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含有约束的两个状态变量系统的转迁集计算*

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摘要: 周期解的分岔广泛存在于实际的非线性动力学系统中. 该文对两个状态变量系统的约束分岔进行了讨论. 在约束条件下系统将产生新的转迁集. 此外,以一个二维系统为例,对含有约束条件和不含有约束条件的分岔特性进行了比较. 所得的结果可以为系统的设计和参数选择提供理论依据.

关　键　词: 分岔; 约束; 奇异性理论; 非线性动力学; 两状态变量

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引　　言

随着非线性科学的发展,其被越来越广泛地应用于工程和技术领域. 一些专家和学者在这方面做了大量的工作,如 Bogoliubov, Mitropolsky, Nayfeh 和 Mook, Smale, Arnold, Golubitsky 等. Chen 和 Landford 将 LS(Liapunov-Schmidt)方法引入非线性分岔的分析之中[1],并对周期解的分岔建立了一套完整的分析方法[2-3].

在研究工程实际问题时,状态变量常常含有约束条件,例如振幅或者浓度不能为负. 对于单自由度周期解的约束分岔,吴志强等做了大量研究[4-7],推动了奇异性理论的发展,为实际的工程和技术提供了理论依据.

对于多自由度系统,会经常研究系统的主共振、内共振和组合共振等情况. 系统的分岔方程往往是多维的. 并且对于实际的振动系统,状态变量经常被约束条件限制,例如在轴系的弯扭耦合振动中,由于轴承半径间隙的存在,弯曲振动的振幅是双边约束;而扭转振动的振幅大于 0,是单边约束. 胡凡努和李养成[8]对多分岔参数和两个状态变量组的奇异性问题做了大量的理论研究. 秦朝红等[9]以内共振悬索为例研究了二维分岔方程的奇异性. 到目前为止,虽然有少量针对两个状态变量奇异性理论的研究,但对于含有约束的两个状态变量的分岔问题的研究还没开展. 在实际问题中,状态变量含有约束的问题常常出现,因此有必要研究两个状态变量含有约束的问题. 为了全面揭示系统的分岔特性,本文在文献[10]的基础上,对奇异性理论进行了推广.

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1 两个状态变量系统的约束分岔的分类

1.1 情况Ⅰ:两个状态变量系统的分岔,其中一个有单边约束,另一个没有约束

对于一个状态变量系统的约束分岔,Wu 等[6]进行了深入的研究. 本文将这种思想推广到了两个状态变量系统的约束分岔研究当中.

引入一般形式的分岔方程:

$$g=\begin{cases} g_1(s,m,\lambda,\alpha)=0, \\ g_2(s,m,\lambda,\alpha)=0, \end{cases} \quad \delta s \geqslant \beta,\ \delta=\pm 1. \tag{1}$$

其中, s 和 m 为两个不同的状态变量, λ 是系统的分岔参数, α 是系统的开折参数.

对于单边约束,为使约束分岔问题转化为非约束分岔问题,引入如下的变换形式:

$$G(x,y,\lambda,\alpha) \square g(s,m,\lambda,\alpha)=\begin{cases} g_1(\delta(x^2+\beta),m,\lambda,\alpha), \\ g_2(\delta(x^2+\beta),m,\lambda,\alpha). \end{cases} \tag{2}$$

从式(1)和式(2)可求得

$$\begin{cases} G_{1x}=g_{1s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=2\delta x g_{1s},\ G_{1xx}=2\delta g_{1s}+4x^2 g_{1ss}, \\ G_{2x}=g_{2s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=2\delta x g_{2s},\ G_{2xx}=2\delta g_{2s}+4x^2 g_{2ss}. \end{cases} \tag{3}$$

将上述关系式代入到分岔集 B [10]可得如下结果:

$$B=B_1\cup B_2,$$

$$B_1=\begin{cases} g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0, \\ g_{1s}g_{2\lambda}-g_{2s}g_{1\lambda}=0, \\ g_{1s}g_{2m}-g_{2s}g_{1m}=0, \\ g_1(s,m,\lambda,\alpha)=0, \\ g_2(s,m,\lambda,\alpha)=0, \\ \quad \delta s\geqslant\beta; \end{cases} \tag{4}$$

$$B_2=\begin{cases} g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0, \\ g_1(\delta\beta,m,\lambda,\alpha)=0, \\ g_2(\delta\beta,m,\lambda,\alpha)=0, \end{cases} \tag{5}$$

其中, B_1 在非约束区间时与没有约束条件的分岔集相同, B_2 是由约束条件引起的约束分岔集,称之为边界引起的约束分岔集.

同理,可以得到滞后集

$$H=H_1\cup H_2,$$

$$H_1=\begin{cases} g_1(s,m,\lambda,\alpha)=0, \\ g_2(s,m,\lambda,\alpha)=0, \\ \dfrac{4(\delta s-\beta)\left(g_{1ss}+g_{1mm}\left(\dfrac{g_{1s}}{g_{1m}}\right)^2-2g_{1sm}\left(\dfrac{g_{1s}}{g_{1m}}\right)\right)+2g_{1s}\delta}{4(\delta s-\beta)\left(g_{2ss}+g_{2mm}\left(\dfrac{g_{1s}}{g_{1m}}\right)^2-2g_{2sm}\left(\dfrac{g_{1s}}{g_{1m}}\right)\right)+2g_{1s}\delta}=\dfrac{g_{1s}}{g_{2s}}=\dfrac{g_{1m}}{g_{2m}}, \\ \qquad \delta s>\beta; \end{cases} \tag{6}$$

$$H_2=\begin{cases}g_{1m}g_{2s}-g_{2m}g_{1s}=0,\\ g_1(\delta\beta,m,\lambda,\alpha)=0,\\ g_2(\delta\beta,m,\lambda,\alpha)=0.\end{cases}\tag{7}$$

通过同样的计算可以得到双极限点集

$$D=D_1\cup D_2,$$

$$D_1=\begin{cases}g_1(s_i,m_i,\lambda,\alpha)=0,\\ g_2(s_i,m_i,\lambda,\alpha)=0,\\ \det(\mathrm{d}G)_{s_i,m_i,\lambda,\alpha}=0,\\ \quad\delta s_i\geqslant\beta,\ i=1,2,\ (s_1,m_1)\neq(s_2,m_2);\end{cases}\tag{8}$$

$$D_2=\begin{cases}g_1(\delta\beta,m_1,\lambda,\alpha)=0,\\ g_2(\delta\beta,m_1,\lambda,\alpha)=0,\\ g_1(s,m_2,\lambda,\alpha)=0,\\ g_2(s,m_2,\lambda,\alpha)=0,\\ \det(\mathrm{d}G)_{s,m_2,\lambda,\alpha}=0,\\ \quad\delta s>\beta,\end{cases}\tag{9}$$

其中 dG 参见文献[10].

1.2 情况Ⅱ:两个状态变量系统的分岔,一个为双边约束,另一个没有约束

考虑一个状态变量的双边约束情况:

$$g(s,m,\lambda,\alpha)=\begin{cases}g_1(s,m,\lambda,\alpha)=0,\\ g_2(s,m,\lambda,\alpha)=0,\end{cases}\qquad \beta_1\leqslant s\leqslant\beta_2.\tag{10}$$

引入如下变换:

$$h(s,x)\ \square\ (s-\beta_1)(s-\beta_2)+x^2=0.\tag{11}$$

为了将双边约束的分岔问题转换为非约束问题,引入如下变换:

$$G(x,y,\lambda,\alpha)\ \square\ g(s,m,\lambda,\alpha)=\begin{cases}g_1(X(x^2,\beta),m,\lambda,\alpha),\\ g_2(X(x^2,\beta),m,\lambda,\alpha),\end{cases}\tag{12}$$

其中 $X(x^2,\beta)=s$ 是 $h(s,x)=0$ 的解.

通过方程(11)和(12)可以得到如下的关系式:

$$\begin{cases}G_{1x}=g_{1s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=\left(-2x\Big/\dfrac{\partial h}{\partial s}\right)g_{1s},\\ G_{2x}=g_{2s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=\left(-2x\Big/\dfrac{\partial h}{\partial s}\right)g_{2s},\\ G_{1xx}=g_{1s}\dfrac{\mathrm{d}^2s}{\mathrm{d}x^2}+g_{1ss}\left(\dfrac{\mathrm{d}s}{\mathrm{d}x}\right)^2=\\ \qquad g_{1s}\left(-2[(2s-\beta_1-\beta_2)^2+4x^2]\Big/\left(\dfrac{\partial h}{\partial s}\right)^3\right)+4x^2\Big/\left(\dfrac{\mathrm{d}h}{\mathrm{d}s}\right)^2g_{1ss},\\ G_{2xx}=g_{2s}\dfrac{\mathrm{d}^2s}{\mathrm{d}x^2}+g_{2ss}\left(\dfrac{\mathrm{d}s}{\mathrm{d}x}\right)^2=\\ \qquad g_{2s}\left(-2[(2s-\beta_1-\beta_2)^2+4x^2]\Big/\left(\dfrac{\partial h}{\partial s}\right)^3\right)+4x^2\Big/\left(\dfrac{\mathrm{d}h}{\mathrm{d}s}\right)^2g_{2ss}.\end{cases}\tag{13}$$

和 1.1 节相似,可以得到如下的转迁集.

分岔集为

$$B = B_1 \cup B_2,$$

$$B_1 = \begin{cases} g_{1m}g_{2\lambda} - g_{2m}g_{1\lambda} = 0, \\ g_{1s}g_{2\lambda} - g_{2s}g_{1\lambda} = 0, \\ g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(s,m,\lambda,\alpha) = 0, \\ g_2(s,m,\lambda,\alpha) = 0, \\ \quad \beta_1 < s < \beta_2; \end{cases} \tag{14}$$

$$B_2 = \begin{cases} g_{1m}g_{2\lambda} - g_{2m}g_{1\lambda} = 0, \\ g_1(\beta_i,m,\lambda,\alpha) = 0, \\ g_2(\beta_i,m,\lambda,\alpha) = 0, \\ \quad i = 1,2; \end{cases} \tag{15}$$

同理,可以得到含双边约束的滞后集

$$H = H_1 \cup H_2,$$

$$H_1 = \begin{cases} g_1(s,m,\lambda,\alpha) = 0, \\ g_2(s,m,\lambda,\alpha) = 0, \\ \Big\{ g_{1s}(-2[(2s-\beta_1-\beta_2)^2 - 4(s-\beta_1)(s-\beta_2)]) - \\ \quad 4(2s-\beta_1-\beta_2)(s-\beta_1)(s-\beta_2) \times \\ \quad \left(g_{1ss} + g_{1sm}\left(\frac{g_{1s}}{g_{1m}}\right)^2 - 2g_{1sm}\left(\frac{g_{1s}}{g_{1m}}\right) \right) \Big\} \Big/ \Big\{ g_{2s}(-2[(2s-\beta_1-\beta_2)^2 - \\ \quad 4(s-\beta_1)(s-\beta_2)]) - 4(2s-\beta_1-\beta_2)(s-\beta_1)(s-\beta_2) \times \\ \quad \left(g_{2ss} + g_{2sm}\left(\frac{g_{1s}}{g_{1m}}\right)^2 - 2g_{2sm}\left(\frac{g_{1s}}{g_{1m}}\right) \right) \Big\} = \frac{g_{1s}}{g_{2s}} = \frac{g_{1m}}{g_{2m}}, \\ \qquad \beta_1 < s < \beta_2; \end{cases} \tag{16}$$

$$H_2 = \begin{cases} g_1(\beta_i,m,\lambda,\alpha) = 0, \\ g_2(\beta_i,m,\lambda,\alpha) = 0, \\ g_{1m}g_{2s} - g_{2m}g_{1s} = 0, \\ \quad i = 1,2. \end{cases} \tag{17}$$

双极限点集为

$$D = D_1 \cup D_2 \cup D_3,$$

$$D_1 = \begin{cases} g_1(s_i,m_i,\lambda,\alpha) = 0, \\ g_2(s_i,m_i,\lambda,\alpha) = 0, \\ \det(\mathrm{d}G)_{s_i,m_i,\lambda,\alpha} = 0, \\ \quad \beta_1 < s < \beta_2,\ i = 1,2,\ (s_1,m_1) \neq (s_2,m_2); \end{cases} \tag{18}$$

$$D_2=\begin{cases}g_1(\beta_1,m_1,\lambda,\alpha)=0,\\ g_2(\beta_1,m_1,\lambda,\alpha)=0,\\ g_1(\beta_2,m_2,\lambda,\alpha)=0,\\ g_2(\beta_2,m_2,\lambda,\alpha)=0;\end{cases}\tag{19}$$

$$D_3=\begin{cases}g_1(\beta_i,m_i,\lambda,\alpha)=0,\\ g_2(\beta_i,m_i,\lambda,\alpha)=0,\\ g_1(s,m,\lambda,\alpha)=0,\\ g_2(s,m,\lambda,\alpha)=0,\\ \det(\mathrm{d}G)_{s,m,\lambda,\alpha}=0,\\ \quad\beta_1<s<\beta_2,\ i=1,2.\end{cases}\tag{20}$$

1.3 情况Ⅲ:两个状态变量系统的分岔,其中一个为分段约束,另一个为非约束

这是一个具有分段约束的分岔方程:

$$G(x,y,\lambda,\alpha)=$$

$$\begin{cases}g_1(s,m,\lambda,\alpha)\\ g_2(s,m,\lambda,\alpha)\end{cases}=\begin{cases}g_{11}(s,m,\lambda,\alpha),\ g_{21}(s,m,\lambda,\alpha), & s\leqslant\beta_1,\\ g_{12}(s,m,\lambda,\alpha),\ g_{22}(s,m,\lambda,\alpha), & \beta_1<s<\beta_2,\\ g_{13}(s,m,\lambda,\alpha),\ g_{23}(s,m,\lambda,\alpha), & s\geqslant\beta_2.\end{cases}\tag{21}$$

我们可以应用1.1节和1.2节得到的结论来推导其转迁集.对于 $s\leqslant\beta_1$ 或者 $s\geqslant\beta_2$ 的情况,可以应用单边约束时的结论;对于 $\beta_1<s<\beta_2$ 的情况,可以应用一个状态变量为双边约束,另一个没有约束情况时的结论.

1.4 情况Ⅳ:两个状态变量系统的分岔,两个状态变量都是单边约束

在这节中,研究两个状态变量都为单边约束的情况:

$$g(s,m,\lambda,\alpha)=0,\ \begin{cases}\delta_1 s=x^2+\beta_1, & x^2\geqslant 0,\ \delta_1=\pm1,\\ \delta_2 m=y^2+\beta_2, & y^2\geqslant 0,\ \delta_2=\pm1.\end{cases}\tag{22}$$

引入如下变换:

$$G(x,y,\lambda,\alpha)\ \square\ g(s,m,\lambda,\alpha)=\begin{cases}g_1(s,m,\lambda,\alpha)=0,\\ g_2(s,m,\lambda,\alpha)=0.\end{cases}\tag{23}$$

由方程(23)可得出下面的关系式:

$$\begin{cases}G_{1x}=g_{1s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=2\delta_1 x g_{1s},\ G_{1y}=g_{1m}\dfrac{\mathrm{d}m}{\mathrm{d}y}=2\delta_2 y g_{1m},\\ G_{2x}=g_{2s}\dfrac{\mathrm{d}s}{\mathrm{d}x}=2\delta_1 x g_{2s},\ G_{2y}=g_{2m}\dfrac{\mathrm{d}m}{\mathrm{d}y}=2\delta_2 y g_{2m},\\ G_{1xx}=2\delta_1 g_{1s}+4x^2 g_{1ss},\ G_{1yy}=2\delta_2 g_{1m}+4y^2 g_{1mm},\\ G_{2xx}=2\delta_1 g_{2s}+4x^2 g_{2ss},\ G_{2yy}=2\delta_2 g_{2m}+4y^2 g_{2mm}.\end{cases}\tag{24}$$

根据方程(24)可得到其转迁集.

分岔集为

$$B=B_1\cup B_2\cup B_3\cup B_4,$$

$$B_1=\begin{cases}g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0,\\ g_{1s}g_{2\lambda}-g_{2s}g_{1\lambda}=0,\\ g_{1s}g_{2m}-g_{2s}g_{1m}=0,\\ g_1(s,m,\lambda,\alpha)=0,\\ g_2(s,m,\lambda,\alpha)=0,\\ \quad\delta_1 s\geqslant\beta_1,\ \delta_2 m\geqslant\beta_2;\end{cases}\tag{25}$$

$$B_2=\begin{cases}g_1(\delta_1\beta_1,\delta_2\beta_2,\lambda,\alpha)=0,\\ g_2(\delta_1\beta_1,\delta_2\beta_2,\lambda,\alpha)=0;\end{cases}\tag{26}$$

$$B_3=\begin{cases}g_{1s}g_{2\lambda}-g_{2s}g_{1\lambda}=0,\\ g_1(s,\delta\beta_2,\lambda,\alpha)=0,\\ g_2(s,\delta\beta_2,\lambda,\alpha)=0,\\ \quad\delta_1 s>\beta_1;\end{cases}\tag{27}$$

$$B_4=\begin{cases}g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0,\\ g_1(\delta\beta_1,m,\lambda,\alpha)=0,\\ g_2(\delta\beta_1,m,\lambda,\alpha)=0,\\ \quad\delta_2 m>\beta_2.\end{cases}\tag{28}$$

B_1 在非约束区间时与没有约束条件的分岔集相同，B_2,B_3,B_4 是由于约束条件引起的，称之为边界条件引起的约束分岔集.

同理，可以得到滞后集

$$H=H_1\cup H_2\cup H_3\cup H_4,$$

$$H_1=\begin{cases}g_1(s,m,\lambda,\alpha)=0,\\ g_2(s,m,\lambda,\alpha)=0,\\ \Big\{2\delta_1 g_{1s}+4(\delta_1 s-\beta_1)g_{1ss}+\\ \quad(2\delta_2 g_{1m}+4(\delta_2 m-\beta_2)g_{1mm})\left(\dfrac{\delta_2 m-\beta_2}{\delta_1 s-\beta_1}\right)\left(\dfrac{g_{1s}}{g_{1m}}\right)^2-\\ \quad 8(\delta_1 s-\beta_1)g_{1sm}\left(\dfrac{g_{1s}}{g_{1m}}\right)\Big\}\Big/\Big\{2\delta_1 g_{2s}+4(\delta_1 s-\beta_1)g_{2ss}+\\ \quad(2\delta_2 g_{2m}+4(\delta_2 m-\beta_2)g_{2mm})\left(\dfrac{\delta_2 m-\beta_2}{\delta_1 s-\beta_1}\right)\left(\dfrac{g_{1s}}{g_{1m}}\right)^2-\\ \quad 8(\delta_1 s-\beta_1)g_{2sm}\left(\dfrac{g_{1s}}{g_{1m}}\right)\Big\}=\dfrac{g_{1s}}{g_{2s}}=\dfrac{g_{1m}}{g_{2m}},\\ \qquad\delta_1 s>\beta_1,\ \delta_2 m>\beta_2;\end{cases}\tag{29}$$

$$H_2=\begin{cases}g_1(\delta_1\beta_1,\delta_2\beta_2,\lambda,\alpha)=0,\\ g_2(\delta_1\beta_1,\delta_2\beta_2,\lambda,\alpha)=0;\end{cases}\tag{30}$$

$$H_3 = \begin{cases} g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(s, \delta\beta_2, \lambda, \alpha) = 0, \\ g_2(s, \delta\beta_2, \lambda, \alpha) = 0, \\ \quad \delta_1 s > \beta_1; \end{cases} \tag{31}$$

$$H_4 = \begin{cases} g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(\delta\beta_1, m, \lambda, \alpha) = 0, \\ g_2(\delta\beta_1, m, \lambda, \alpha) = 0, \\ \quad \delta_2 m > \beta_2. \end{cases} \tag{32}$$

双极限点集为

$$D = D_1 \cup D_2 \cup D_3 \cup D_4,$$

$$D_1 = \begin{cases} g_1(s_i, m_i, \lambda, \alpha) = 0, \\ g_2(s_i, m_i, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_i, m_i, \lambda, \alpha} = 0, \\ \quad i = 1,2,\ \delta_1 s \geqslant \beta_1,\ \delta_2 m \geqslant \beta_2,\ (s_1, m_1) \neq (s_2, m_2); \end{cases} \tag{33}$$

$$D_2 = \begin{cases} g_1(\delta_1\beta_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_2(\delta_1\beta_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s, m, \lambda, \alpha} = 0, \\ \quad \delta_1 s > \beta_1,\ \delta_2 m > \beta_2; \end{cases} \tag{34}$$

$$D_3 = \begin{cases} g_1(s_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_2(s_1, \delta_2\beta_2, \lambda, \alpha) = 0, \\ g_1(s_2, m, \lambda, \alpha) = 0, \\ g_2(s_2, m, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_2, m, \lambda, \alpha} = 0, \\ \quad \delta_1 s_i > \beta_1,\ \delta_2 m > \beta_2,\ i = 1,2; \end{cases} \tag{35}$$

$$D_4 = \begin{cases} g_1(\delta_1\beta_1, m_1, \lambda, \alpha) = 0, \\ g_2(\delta_1\beta_1, m_1, \lambda, \alpha) = 0, \\ g_1(s_2, m_2, \lambda, \alpha) = 0, \\ g_2(s_2, m_2, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_2, m_2, \lambda, \alpha} = 0, \\ \quad \delta_1 s_2 > \beta_1,\ \delta_2 m_i > \beta_2,\ i = 1,2. \end{cases} \tag{36}$$

1.5 情况Ⅴ:两个状态变量系统的分岔,两个状态变量都是双边约束

在这节里,考虑两个状态变量都是双边约束的情况,

$$\begin{cases} G(x,y,\lambda,\alpha) \square g(s,m,\lambda,\alpha) = \begin{cases} g_1(s,m,\lambda,\alpha) = 0, \\ g_2(s,m,\lambda,\alpha) = 0, \end{cases} \quad \begin{cases} \beta_1 \leqslant s \leqslant \beta_2, \\ \beta_3 \leqslant m \leqslant \beta_4, \end{cases} \\ h_1(s,x) \square (s-\beta_1)(s-\beta_2) + x^2 = 0, \\ h_2(s,x) \square (m-\beta_3)(m-\beta_4) + y^2 = 0. \end{cases} \tag{37}$$

通过方程(37)可以得到以下关系式:

$$\begin{cases} G_{1x} = g_{1s}\dfrac{\mathrm{d}s}{\mathrm{d}x} = \left(-2x\Big/\dfrac{\partial h_1}{\partial s}\right)g_{1s}, \\ G_{2x} = g_{2s}\dfrac{\mathrm{d}s}{\mathrm{d}x} = \left(-2x\Big/\dfrac{\partial h_1}{\partial s}\right)g_{2s}, \\ G_{1xx} = g_{1s}\dfrac{\mathrm{d}^2 s}{\mathrm{d}x^2} + g_{1ss}\left(\dfrac{\mathrm{d}s}{\mathrm{d}x}\right)^2 = \\ \qquad g_{1s}\left(-2[(2s-\beta_1-\beta_2)^2 + 4x^2]\Big/\left(\dfrac{\partial h_1}{\partial s}\right)^3\right) + 4x^2\Big/\left(\dfrac{\mathrm{d}h_1}{\mathrm{d}s}\right)^2 g_{1ss}, \\ G_{2xx} = g_{2s}\dfrac{\mathrm{d}^2 s}{\mathrm{d}x^2} + g_{2ss}\left(\dfrac{\mathrm{d}s}{\mathrm{d}x}\right)^2 = \\ \qquad g_{2s}\left(-2[(2s-\beta_1-\beta_2)^2 + 4x^2]\Big/\left(\dfrac{\partial h_1}{\partial s}\right)^3\right) + 4x^2\Big/\left(\dfrac{\mathrm{d}h_1}{\mathrm{d}s}\right)^2 g_{2ss}, \end{cases} \tag{38}$$

$$\begin{cases} G_{1y} = g_{1m}\dfrac{\mathrm{d}m}{\mathrm{d}y} = \left(-2y\Big/\dfrac{\partial h_2}{\partial m}\right)g_{1m}, \\ G_{2y} = g_{2m}\dfrac{\mathrm{d}m}{\mathrm{d}y} = \left(-2y\Big/\dfrac{\partial h_2}{\partial m}\right)g_{2m}, \\ G_{1yy} = g_{1m}\dfrac{\mathrm{d}^2 m}{\mathrm{d}y^2} + g_{1mm}\left(\dfrac{\mathrm{d}m}{\mathrm{d}y}\right)^2 = \\ \qquad g_{1m}\left(-2[(2m-\beta_3-\beta_4)^2 + 4y^2]\Big/\left(\dfrac{\partial h_2}{\partial m}\right)^3\right) + 4y^2\Big/\left(\dfrac{\mathrm{d}h_2}{\mathrm{d}m}\right)^2 g_{1mm}, \\ G_{2yy} = g_{2s}\dfrac{\mathrm{d}^2 m}{\mathrm{d}y^2} + g_{2ss}\left(\dfrac{\mathrm{d}m}{\mathrm{d}y}\right)^2 = \\ \qquad g_{2m}\left(-2[(2m-\beta_3-\beta_4)^2 + 4y^2]\Big/\left(\dfrac{\partial h_2}{\partial m}\right)^3\right) + 4y^2\Big/\left(\dfrac{\mathrm{d}h_2}{\mathrm{d}m}\right)^2 g_{2mm}, \end{cases} \tag{39}$$

可得到其分岔集如下:

$$B = B_1 \cup B_2 \cup B_3 \cup B_4,$$

$$B_1 = \begin{cases} g_{1m}g_{2\lambda} - g_{2m}g_{1\lambda} = 0, \\ g_{1s}g_{2\lambda} - g_{2s}g_{1\lambda} = 0, \\ g_{1s}g_{2m} - g_{2s}g_{1m} = 0, \\ g_1(s,m,\lambda,\alpha) = 0, \\ g_2(s,m,\lambda,\alpha) = 0, \\ \quad \beta_1 \leqslant s \leqslant \beta_2,\ \beta_3 \leqslant m \leqslant \beta_4; \end{cases} \tag{40}$$

$$B_2=\begin{cases}g_1(\beta_i,\beta_j,\lambda,\alpha)=0,\\ g_2(\beta_i,\beta_j,\lambda,\alpha)=0,\\ \quad i=1,2,\ j=3,4;\end{cases}\tag{41}$$

$$B_3=\begin{cases}g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0,\\ g_1(\beta_i,m,\lambda,\alpha)=0,\\ g_2(\beta_i,m,\lambda,\alpha)=0,\\ \quad \beta_3<m<\beta_4,\ i=1,2;\end{cases}\tag{42}$$

$$B_4=\begin{cases}g_{1s}g_{2\lambda}-g_{2s}g_{1\lambda}=0,\\ g_1(s,\beta_j,\lambda,\alpha)=0,\\ g_2(s,\beta_j,\lambda,\alpha)=0,\\ \quad \beta_1<s<\beta_2,\ j=3,4;\end{cases}\tag{43}$$

滞后集为

$$H=H_1\cup H_2\cup H_3\cup H_4,$$

$$H_1=\begin{cases}g_1(s,m,\lambda,\alpha)=0,\\ g_2(s,m,\lambda,\alpha)=0,\\ \dfrac{G_{1xx}\left(\dfrac{G_{1y}}{G_{1x}}\right)^2+G_{1yy}+2G_{1xy}\left(-\dfrac{G_{1y}}{G_{1x}}\right)}{G_{2xx}\left(\dfrac{G_{1y}}{G_{1x}}\right)^2+G_{2yy}+2G_{2xy}\left(-\dfrac{G_{1y}}{G_{1x}}\right)}=\dfrac{G_{1x}}{G_{2x}}=\dfrac{G_{1y}}{G_{2y}},\\ \dfrac{G_{1y}}{G_{1x}}=\dfrac{\left(-2y\Big/\dfrac{\partial h_2}{\partial m}\right)g_{1m}}{\left(-2x\Big/\dfrac{\partial h_1}{\partial s}\right)g_{1s}},\\ \dfrac{\partial h_2}{\partial m}=2m-\beta_3-\beta_4,\\ \dfrac{\partial h_1}{\partial s}=2s-\beta_1-\beta_2,\\ \quad \beta_1\leqslant s\leqslant\beta_2,\ \beta_3\leqslant m\leqslant\beta_4;\end{cases}\tag{44}$$

$$H_2=\begin{cases}g_1(\beta_i,\beta_j,\lambda,\alpha)=0,\\ g_2(\beta_i,\beta_j,\lambda,\alpha)=0,\\ \quad i=1,2,\ j=3,4;\end{cases}\tag{45}$$

$$H_3=\begin{cases}g_{1s}g_{2m}-g_{2s}g_{1m}=0,\\ g_1(s,\beta_j,\lambda,\alpha)=0,\\ g_2(s,\beta_j,\lambda,\alpha)=0,\\ \quad \beta_1<s<\beta_2,\ j=3,4;\end{cases}\tag{46}$$

$$H_4=\begin{cases}g_{1s}g_{2m}-g_{2s}g_{1m}=0,\\ g_1(\beta_i,m,\lambda,\alpha)=0,\\ g_2(\beta_i,m,\lambda,\alpha)=0,\\ \quad \beta_3<m<\beta_4,\ i=1,2;\end{cases}\tag{47}$$

双极限点集为

$$D = D_1 \cup D_2 \cup D_3 \cup D_4,$$

$$D_1 = \begin{cases} g_1(s_i, m_i, \lambda, \alpha) = 0, \\ g_2(s_i, m_i, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_i, m_i, \lambda, \alpha} = 0, \\ \quad i = 1, 2,\ \beta_1 \leqslant s \leqslant \beta_2, \\ \quad \beta_3 \leqslant m \leqslant \beta_4,\ (s_1, m_1) \neq (s_2, m_2); \end{cases} \tag{48}$$

$$D_2 = \begin{cases} g_1(\beta_i, \beta_j, \lambda, \alpha) = 0, \\ g_2(\beta_i, \beta_j, \lambda, \alpha) = 0, \\ g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s, m, \lambda, \alpha} = 0, \\ \quad i = 1, 2,\ j = 3, 4; \end{cases} \tag{49}$$

$$D_3 = \begin{cases} g_1(s_1, \beta_j, \lambda, \alpha) = 0, \\ g_2(s_1, \beta_j, \lambda, \alpha) = 0, \\ g_1(s_2, m_2, \lambda, \alpha) = 0, \\ g_2(s_2, m_2, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_2, m_2, \lambda, \alpha} = 0, \\ \quad \beta_1 < s < \beta_2,\ \beta_3 < m_2 < \beta_4,\ j = 3, 4; \end{cases} \tag{50}$$

$$D_4 = \begin{cases} g_1(\beta_i, m_1, \lambda, \alpha) = 0, \\ g_2(\beta_i, m_1, \lambda, \alpha) = 0, \\ g_1(s_2, m_2, \lambda, \alpha) = 0, \\ g_2(s_2, m_2, \lambda, \alpha) = 0, \\ \det(\mathrm{d}G)_{s_2, m_2, \lambda, \alpha} = 0, \\ \quad \beta_1 < s_2 < \beta_2,\ \beta_3 < m < \beta_4,\ i = 1, 2. \end{cases} \tag{51}$$

1.6 情况Ⅵ:两个状态变量系统的分岔,其中一个是单边约束,另一个为双边约束

在这节中考虑两个状态变量系统的分岔,其中一个是单边约束,另一个为双边约束:

$$G(x, y, \lambda, \alpha) \square g(s, m, \lambda, \alpha) = $$

$$\begin{cases} \delta s = x^2 + \beta_0, & x^2 > 0,\ \delta = \pm 1, \\ h(m, x) \square (m - \beta_1)(m - \beta_2) + y^2 = 0, & \beta_1 \leqslant m \leqslant \beta_2. \end{cases} \tag{52}$$

可求得其分岔集为

$$B = B_1 \cup B_2 \cup B_3 \cup B_4,$$

$$B_1 = \begin{cases} g_{1m} g_{2\lambda} - g_{2m} g_{1\lambda} = 0, \\ g_{1s} g_{2\lambda} - g_{2s} g_{1\lambda} = 0, \\ g_{1s} g_{2m} - g_{2s} g_{1m} = 0, \\ g_1(s, m, \lambda, \alpha) = 0, \\ g_2(s, m, \lambda, \alpha) = 0, \\ \quad \delta s \geqslant \beta_0,\ \beta_1 \leqslant m \leqslant \beta_2; \end{cases} \tag{53}$$

$$B_2=\begin{cases}g_1(\delta\beta_0,\beta_i,\lambda,\alpha)=0,\\g_2(\delta\beta_0,\beta_i,\lambda,\alpha)=0,\\\quad i=1,2;\end{cases}\tag{54}$$

$$B_3=\begin{cases}g_{1s}g_{2\lambda}-g_{2s}g_{1\lambda}=0,\\g_1(s,\beta_i,\lambda,\alpha)=0,\\g_2(s,\beta_i,\lambda,\alpha)=0,\\\quad \delta s\geqslant\beta_0,\ i=1,2;\end{cases}\tag{55}$$

$$B_4=\begin{cases}g_{1m}g_{2\lambda}-g_{2m}g_{1\lambda}=0,\\g_1(\delta\beta_0,m,\lambda,\alpha)=0,\\g_2(\delta\beta_0,m,\lambda,\alpha)=0,\\\quad \beta_1<m<\beta_2;\end{cases}\tag{56}$$

滞后集为

$$H=H_1\cup H_2\cup H_3\cup H_4,$$

$$H_1=\begin{cases}g_1(s,m,\lambda,\alpha)=0,\\g_2(s,m,\lambda,\alpha)=0,\\\dfrac{G_{1xx}\left(\dfrac{G_{1y}}{G_{1x}}\right)^2+G_{1yy}+2G_{1xy}\left(-\dfrac{G_{1y}}{G_{1x}}\right)}{G_{2xx}\left(\dfrac{G_{1y}}{G_{1x}}\right)^2+G_{2yy}+2G_{2xy}\left(-\dfrac{G_{1y}}{G_{1x}}\right)}=\dfrac{G_{1x}}{G_{2x}}=\dfrac{G_{1y}}{G_{2y}},\\\dfrac{G_{1y}}{G_{1x}}=\dfrac{\left(-2y\Big/\dfrac{\partial h_2}{\partial m}\right)g_{1m}}{2xg_{1s}\delta},\\\dfrac{\partial h_2}{\partial m}=2m-\beta_1-\beta_2,\\\quad \delta s\geqslant\beta_0,\ \beta_1\leqslant m\leqslant\beta_2;\end{cases}\tag{57}$$

$$H_2=\begin{cases}g_1(\delta\beta_0,\beta_i,\lambda,\alpha)=0,\\g_2(\delta\beta_0,\beta_i,\lambda,\alpha)=0,\\\quad i=1,2;\end{cases}\tag{58}$$

$$H_3=\begin{cases}g_{1s}g_{2m}-g_{2s}g_{1m}=0,\\g_1(s,\beta_i,\lambda,\alpha)=0,\\g_2(s,\beta_i,\lambda,\alpha)=0,\\\quad \delta s>\beta_0,\ i=1,2;\end{cases}\tag{59}$$

$$H_4=\begin{cases}g_{1s}g_{2m}-g_{2s}g_{1m}=0,\\g_1(\delta\beta_0,m,\lambda,\alpha)=0,\\g_2(\delta\beta_0,m,\lambda,\alpha)=0,\\\quad \beta_1<m<\beta_2;\end{cases}\tag{60}$$

双极限点集为

$$D=D_1\cup D_2\cup D_3\cup D_4,$$

$$D_1=\begin{cases}g_1(s_i,m_i,\lambda,\alpha)=0,\\ g_2(s_i,m_i,\lambda,\alpha)=0,\\ \det(\mathrm{d}G)_{s_i,m_i,\lambda,\alpha}=0,\\ \quad \delta s\geqslant\beta_0,\ \beta_1\leqslant m\leqslant\beta_2,\ i=1,2;\end{cases}\tag{61}$$

$$D_2=\begin{cases}g_1(\delta\beta_0,\beta_i,\lambda,\alpha)=0,\\ g_2(\delta\beta_0,\beta_i,\lambda,\alpha)=0,\\ g_1(s,m,\lambda,\alpha)=0,\\ g_2(s,m,\lambda,\alpha)=0,\\ \det(\mathrm{d}G)_{s,m,\lambda,\alpha}=0,\\ \quad \delta s>\beta_0,\ \beta_1<m<\beta_2,\ i=1,2;\end{cases}\tag{62}$$

$$D_3=\begin{cases}g_1(s_1,\beta_i,\lambda,\alpha)=0,\\ g_2(s_1,\beta_i,\lambda,\alpha)=0,\\ g_1(s_2,m,\lambda,\alpha)=0,\\ g_2(s_2,m,\lambda,\alpha)=0,\\ \det(\mathrm{d}G)_{s_2,m,\lambda,\alpha}=0,\\ \quad \delta s>\beta_0,\ \beta_1<m<\beta_2,\ i=1,2;\end{cases}\tag{63}$$

$$D_4=\begin{cases}g_1(\delta\beta_0,m_1,\lambda,\alpha)=0,\\ g_2(\delta\beta_0,m_1,\lambda,\alpha)=0,\\ g_1(s,m_2,\lambda,\alpha)=0,\\ g_2(s,m_2,\lambda,\alpha)=0,\\ \det(\mathrm{d}G)_{s,m_2,\lambda,\alpha}=0,\\ \quad \delta s>\beta_0,\ \beta_1<m<\beta_2.\end{cases}\tag{64}$$

1.7　情况Ⅶ:两个状态变量系统的分岔,两个变量都为分段约束

对于两个状态变量都为分段约束的情况,可应用前几节的结论. 例如下面这个含有分段约束的分岔方程:

$$G(x,y,\lambda,\alpha)=\begin{cases}g_1(s,m,\lambda,\alpha)\\ g_2(s,m,\lambda,\alpha)\end{cases}=$$

$$\begin{cases}g_{11}(s,m,\lambda,\alpha),g_{21}(s,m,\lambda,\alpha), & s\leqslant\beta_1,\ m\leqslant\gamma_1,\\ g_{12}(s,m,\lambda,\alpha),g_{22}(s,m,\lambda,\alpha), & \beta_1<s<\beta_2,\ \gamma_1<m<\gamma_2,\\ g_{13}(s,m,\lambda,\alpha),g_{23}(s,m,\lambda,\alpha), & \beta_1<s<\beta_2,\ m>\gamma_2,\end{cases}\tag{65}$$

对于 $s\leqslant\beta_1$, $m\leqslant\gamma_1$ 的情况,可应用两个状态变量都为单边约束情况时得到的结论. 对于 $\beta_1<s<\beta_2$, $\gamma_1<m<\gamma_2$ 的情况,可应用两个状态变量都为双边约束情况时得到的结论. 对于 $\beta_1<s<\beta_2$, $m>\gamma_2$ 的情况,可应用一个状态变量为单边约束,另一个为双边约束情况时得到的结论.

2 二维分岔系统的奇异性分析

2.1 非约束系统的奇异性分析

考虑下面的二维分岔方程[11]：

$$\begin{cases} g_1 = s^2 - m^2 - \lambda + \alpha m, \\ g_2 = m^2 - \lambda + \beta s, \end{cases} \tag{66}$$

其中，s，m 是状态变量，λ 是分岔参数，α，β 是开折参数.

可求得方程(66)的转迁集为

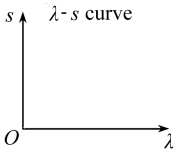
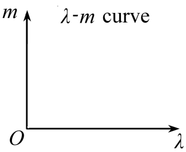
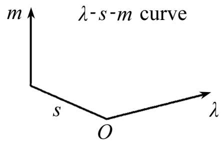
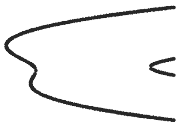
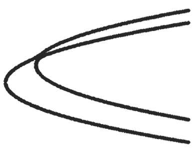
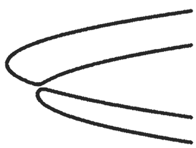
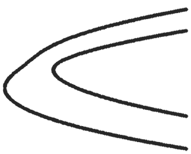
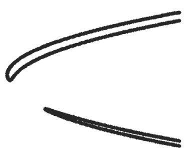
$$B = \left\{ \frac{\alpha^2}{2} - \beta^2 = 0 \right\}, \tag{67}$$

$$\begin{cases} H_1 = \{ \beta(2\alpha + 3\beta + 3(\beta(4\alpha + \beta))^{1/2}) = 0 \}, \\ H_2 = \{ \beta(2\alpha + 3\beta - 3(\beta(4\alpha + \beta))^{1/2}) = 0 \}, \end{cases} \tag{68}$$

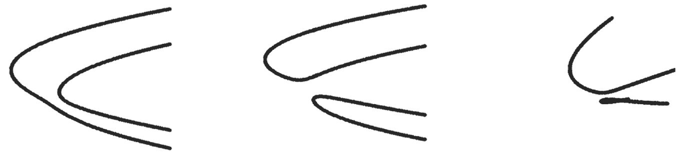
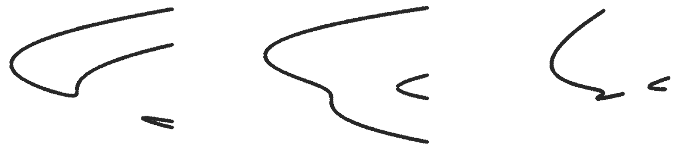
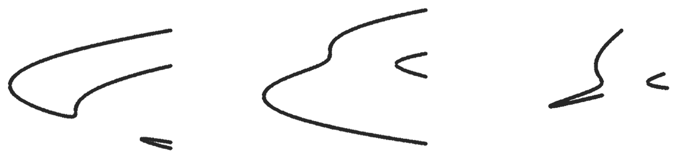
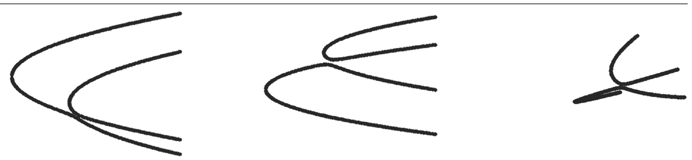
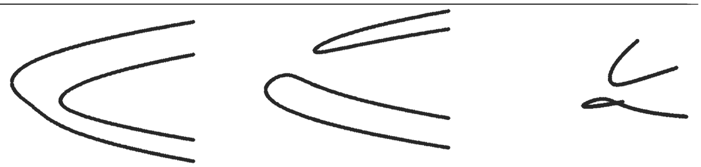
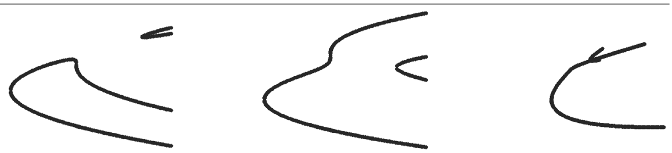
如图 1 所示. 不同区间内的分岔图如表 1 所示.

表 1 方程(66)在不同保持域内的分岔图

Table 1 The bifurcation diagrams in different persistent regions of system (66)

persistent regions	λ-s curve 	λ-m curve 	λ-s-m curve 
(1)			
(2)			
(3)			
(4)			

续表 1

(5)	
(6)	
(7)	
(8)	
(9)	
(10)	
(11)	
(12)	

从图 1 中可以看出,非约束分岔系统的转迁集将参数平面分为 12 个区域. 从表 1 中可以得出 12 种分岔模式,例如在区间(2)有分岔现象发生;区间(1)和(6)有滞后现象发生. 这些结论可以为系统的动力学设计和参数选择提供指导.

2.2 约束系统的奇异性分析,其中一个状态变量为单边约束,另一个为双边约束

含有约束的分岔方程如下:

$$\begin{cases} g_1 = s^2 - m^2 - \lambda + \alpha m; \\ g_2 = m^2 - \lambda + \beta s; \\ \quad s \geqslant 1,\ 1 \leqslant m \leqslant 6; \end{cases} \tag{69}$$

令

$$\begin{cases} s = x^2 + 1, \\ h = (m-1)(m-6) + y^2. \end{cases} \tag{70}$$

可求得该二维系统(其中一个状态变量为单边约束,另一个为双边约束)的转迁集为方程(71)和(72),如图 2 所示.

$$\begin{cases} B_1 = \left\{ \dfrac{\alpha^2}{2} - \beta^2 = 0 \right\},\ B_{21} = \{ \alpha - \beta - 1 = 0 \}, \\ B_{22} = \{ 6\alpha - \beta - 71 = 0 \},\ B_{31} = \left\{ -\dfrac{\beta^2}{4} + \alpha - 2 = 0 \right\}, \\ B_{32} = \left\{ -\dfrac{\beta^2}{4} + 6\alpha - 72 = 0 \right\},\ B_4 = \left\{ \dfrac{\alpha^2}{8} - \beta + 1 = 0 \right\}. \end{cases} \tag{71}$$

$$\begin{cases} H_1 = \left\{ \dfrac{\alpha^2 \beta}{(\beta+2)^2} - \beta + 1 = 0 \right\},\ H_{21} = \{ \alpha - \beta - 1 = 0 \}, \\ H_{22} = \{ 6\alpha - \beta - 71 = 0 \},\ H_3 = \{ (\alpha - 2)(\alpha\beta^2 - 6\beta^2 + 16) = 0 \}, \\ H_4 = \left\{ -\dfrac{\alpha^2 \beta}{(\beta+2)^2} + \beta - 1 = 0 \right\}, \end{cases} \tag{72}$$

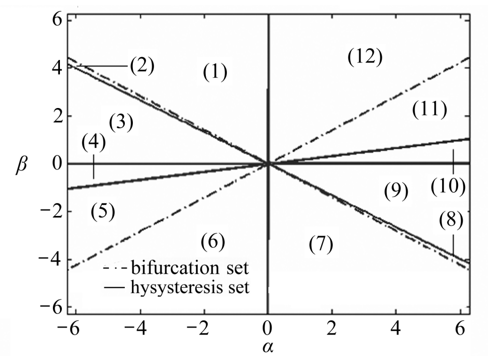


图 1 方程(66)的转迁集

Fig. 1 Transition sets of system (66)

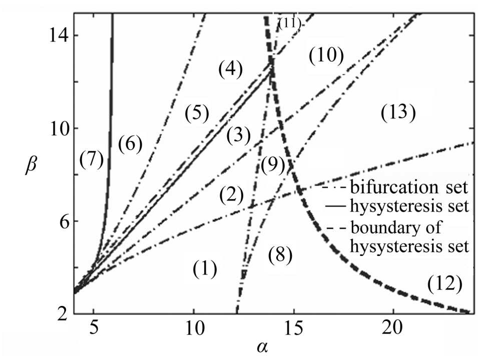


图 2 方程(69)的转迁集

Fig. 2 Transition sets of system (69)

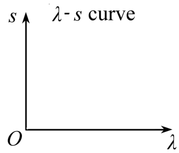
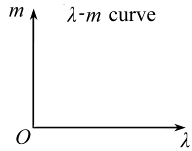
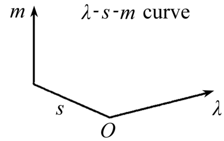
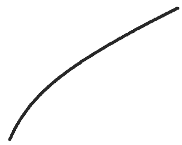
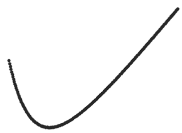
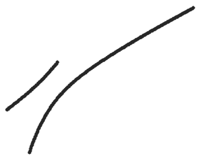
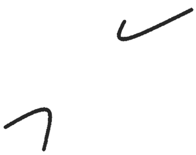
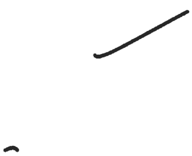
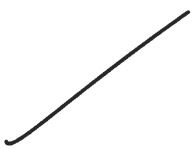
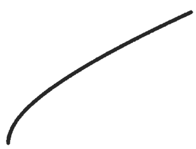
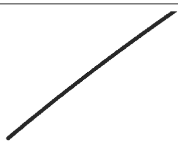
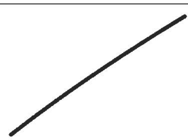
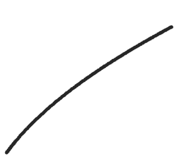
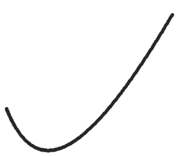
在图 2 中,滞后集的有效范围在边界线的左侧.

从图 2 可以看出在约束情况下,参数平面被转迁集分为 13 个区间. 从表 2 中可以得出 13 种分岔模式,在区间(12)中无解;在区间(1)、(6)、(7)和(11)中,对于不同的分岔参数系统只有单个解分支;在区间(2)中某些分岔参数对应于多个解分支.

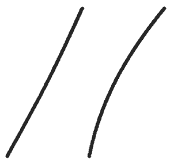
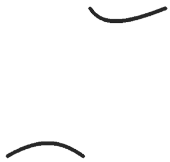
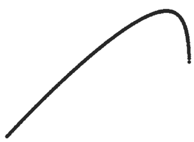
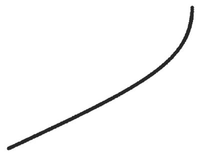
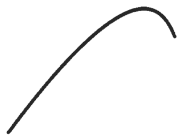
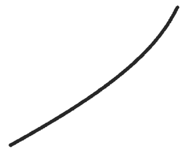
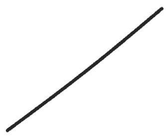
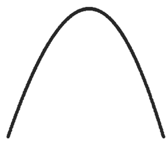
将非约束系统的分岔特性和约束系统的分岔特性进行了比较,发现当分岔参数在 $24 \geqslant \alpha \geqslant 4$, $\beta \geqslant 2$ 中时,非约束系统有 3 种分岔模式,但相应的约束系统有 13 种分岔模式. 因此约束条件引起了 10 种新的分岔模式,这对实际工程具有指导意义.

表 2　方程(69)在不同保持域内的分岔图

Table 2　The bifurcation diagrams in different persistent regions of system (69)

persistent regions			
(1)			
(2)			
(3)			
(4)			
(5)			
(6)			
(7)			
(8)			

续表 2

(9)			
(10)			
(11)			
(12)	none	none	none
(13)			

3 结　　论

本文将一个状态变量的约束分岔理论推广到两个状态变量的约束分岔分析之中,得出了两个状态变量约束系统的转迁集.分岔集 B_1 和双极限点集 D_1 在非边界区间时与非约束系统的分岔集和双极限点集相同,滞后集在非边界区间时与非约束系统的滞后集不相同, B_2,B_3,B_4,H_2,H_3,H_4,D_2,D_3,D_4 是由于边界条件引起的转迁集,称之为边界转迁集.

以一个二维分岔系统为例,研究了系统的分岔特性.在非约束的情况下,转迁集将参数平面分为 12 个不同的区间.在区间(2)中存在分岔现象;在区间(1)和(6)中存在滞后现象,这可能引起结构的破坏.在约束情况下,转迁集将约束系统的参数平面分为 13 个区间.在区间(12)中无解;在区间(1)、(6)、(7)和(11)中,对于不同的分岔参数系统只有一个解;在区间(2)中某些分岔参数对应于多个解分支.

将非约束系统的分岔特性和约束系统的分岔特性进行了比较,发现当分岔参数在 $24 \geqslant \alpha \geqslant 4$,$\beta \geqslant 2$ 中时,非约束系统有 3 种分岔模式,但相应的约束系统有 13 种分岔模式.因此约束条件引起了 10 种新的分岔模式,这对工程实际具有指导意义.

以上结论可以为系统的动力学分析和设计提供指导.

参考文献(References):

[1] Chen Y S, Langford W F. The subharmonic bifurcation solution of nonlinear Mathieu's equation and Euler dynamically buckling problems[J]. *Acta Mechanica Sinica*, 1988, **20**(6): 522-632.

[2] 陈予恕, 丁千. C-L 方法及其在工程非线性动力学问题中的应用[J]. 应用数学和力学, 2001, **22**(2): 127-134. (CHEN Yu-shu, DING Qian. C-L method and its application to engineering

nonlinear dynamical problems[J]. *Applied Mathematics and Mechanics*(*English Edition*), 2001, **22**(2): 144-153.)

[3] Chen Y S, Xu J. Universal classification of bifurcating solutions to a primary parametric resonance in van der Pol-Duffing-Mathicu's systems[J]. *J Science in China*, 1996, **39**(4): 405-417.

[4] 吴志强, 陈予恕. 具有单边约束的基本分岔问题的新分岔模式[J]. 应用数学和力学, 2001, **22**(11): 1135-1141.(WU Zhi-qing, CHEN Yu-shu. New bifurcation patterns in elementary bifurcation problems with single-side constraint[J]. *Applied Mathematics and Mechanics*(*English Edition*), 2001, **22**(11): 1260-1267.)

[5] 吴志强, 陈予恕. 含约束非线性动力系统的分岔分类[J]. 应用数学和力学, 2002, **23**(5): 477-482.(WU Zhi-qing, CHEN Yu-shu. Classification of bifurcations for nonlinear dynamical problems with constraints[J]. *Applied Mathematics and Mechanics*(*English Edition*), 2002, **23**(5): 535-541.)

[6] Wu Z Q, Yu P, Wang K Q. Bifurcation analysis on a self-excited hysteretic system[J]. *International Journal of Bifurcation and Chaos*, 2004, **14**(8): 2825-2842.

[7] 吴志强, 丁然, 陈予恕. 约束含参分岔问题的分类[J]. 应用数学和力学, 2010, **31**(2): 127-133. (WU Zhi-qing, DING Ran, CHEN Yu-shu. Classification of parametrically constrained bifurcations[J]. *Applied Mathematics and Mechanics*(*English Edition*), 2010, **31**(2): 135-142.)

[8] 胡凡努, 李养成. 关于两个状态变量组的等变分歧问题的通用开折[J]. 数学理论与应用, 2000, **20**(3): 50-57.(HU Fan-nu, LI Yang-cheng. Versal unfolding of equivariant bifurcation problems with two types of state variables[J]. *Mathematical Theory and Application*, 2000, **20**(3): 50-57. (in Chinese))

[9] 秦朝红, 陈予恕, 李军. 1∶1 内振悬索的二维奇异性分析[J]. 应用数学和力学, 2010, **31**(2): 134-142.(QIN Zhao-hong, CHEN Yu-shu, LI Jun. singularity analysis of a two-dimensional elastic cable with 1∶1 internal resonance[J]. *Applied Mathematics and Mechanics*(*English Edition*), 2010, **31**(2): 143-150.)

[10] Golubitsky M, Schaeffer D G. *Singularities and Groups in Bifurcation Theory*(Ⅰ)[M]. NY: Spring-Verlag, 1985.

[11] Chen Y S, Qin Z H. Singular analysis of two-dimensional bifurcation system[J]. *Science China Technological Sciences*, 2010, **53**(3): 608-611.

Transition Sets of Bifurcations of Dynamical System With Two State Variables With Constraints

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Abstract: Bifurcation of periodic solutions widely exists in nonlinear dynamical systems. Categories of bifurcations of systems with two state variables with different types of constraints were discussed where some new types of transition sets were added. Additionally, the bifurcation properties of two-dimensional systems without constraints were compared with the ones with constraints. The results obtained can be used by engineers for the choice of the structural parameters of the system.

Key words: bifurcation; constraint; singularity theory; nonlinear dynamics; two state variables