty}^{\infty}a_k\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\mathrm{e}^{\mathrm{i}k\omega_0 t},\qquad \text{当 } z=0, \end{cases} \tag{24}$$

$$F(\infty,t)=0, \tag{25}$$

上面略去了上角的星号.

利用 Fourier 变换法,可写为

$$\psi(z,\omega)=\int_{-\infty}^{\infty}u(z,t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t, \tag{26}$$

$$u(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(z,\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\,\omega \tag{27}$$

和

$$\int_{-\infty}^{\infty}\frac{\mathrm{D}^{\beta}}{\mathrm{D}t^{\beta}}[u(z,t)]\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t=(\mathrm{i}\,\omega)^{\beta}\psi(z,\omega), \tag{28}$$

$$(\mathrm{i}\,\omega)^{\beta}=|\omega|^{\beta}\mathrm{e}^{(\mathrm{i}\beta\pi/2)\operatorname{sign}\omega}=|\omega|^{\beta}\left(\cos\frac{\beta\pi}{2}+\mathrm{i}\ \operatorname{sign}\omega\sin\frac{\beta\pi}{2}\right). \tag{29}$$

方程(24)和(25)组成的方程组的解为

$$F(z,t)=\sum_{k=-\infty}^{\infty}\frac{a_k(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\mathrm{e}^{-m_k z+\mathrm{i}(k\omega_0 t-n_k z)}}{(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})+\gamma(m_k+\mathrm{i}n_k)(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})}, \tag{30}$$

这里

$$m_k=\sqrt{\frac{\sqrt{L_{\mathrm{r}}^2+L_{\mathrm{i}}^2}+L_{\mathrm{r}}}{2}},\ n_k=\sqrt{\frac{\sqrt{L_{\mathrm{r}}^2+L_{\mathrm{i}}^2}-L_{\mathrm{r}}}{2}}, \tag{31}$$

$$L_{\mathrm{r}}=\frac{1}{K}+\left[M^2\left\{\left(1+\lambda^{\alpha}|k\omega_0|^{\alpha}\cos\frac{\alpha\pi}{2}\right)\left(1+\theta^{\beta}|k\omega_0|^{\beta}\cos\frac{\beta\pi}{2}\right)+\right.\right.$$

$$\lambda^{\alpha}\theta^{\beta}|k\omega_0|^{\alpha+\beta}(\mathrm{sign}(k\omega_0))^2\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}\Big\} -$$
$$(k\omega_0+2\Omega)\mathrm{sign}(k\omega_0)\Big\{\lambda^{\alpha}|k\omega_0|^{\alpha}\sin\frac{\alpha\pi}{2}\Big(1+\theta^{\beta}|k\omega_0|^{\beta}\cos\frac{\beta\pi}{2}\Big) -$$
$$\theta^{\beta}|k\omega_0|^{\beta}\sin\frac{\beta\pi}{2}\Big(1+\lambda^{\alpha}|k\omega_0|^{\alpha}\cos\frac{\alpha\pi}{2}\Big)\Big\}\Big]\Big/\Big[\Big(1+\theta^{\beta}|k\omega_0|^{\beta}\cos\frac{\beta\pi}{2}\Big)^2 +$$
$$\Big(\theta^{\beta}|k\omega_0|^{\beta}\mathrm{sign}(k\omega_0)\sin\frac{\beta\pi}{2}\Big)^2\Big], \tag{32}$$

$$L_{\mathrm{i}} = \Big[(k\omega_0+2\Omega)\Big\{\Big(1+\lambda^{\alpha}|k\omega_0|^{\alpha}\cos\frac{\alpha\pi}{2}\Big)\Big(1+\theta^{\beta}|k\omega_0|^{\beta}\cos\frac{\beta\pi}{2}\Big) +$$
$$\lambda^{\alpha}\theta^{\beta}|k\omega_0|^{\alpha+\beta}(\mathrm{sign}(k\omega_0))^2\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}\Big\} -$$
$$M^2\mathrm{sign}(k\omega_0)\Big\{\theta^{\beta}|k\omega_0|^{\beta}\sin\frac{\beta\pi}{2}\Big(1+\lambda^{\alpha}|k\omega_0|^{\alpha}\cos\frac{\alpha\pi}{2}\Big) -$$
$$\lambda^{\alpha}|k\omega_0|^{\alpha}\sin\frac{\alpha\pi}{2}\Big(1+\theta^{\beta}|k\omega_0|^{\beta}\cos\frac{\beta\pi}{2}\Big)\Big\}\Big]\Big/\Big[\Big(1+\theta^{\beta}|k\omega_0|^{\beta}\cos\frac{\beta\pi}{2}\Big)^2 +$$
$$\Big(\theta^{\beta}|k\omega_0|^{\beta}\mathrm{sign}(k\omega_0)\sin\frac{\beta\pi}{2}\Big)^2\Big]. \tag{33}$$

值得注意的是,结果方程(30)对应于平板的一般周期振荡.作为一个特殊的振荡,流场由适当选择的 Fourier 系数 a_k 得到,就可以给出由不同的平板振荡所引起的流场.例如,流场 $F_j(j=1,2,\cdots,5)$ 分别由如下 5 种振荡:

$$\exp(\mathrm{i}\,\omega_0 t),\ \cos\omega_0 t,\ \sin\omega_0 t,\ \begin{cases}1, & |t|<T_1\\ 0, & T_1<|t|<T_0/2\end{cases},\ \sum_{k=-\infty}^{\infty}\delta(t-kT_0)$$

引起时,流场的表达式分别为

$$F_1(z,t)=\frac{(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\mathrm{e}^{-m_1z+\mathrm{i}(\omega_0t-n_1z)}}{(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})+\gamma(m_1+\mathrm{i}n_1)(1+\theta^{\beta}(\mathrm{i}\,\omega_0)^{\beta})}, \tag{34}$$

$$F_2(z,t)=\frac{1}{2}\Big[\frac{(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\mathrm{e}^{-m_1z+\mathrm{i}(\omega_0t-n_1z)}}{(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})+\gamma(m_1+\mathrm{i}n_1)(1+\theta^{\beta}(\mathrm{i}\,\omega_0)^{\beta})}+$$
$$\frac{(1+\lambda^{\alpha}(-\mathrm{i}\,\omega_0)^{\alpha})\mathrm{e}^{-m_{-1}z-\mathrm{i}(\omega_0t+n_{-1}z)}}{(1+\lambda^{\alpha}(-\mathrm{i}\,\omega_0)^{\alpha})+\gamma(m_{-1}+\mathrm{i}n_{-1})(1+\theta^{\beta}(-\mathrm{i}\,\omega_0)^{\beta})}\Big], \tag{35}$$

$$F_3(z,t)=\frac{1}{2\mathrm{i}}\Big[\frac{(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\mathrm{e}^{-m_1z+\mathrm{i}(\omega_0t-n_1z)}}{(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})+\gamma(m_1+\mathrm{i}n_1)(1+\theta^{\beta}(\mathrm{i}\,\omega_0)^{\beta})}-$$
$$\frac{(1+\lambda^{\alpha}(-\mathrm{i}\,\omega_0)^{\alpha})\mathrm{e}^{-m_{-1}z-\mathrm{i}(\omega_0t+n_{-1}z)}}{(1+\lambda^{\alpha}(-\mathrm{i}\,\omega_0)^{\alpha})+\gamma(m_{-1}+\mathrm{i}n_{-1})(1+\theta^{\beta}(-\mathrm{i}\,\omega_0)^{\beta})}\Big], \tag{36}$$

$$F_4(z,t)=\sum_{k=-\infty}^{\infty}\frac{\sin k\omega_0T_1}{k\pi}\big[(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\mathrm{e}^{-m_kz+\mathrm{i}(k\omega_0t-n_kz)}\big]/\big[(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})+$$
$$\gamma\{(m_k+\mathrm{i}n_k)(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})\}\big],\qquad k\neq 0, \tag{37}$$

$$F_5(z,t)=\frac{1}{T_0}\sum_{k=-\infty}^{\infty}\frac{(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\mathrm{e}^{-m_kz+\mathrm{i}(k\omega_0t-n_kz)}}{(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})+\gamma(m_k+\mathrm{i}n_k)(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})}. \tag{38}$$

4 两平板间的周期流动

本节处理相距 d 的两块刚性平板间的流动.由式(24)和(25)定义的无量纲数学问题,得

到

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)F(z,t)+\gamma\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial F}{\partial z}=0,\qquad \text{当 } z=1, \tag{39}$$

其中

$$\begin{cases} z^{*}=\dfrac{z}{d},\ F^{*}=\dfrac{F}{U_0},\ t^{*}=\dfrac{t}{d^2/\nu},\ \omega_0^{*}=\dfrac{\omega_0}{\nu/d^2}, \\ \lambda^{*}=\dfrac{\lambda}{d^2/\nu},\ \theta^{*}=\dfrac{\theta}{d^2/\nu},\ \Omega^{*}=\dfrac{\Omega}{\nu/d^2}, \\ M^{*2}=\dfrac{\sigma B_0^2}{\mu/d^2},\ \dfrac{1}{K}=\dfrac{\phi}{k/d^2},\ \gamma^{*}=\dfrac{\gamma}{d}. \end{cases} \tag{40}$$

其解为

$$\begin{aligned} F(z,t)=\sum_{k=-\infty}^{\infty}a_k\Big[&(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\{(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\sinh\beta_k(1-z)+ \\ &\gamma(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})\beta_k\cosh\beta_k(1-z)\}\Big/\{2\gamma(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})\times \\ &(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\beta_k\cosh\beta_k+ \\ &\sinh\beta_k[\gamma^2(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})^2\beta_k^2+(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})^2]\}\Big]\mathrm{e}^{\mathrm{i}k\omega_0 t}, \end{aligned} \tag{41}$$

$$\begin{aligned} \beta_k^2=\Big[&\mathrm{i}(k\omega_0+2\Omega)(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})+\frac{1}{K}(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})+ \\ &M^2(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\Big]\Big/[1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta}]. \end{aligned} \tag{42}$$

由5个特定振荡(见上节)引起的流场为

$$\begin{aligned} F_1(z,t)=&(1+\lambda^{\alpha}(i\omega_0)^{\alpha})\{(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\sinh\beta_1(1-z)+ \\ &\gamma(1+\theta^{\beta}(\mathrm{i}\,\omega_0)^{\beta})\beta_1\cosh\beta_1(1-z)\}\mathrm{e}^{\mathrm{i}\omega_0 t}\Big/\{2\gamma(1+\theta^{\beta}(\mathrm{i}\,\omega_0)^{\beta})\times \\ &(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\beta_1\cosh\beta_1+\sinh\beta_1[\gamma^2(1+\theta^{\beta}(\mathrm{i}\omega_0)^{\beta})^2\beta_1^2+ \\ &(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})^2]\}, \end{aligned} \tag{43}$$

$$\begin{aligned} F_2(z,t)=\frac{1}{2}\Big\{\Big[&(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\{(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\sinh\beta_1(1-z)+ \\ &\gamma(1+\theta^{\beta}(\mathrm{i}\,\omega_0)^{\beta})\beta_1\cosh\beta_1(1-z)\}\Big/\{2\gamma(1+\theta^{\beta}(\mathrm{i}\,\omega_0)^{\beta})\times \\ &(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\beta_1\cosh\beta_1+ \\ &\sinh\beta_1[\gamma^2(1+\theta^{\beta}(\mathrm{i}\,\omega_0)^{\beta})^2\beta_1^2+(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})^2]\}\Big]\mathrm{e}^{\mathrm{i}\omega_0 t}+ \\ \Big[&(1+\lambda^{\alpha}(-\mathrm{i}\,\omega_0)^{\alpha})\{(1+\lambda^{\alpha}(-\mathrm{i}\,\omega_0)^{\alpha})\sinh\beta_{-1}(1-z)+ \\ &\gamma(1+\theta^{\beta}(-\mathrm{i}\,\omega_0)^{\beta})\beta_{-1}\cosh\beta_{-1}(1-z)\}\Big/\{2\gamma(1+\theta^{\beta}(-\mathrm{i}\,\omega_0)^{\beta})\times \\ &(1+\lambda^{\alpha}(-\mathrm{i}\,\omega_0)^{\alpha})\beta_{-1}\cosh\beta_{-1}+ \\ &\sinh\beta_{-1}[\gamma^2(1+\theta^{\beta}(-\mathrm{i}\,\omega_0)^{\beta})^2\beta_{-1}^2+(1+\lambda^{\alpha}(-\mathrm{i}\,\omega_0)^{\alpha})^2]\}\Big]\mathrm{e}^{-\mathrm{i}\omega_0 t}\Big\}, \end{aligned} \tag{44}$$

$$F_3(z,t)=\frac{1}{2\mathrm{i}}\Big\{\Big[(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\{(1+\lambda^{\alpha}(\mathrm{i}\,\omega_0)^{\alpha})\sinh\beta_1(1-z)+$$

$$\gamma(1+\theta^{\beta}(\mathrm{i}\omega_0)^{\beta})\beta_1\cosh\beta_1(1-z)\}\Big/\{2\gamma(1+\theta^{\beta}(\mathrm{i}\omega_0)^{\beta})\times$$

$$(1+\lambda^{\alpha}(\mathrm{i}\omega_0)^{\alpha})\beta_1\cosh\beta_1+$$

$$\sinh\beta_1[\gamma^2(1+\theta^{\beta}(\mathrm{i}\omega_0)^{\beta})^2\beta_1^2+(1+\lambda^{\alpha}(\mathrm{i}\omega_0)^{\alpha})^2]\}\Big]\mathrm{e}^{\mathrm{i}\omega_0 t}-$$

$$\Big[(1+\lambda^{\alpha}(-\mathrm{i}\omega_0)^{\alpha})\{(1+\lambda^{\alpha}(-\mathrm{i}\omega_0)^{\alpha})\sinh\beta_{-1}(1-z)+$$

$$\gamma(1+\theta^{\beta}(-\mathrm{i}\omega_0)^{\beta})\beta_{-1}\cosh\beta_{-1}(1-z)\}\Big/\{2\gamma(1+\theta^{\beta}(-\mathrm{i}\omega_0)^{\beta})\times$$

$$(1+\lambda^{\alpha}(-\mathrm{i}\omega_0)^{\alpha})\beta_{-1}\cosh\beta_{-1}+$$

$$\sinh\beta_{-1}[\gamma^2(1+\theta^{\beta}(-\mathrm{i}\omega_0)^{\beta})^2\beta_{-1}^2+(1+\lambda^{\alpha}(-\mathrm{i}\omega_0)^{\alpha})^2]\}\Big]\mathrm{e}^{-\mathrm{i}\omega_0 t}\Big\},\tag{45}$$

$$F_4(z,t)=\sum_{k=-\infty}^{\infty}\frac{\sin k\omega_0T_1}{k\pi}\Big[(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\{(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\sinh\beta_k(1-z)+$$

$$\gamma(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})\beta_k\cosh\beta_k(1-z)\}\Big/\{2\gamma(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})\times$$

$$(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\beta_k\cosh\beta_k+\sinh\beta_k[\gamma^2(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})^2\beta_k^2+$$

$$(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})^2]\}\Big]\mathrm{e}^{\mathrm{i}k\omega_0 t},\qquad k\neq 0,\tag{46}$$

$$F_5(z,t)=\frac{1}{T_0}\sum_{k=-\infty}^{\infty}\Big[(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\{(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\sinh\beta_k(1-z)+$$

$$\gamma(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})\beta_k\cosh\beta_k(1-z)\}\mathrm{e}^{\mathrm{i}k\omega_0 t}\Big/\{2\gamma(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})\times$$

$$(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})\beta_k\cosh\beta_k+\sinh\beta_k[\gamma^2(1+\theta^{\beta}(\mathrm{i}k\omega_0)^{\beta})^2\beta_k^2+$$

$$(1+\lambda^{\alpha}(\mathrm{i}k\omega_0)^{\alpha})^2]\}\Big].\tag{47}$$

5 Poiseuille 流

考虑两块无限大的、相距 $2h$ 的固定平板间的流动. 流动是由于如下周期压力梯度所引起:

$$\frac{\partial\hat{p}}{\partial x}+\mathrm{i}\frac{\partial\hat{p}}{\partial y}=\rho Q_0\mathrm{e}^{\mathrm{i}\omega_0 t}.\tag{48}$$

流动问题的控制方程为

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left(\frac{\partial F}{\partial t}+2\mathrm{i}\Omega F\right)=$$

$$-\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)Q_0\mathrm{e}^{\mathrm{i}\omega_0 t}+\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial^2F}{\partial z^2}-$$

$$M^2\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)F-\frac{1}{K}\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)F,\tag{49}$$

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)F(z,t)+\gamma\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial F}{\partial z}=0,\qquad \text{当 } z=1,\tag{50}$$

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)F(z,t)-\gamma\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial F}{\partial z}=0,\qquad \text{当 } z=-1,\tag{51}$$

其中

$$\begin{cases} z^* = \dfrac{z}{\upsilon/(h/t)},\ F^* = \dfrac{F}{h/t},\ t^* = \dfrac{t}{\nu/(h/t)^2},\ \omega_0^* = \dfrac{\omega_0}{(h/t)^2/\upsilon}, \\ \lambda^* = \dfrac{\lambda}{\nu/(h/t)^2},\ \theta^* = \dfrac{\theta}{\nu/(h/t)^2},\ \dfrac{1}{K} = \dfrac{\phi}{k(h/t)^2/\upsilon^2},\ \Omega^* = \dfrac{\Omega}{(h/t)^2/\upsilon}, \\ M^{*2} = \dfrac{\sigma B_0^2}{\rho^2(h/t)^2/\mu},\ \gamma^* = \dfrac{\gamma}{\upsilon/(h/t)},\ Q_0^* = \dfrac{Q_0}{(h/t)^3/\upsilon}. \end{cases} \tag{52}$$

问题的解为

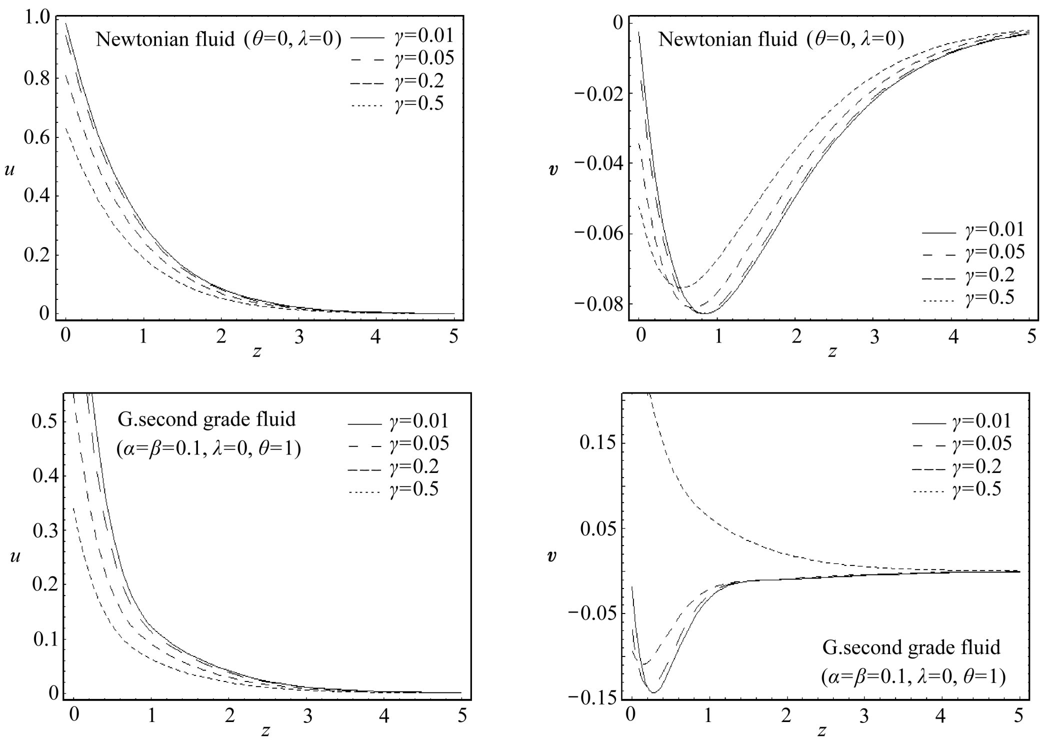
$$\begin{aligned} F(z,t) = {} & \frac{Q_0[(1+\lambda^\alpha(\mathrm{i}\,\omega_0)^\alpha)/(1+\theta^\beta(\mathrm{i}\,\omega_0)^\beta)]}{(\beta_1)^2} \times \\ & \{[(1+\lambda^\alpha(\mathrm{i}\,\omega_0)^\alpha)\{\cosh(\beta_1)z - \cosh(\beta_1)\} - \\ & \gamma(1+\theta^\beta(\mathrm{i}\omega_0)^\beta)(\beta_1)\sinh(\beta_1)]/[(1+\lambda^\alpha(\mathrm{i}\,\omega_0)^\alpha)\cosh(\beta_1) + \\ & \gamma(1+\theta^\beta(\mathrm{i}\,\omega_0)^\beta)(\beta_1)\sinh(\beta_1)]\}\,\mathrm{e}^{\mathrm{i}\omega_0 t}, \end{aligned} \tag{53}$$

其中

$$\beta_1 = \beta_k|_{k=1}. \tag{54}$$

6　图形结果及其讨论

本节介绍上面 3 种流动的特色. 特别强调无滑移和有滑移条件下不同的结果. 为此目的用图 1 至图 3 显示. 对 5 种流体模型流动情况下的数值由表列出. 表中列出了 5 种流体模型(即 Newton 流、广义二阶(G. second grade)流、广义 Maxwell(G. Maxwell)流、Oldroyd-B 流和广义 Oldroyd-B(G. Oldroyd-B)流)速度间的比较. 此外, 所有图形的左列表示 u 的变化, 而右列表示 v 的变化. 注意, 在一般周期振荡情况下, 仅勾画出 $\cos\omega_0 t$ 引起的流动.



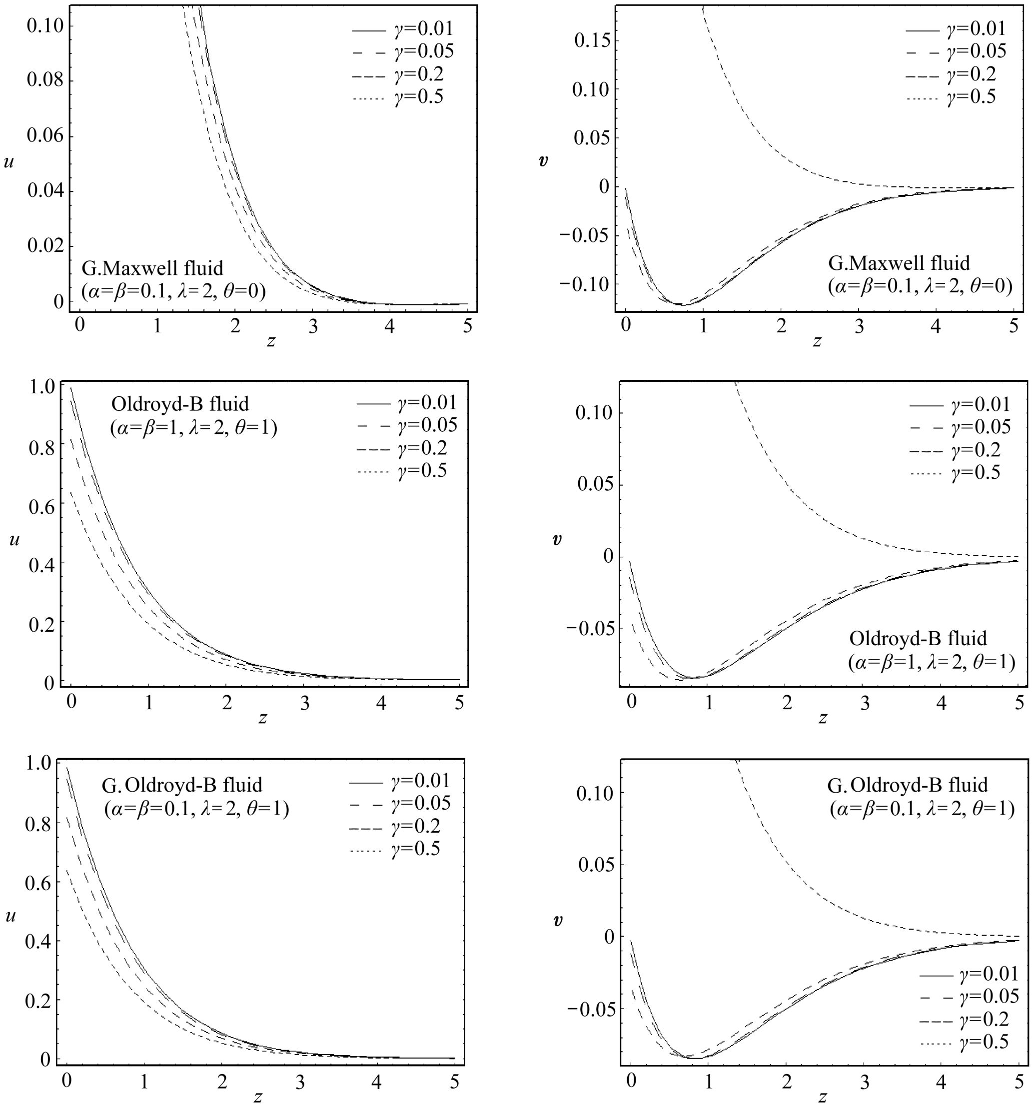
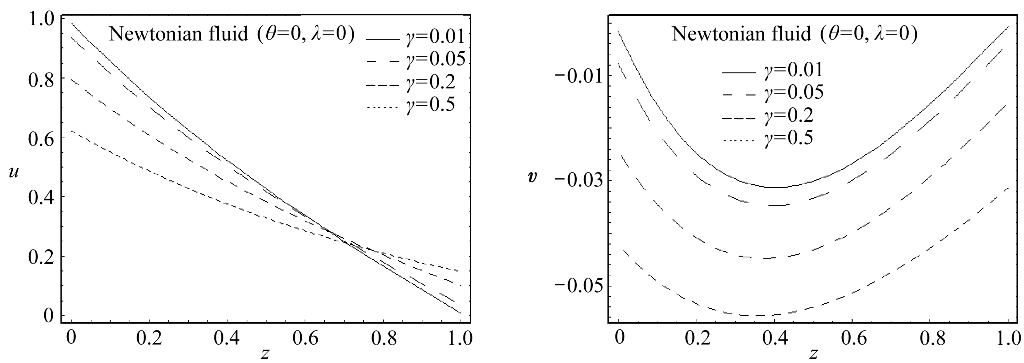


图 1 当 $M = t = 0.5$, $\omega_0 = 0.1$, $\Omega = 0.3$ 固定,作一般周期振荡时, γ 的变化对速度的影响

Fig. 1 The variation of γ on the velocity parts for general periodic oscillations when $M = t = 0.5$, $\omega_0 = 0.1$, $\Omega = 0.3$ are fixed



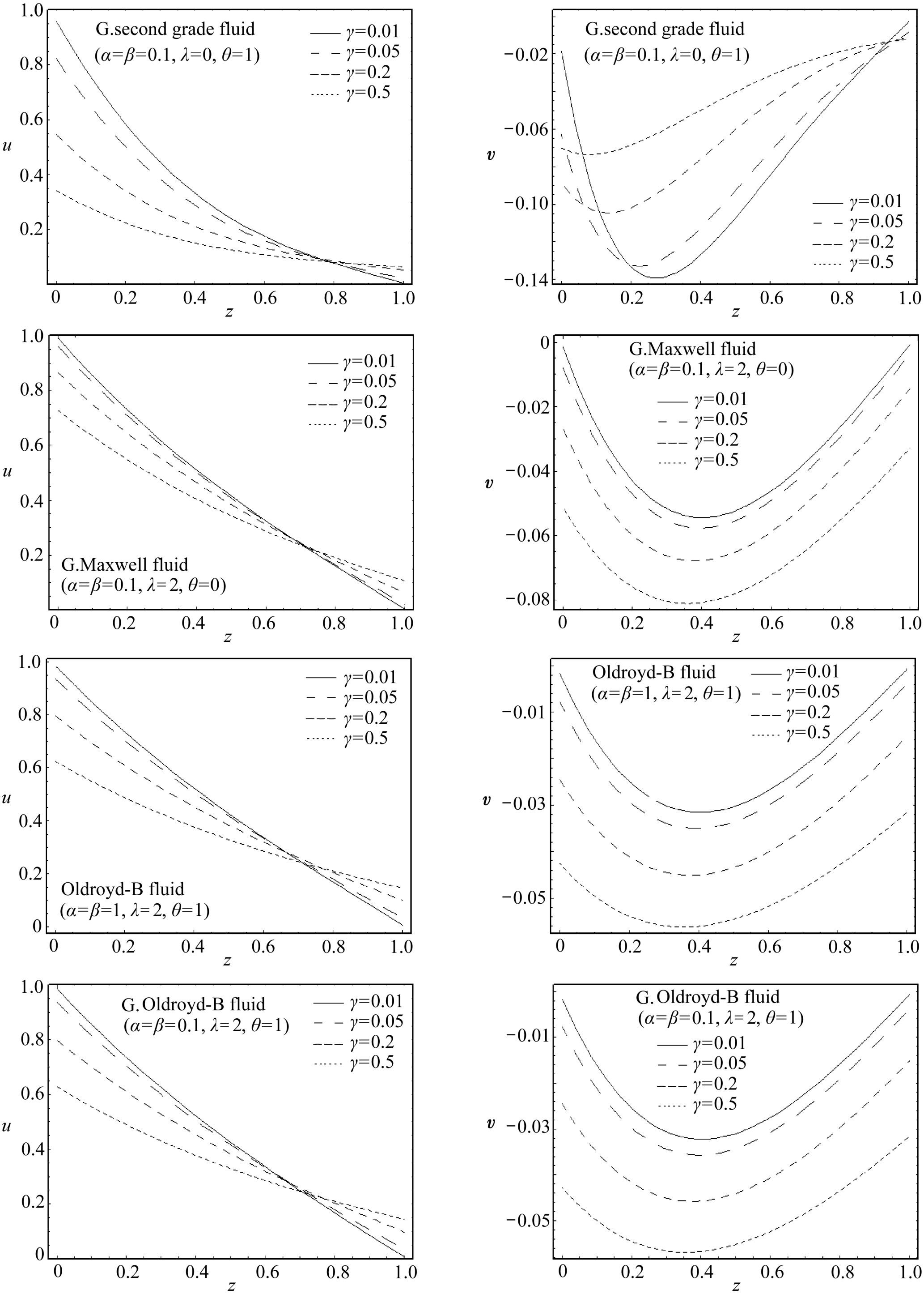
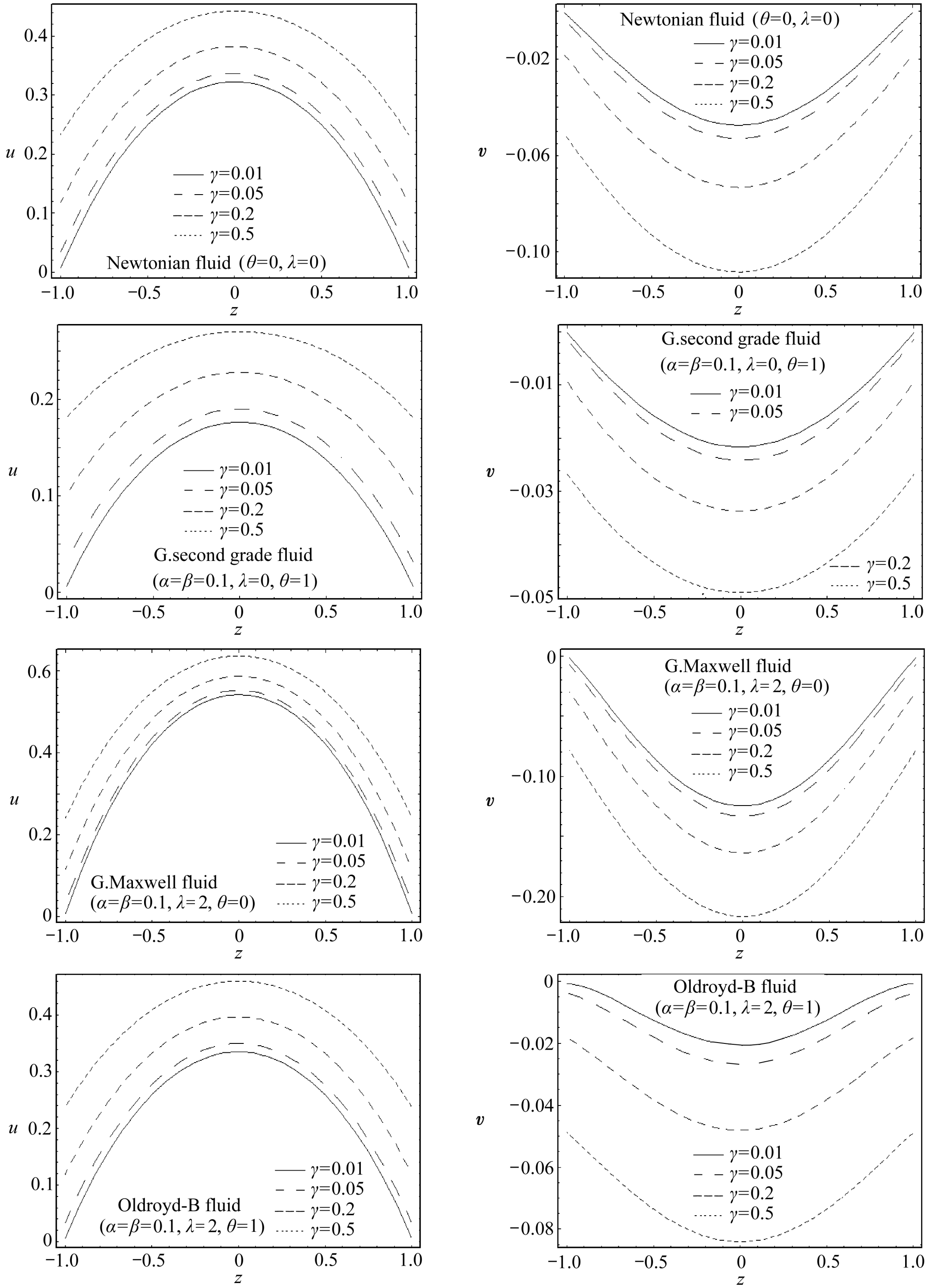


图 2　当 $M = t = 0.5$，$\omega_0 = 0.1$，$\Omega = 0.3$ 固定，两平板间作周期振荡时，γ 的变化对速度的影响

Fig. 2　The variation of γ on the velocity parts for periodic flow between two plates when $M = t = 0.5$，$\omega_0 = 0.1$，$\Omega = 0.3$ are fixed

图 1 示出作一般周期振荡时，u，v 的变化. 对所有 5 种类型流体来说，当滑移参数 γ 增大时，u 减小. 然而，v 值随着滑移参数 γ 的增大而增大. 图 2 给出了 γ 对两平行板间 $\cos\omega_0 t$ 振荡

所引起流动的影响. 由图 2 可见,速度 u 先随着 γ 的增加而减小,接着随着 γ 的增加而增大. 可是,除了广义二阶流体模型,v 值随着 γ 的增大而减小. 图 3 给出了 γ 对 Poiseuille 流动的影响. 图 3 表明,对所有流体模型,速度 u 都随着 γ 的增大而增大. 除广义 Oldroyd-B 流体模型,其他流体模型的速度 v,都随着 γ 的增大而减少.



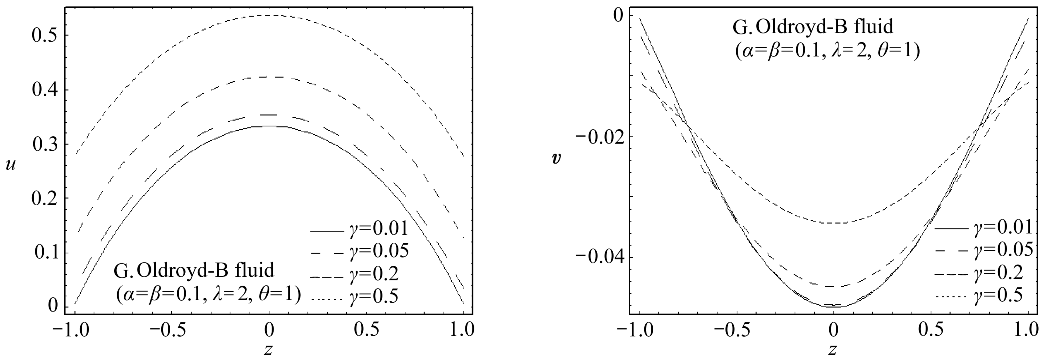


图 3　当 $M = t = 0.5$, $\omega_0 = 0.1$, $\Omega = 0.3$, $Q_0 = -1$ 固定, γ 的变化对 Poiseuille 流动速度的影响

Fig. 3　The variation of γ on the velocity parts for Poiseuille flow when $M = t = 0.5$, $\omega_0 = 0.1$, $\Omega = 0.3$, $Q_0 = -1$ are fixed

表 1　在滑移条件下作一般周期振荡时,流动速度的比较

Table 1　General periodic oscillations under slip condition

($\omega_0 = 0.1$, $\Omega = 0.3$, $M = t = 0.5$, $z = 0.5$, $\gamma = 0.5$)

type of fluid	rheological parameters	u	v
Newtonian fluid	$\lambda = 0$, $\theta = 0$	0.348 290	−0.075 379 6
G. second grade fluid	$(\alpha = \beta = 0.1)\lambda = 0$, $\theta = 1$	0.124 371	−0.044 012 6
G. Maxwell fluid	$(\alpha = \beta = 0.1)\lambda = 2$, $\theta = 0$	0.371 511	−0.115 682 0
Oldroyd-B fluid	$(\alpha = \beta = 1)\lambda = 2$, $\theta = 1$	0.349 637	−0.083 289 6
G. Oldroyd-B fluid	$(\alpha = \beta = 0.1)\lambda = 2$, $\theta = 1$	0.350 449	−0.077 400 5

表 2　滑移条件下两平板间作周期流动时,流动速度的比较

Table 2　Periodic flow between two plates under slip condition

($\omega_0 = 0.1$, $\Omega = 0.3$, $M = t = 0.5$, $z = 0.5$, $\gamma = 0.5$)

type of fluid	rheological parameters	u	v
Newtonian fluid	$\lambda = 0$, $\theta = 0$	0.327 715	−0.054 070 0
G. second grade fluid	$(\alpha = \beta = 0.1)\lambda = 0$, $\theta = 1$	0.126 460	−0.040 294 5
G. Maxwell fluid	$(\alpha = \beta = 0.1)\lambda = 2$, $\theta = 0$	0.342 603	−0.077 911 4
Oldroyd-B fluid	$(\alpha = \beta = 1)\lambda = 2$, $\theta = 1$	0.328 915	−0.054 598 5
G. Oldroyd-B fluid	$(\alpha = \beta = 0.1)\lambda = 2$, $\theta = 1$	0.329 016	−0.055 197 2

表 3　滑移条件下 Poiseuille 流动时速度的比较

Table 3　Poiseuille flow under slip condition

($Q_0 = -1$, $\omega_0 = 0.1$, $\Omega = 0.3$, $M = t = 0.5$, $z = 0.5$, $\gamma = 0.5$)

type of fluid	rheological parameters	u	v
Newtonian fluid	$\lambda = 0$, $\theta = 0$	0.394 565	−0.093 037 9
G. second grade fluid	$(\alpha = \beta = 0.1)\lambda = 0$, $\theta = 1$	0.249 828	−0.043 416 2
G. Maxwell fluid	$(\alpha = \beta = 0.1)\lambda = 2$, $\theta = 0$	0.548 910	−0.177 590 0
Oldroyd-B fluid	$(\alpha = \beta = 1)\lambda = 2$, $\theta = 1$	0.409 047	−0.073 581 4
G. Oldroyd-B fluid	$(\alpha = \beta = 0.1)\lambda = 2$, $\theta = 1$	0.477 628	−0.026 381 6

表 1 至表 3 给出滑移条件下,5 种流体模型在上面考虑的 3 类问题中,速度分量的比较. 表 1 和表 2 给出了作 $\cos\omega_0 t$ 振荡时,速度分量的比较. 表 1 表明,广义 Maxwell 流体的速度 u 和 v 量值上最大;广义二阶流体的速度 u 和 v 量值上最小. 表 2 表明,广义 Maxwell 流体的速度

u 和 v 量值上最大；广义二阶流体的速度 u 和 v 量值上最小．表 3 显示出 Poiseuille 流体速度分量的比较．我们发现，广义 Maxwell 流体的速度 u 和 v 量值上最大．然而，广义二阶流体的速度 u 最小；广义 Oldroyd-B 流体的速度 v 量值上最小．

表 4 在无滑移条件下作一般周期振荡时，流动速度的比较

Table 4 General periodic oscillations under no-slip condition

$(\omega_0 = 0.1,\ \Omega = 0.3,\ M = t = 0.5,\ z = 0.5,\ \gamma = 0)$

type of fluid	rheological parameters	u	v
Newtonian fluid	$\lambda = 0,\ \theta = 0$	0.558 030	−0.073 264 0
G. second grade fluid	$(\alpha = \beta = 0.1)\lambda = 0,\ \theta = 1$	0.308 590	−0.117 361 0
G. Maxwell fluid	$(\alpha = \beta = 0.1)\lambda = 2,\ \theta = 0$	0.514 386	−0.112 631 0
Oldroyd-B fluid	$(\alpha = \beta = 1)\lambda = 2,\ \theta = 1$	0.558 487	−0.074 289 3
G. Oldroyd-B fluid	$(\alpha = \beta = 0.1)\lambda = 2,\ \theta = 1$	0.556 351	−0.075 137 7

表 5 无滑移条件下两平板间作周期流动时，流动速度的比较

Table 5 Periodic flow between two plates under no-slip condition

$(\omega_0 = 0.1,\ \Omega = 0.3,\ M = t = 0.5,\ z = 0.5,\ \gamma = 0)$

type of fluid	rheological parameters	u	v
Newtonian fluid	$\lambda = 0,\ \theta = 0$	0.428 929	−0.029 219 5
G. second grade fluid	$(\alpha = \beta = 0.1)\lambda = 0,\ \theta = 1$	0.253 374	−0.011 105 0
G. Maxwell fluid	$(\alpha = \beta = 0.1)\lambda = 2,\ \theta = 0$	0.415 805	−0.051 478 4
Oldroyd-B fluid	$(\alpha = \beta = 1)\lambda = 2,\ \theta = 0$	0.429 314	−0.029 479 5
G. Oldroyd-B fluid	$(\alpha = \beta = 0.1)\lambda = 2,\ \theta = 1$	0.428 455	−0.031 030 0

表 6 无滑移条件下 Poiseuille 流动时速度的比较

Table 6 Poiseuille flow under no-slip condition

$(Q_0 = -1,\ \omega_0 = 0.1,\ \Omega = 0.3,\ M = t = 0.5,\ z = 0.5,\ \gamma = 0)$

type of fluid	rheological parameters	u	v
Newtonian fluid	$\lambda = 0,\ \theta = 0$	0.245 260	−0.032 171 5
G. second grade fluid	$(\alpha = \beta = 0.1)\lambda = 0,\ \theta = 1$	0.133 238	−0.015 270 7
G. Maxwell fluid	$(\alpha = \beta = 0.1)\lambda = 2,\ \theta = 0$	0.418 020	−0.084 386 0
Oldroyd-B fluid	$(\alpha = \beta = 1)\lambda = 2,\ \theta = 1$	0.254 642	−0.011 232 6
G. Oldroyd-B fluid	$(\alpha = \beta = 0.1)\lambda = 2,\ \theta = 1$	0.252 292	−0.033 803 3

表 4 至表 6 显示出 3 种振荡在无滑移条件时速度的变化．表 4 表明，Oldroyd-B 流体的速度 u 最大；广义二阶流体的速度 u 最小．速度 v 量值上广义二阶流体最大；Newton 流体最小．表 5 表明，Oldroyd-B 流体的速度 u 最大；广义二阶流体的速度 u 最小．可是，速度 v 数值上广义 Maxwell 流体最大；广义二阶流体最小．表 6 给出了 Poiseuille 流动时，广义 Maxwell 流体的速度 u 最大；广义二阶流体的速度 u 最小．然而，速度 v 数值上广义 Maxwell 流体最大；Oldroyd-B 流体最小．

7 结　　论

分析了 3 种振荡流动，并得到封闭形式的解．在一般 Oldroyd-B 流体中，介绍了剪应力滑移条件的使用．该条件至今还没有被引入经典型的（Maxwell，Oldroyd-B，Burgers）流体．在最终结果中清楚地给出了滑移参数的贡献．对 5 种流体模型的速度分量进行了比较．本文的主要结论如下：

1）有滑移存在时，x 方向的速度分量，小于无滑移情况.

2）有滑移存在时，y 方向速度分量的量值，大于无滑移情况.

3）在 Poiseuille 流动中，有滑移时速度 u 和 v 的量值，比无滑移情况要大得多.

4）当滑移可以忽略不计时，可将所考虑 5 种流体模型的数学结果的滑移参数取为 0.

5）有滑移存在时，广义 Maxwell 流体的 u 和 v 的量值最大.

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Exact Solutions in Generalized Oldroyd-B Fluid

T. Hayat[1,2], Sahrish Zaib[1], S. Asghar[3], Awatif A. Hendi[2]

(1. *Department of Mathematics*, *Quaid-i-Azam University* 45320, *Islamabad* 44000, *Pakistan*;

2. *Department of Physics*, *Faculty of Science*, *King Saud University*, *P. O. Box* 1846, *Riyadh* 11321, *Saudi Arabia*;

3. *Department of Mathematics*, *COMSATS Institute of Information Technology*, *Islamabad* 44000, *Pakistan*)

Abstract: The influence of slip condition on the magnetohydrodynamic (MHD) and rotating flow of a generalized Oldroyd-B fluid occupying a porous space was investigated. Fractional calculus approach was used in the mathematical modeling. Three illustrative examples induced by plate oscillations and periodic pressure gradient were considered and the exact solutions in each case was derived. Comparison was provided between the results of slip and no-slip conditions. The influence of slip was highlighted on the velocity profile by displaying graphs.

Key words: slip conditions; exact solutions; fractional calculus