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加热下分数阶广义二阶流体的 Stokes 第一问题的高阶数值方法*

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摘要： 针对一类带 Dirichlet 边值条件和初值条件的加热下分数阶广义二阶流体的 Stokes 第一问题,提出了一种新的高阶隐式数值格式.应用 Fourier 分析方法和矩阵方法研究了该格式的稳定性、可解性及收敛性.也进一步给出一个时间误差阶更高的改进的隐式格式.最后通过两个数值算例验证了格式的理论分析是有效可靠的.

关　键　词： 分数阶 Stokes 问题；　隐式差分格式；　可解性；　稳定性；　收敛性

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引　　言

本文研究求解如下加热下分数阶广义二阶流体的 Stokes 第一问题的高阶数值方法：

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[\kappa_1 \frac{\partial^2 u(x,t)}{\partial x^2}\right] + \kappa_2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$

$$0 \leqslant x \leqslant L;\ 0 < t \leqslant T, \tag{1}$$

并满足初值条件：

$$u(x,0) = \varphi(x), \qquad 0 \leqslant x \leqslant L \tag{2}$$

及 Dirichlet 边值条件：

$$u(0,t) = \phi_1(t), \qquad 0 < t \leqslant T, \tag{3}$$

$$u(L,t) = \phi_2(t), \qquad 0 < t \leqslant T, \tag{4}$$

其中，$0 < \gamma < 1$，常数 κ_1，$\kappa_2 > 0$，函数 $f(x,t) \in C^1([0,T])$,记号 ${}_0D_t^{1-\gamma}$ 表示 Riemann-Liouville 分数阶导数[1]，其定义为

$${}_0D_t^{1-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(\tau)(t-\tau)^{\gamma-1}\mathrm{d}\tau. \tag{5}$$

Stokes 第一问题(FOSFP)广泛应用于生物流变学、化学、地球物理学和石油工业等科学工程领域[2-5].近年来,许多学者对 FOSFP 本身理论进行了研究,获得了众多有意义的成果,这些工作可以参见 Tan 等[6]、Hayat 等[7]、Devakara 等[8]、Salah 等[9]、Ezzat 等[10]、Shen 等[11]等人的论文.然而,由于这类问题的仅在一些特殊的情形下才能获得其理论解.因此,研究求解 FOSFP

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的数值方法显得非常重要.

分数阶微分方程的数值方法近年来成为了研究的热门,也获得了丰富的研究成果,最具代表性的工作可参见文献[1,12-15,22]等. 2009 年,Chen 等[12]分别构造了求解方程(1)的显式和隐式数值格式并通过 Fourier 方法给出了其稳定性和收敛性结果. 其结果表明这些格式具有收敛阶 $O(\tau+h^2)$, 这里 τ 和 h 分别为时间和空间步长. 同年,Wu[16]也给出了求解这类问题的另一数值格式,该格式的收敛阶为 $O(\tau+h^2)$. 更进一步,在 2011 年,Chen 等[15]研究了求解变阶的非线性 FOSFP 的数值方法,并构造了两类数值格式,它们的收敛阶分别为 $O(\tau+h^4)$ 和 $O(\tau^2+h^4)$.

本文的目的是利用紧致有限差分算子构造出新的求解加热下分数阶广义二阶流体的 Stokes 问题(1)的高精度隐式有限差分格式.

1 高阶隐式有限差分格式(IFDS)的构造

为方便计, 我们引进如下记号:

$$\Omega=\{(x,t)\mid 0\leqslant x\leqslant L,\ 0\leqslant t\leqslant T\},$$

$$U(\Omega)=\left\{u(x,t)\ \middle|\ \frac{\partial^4 u(x,t)}{\partial x^4},\ \frac{\partial^4 u(x,t)}{\partial x^2\partial t^2},\ \frac{\partial^5 u(x,t)}{\partial x^4\partial t}\in C(\Omega)\right\},$$

$$\delta_x^2 u(x,t)=u(x+h,t)+u(x-h,t)-2u(x,t),$$

$$Lu(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2},\ \mu_1=\frac{\kappa_1\tau^\gamma}{\Gamma(1+\gamma)h^2},\ \mu_2=\frac{\kappa_2\tau}{h^2},$$

$$v(x,t)=u(x,t+\tau)-u(x,t).$$

对于给定的实数 α, β,定义 $(M-1)\times(M-1)$ 矩阵 $\boldsymbol{W}(\alpha,\beta)$ 如下:

$$\boldsymbol{W}(\alpha,\beta)=\begin{bmatrix}\alpha & \beta & & 0\\ \beta & \alpha & \ddots & \\ & \ddots & \ddots & \beta\\ 0 & & \beta & \alpha\end{bmatrix}.$$

此外,常数 C 在不同位置代表不同的正常数. 本文恒设函数 $u(x,t)\in U(\Omega)$.

令 $t_k=k\tau$, $k=0,1,\cdots,N$, $x_j=jh$, $j=0,1,\cdots$, M 分别为时间和空间的一个等距剖分, 其中 $\tau=T/N$ 和 $h=L/M$ 分别表示时间步长与空间步长.

方程(1)两端从 t_k 到 t_{k+1} 进行积分 $(k=0,1,\cdots,N-1)$ 得

$$\begin{aligned}u(x_j,t_{k+1})=&u(x_j,t_k)+\frac{\kappa_1}{\Gamma(\gamma)}\int_0^{t_{k+1}}\frac{Lu(x_j,\eta)}{(t_{k+1}-\eta)^{1-\gamma}}\mathrm{d}\eta-\frac{\kappa_1}{\Gamma(\gamma)}\int_0^{t_k}\frac{Lu(x_j,\eta)}{(t_k-\eta)^{1-\gamma}}\mathrm{d}\eta+\\ &\kappa_2\int_{t_k}^{t_{k+1}}Lu(x_j,\eta)\mathrm{d}\eta+\int_{t_k}^{t_{k+1}}f(x_j,t)\mathrm{d}t=\\ &u(x_j,t_k)+\frac{\kappa_1}{\Gamma(\gamma)}\int_0^{\tau}\frac{Lu(x_j,\eta)}{(t_{k+1}-\eta)^{1-\gamma}}\mathrm{d}\eta+\frac{\kappa_1}{\Gamma(\gamma)}\int_0^{t_k}\frac{Lv(x_j,\eta)}{(t_k-\eta)^{1-\gamma}}\mathrm{d}\eta+\\ &\kappa_2\int_{t_k}^{t_{k+1}}Lu(x_j,\eta)\mathrm{d}\eta+\int_{t_k}^{t_{k+1}}f(x_j,t)\mathrm{d}t=\\ &u(x_j,t_k)+I_1+I_2+I_3+I_4,\end{aligned}\tag{6}$$

其中

$$I_1=\frac{\kappa_1}{\Gamma(\gamma)}\int_0^{\tau}\frac{Lu(x_j,\eta)}{(t_{k+1}-\eta)^{1-\gamma}}\mathrm{d}\eta,\ I_2=\frac{\kappa_1}{\Gamma(\gamma)}\int_0^{t_k}\frac{Lv(x_j,\eta)}{(t_k-\eta)^{1-\gamma}}\mathrm{d}\eta,$$

$$I_3=\kappa_2\int_{t_k}^{t_{k+1}}Lu(x_j,\eta)\,\mathrm{d}\eta,\ I_4=\int_{t_k}^{t_{k+1}}f(x_j,t)\,\mathrm{d}t.$$

本文将应用 Cui[17]给出的如下紧致有限差分算子来逼近二阶导数 $\partial^2 u/\partial x^2$：

$$\frac{\delta_x^2}{(1+\delta_x^2/12)h^2}u(x_j,t_k)=\frac{\partial^2 u(x_j,t_k)}{\partial x^2}-\frac{1}{240}\frac{\partial^4 u(x_j,t_k)}{\partial x^4}h^4+O(h^6). \tag{7}$$

引理 1 令 $b_j=(j+1)^\gamma-j^\gamma,\ j=0,1,\cdots,N$,则序列 $\{b_j\}$ 满足：

(i) $b_0=1$;

(ii) $0<b_{k+1}<b_k\leqslant 1,\qquad k=0,1,2,\cdots,N$;

(iii) $\tau\leqslant Cb_k\tau^\gamma$.

引理 1 的证明参见文献[18].

下面分别讨论 I_1,I_2,I_3 和 I_4. 首先对于 I_1, 我们有

$$\begin{aligned}
I_1&=\frac{\kappa_1}{\Gamma(\gamma)}\int_0^\tau\frac{Lu(x_j,\tau)}{(t_{k+1}-\eta)^{1-\gamma}}\mathrm{d}\eta+R_{11}=\\
&\frac{\kappa_1 Lu(x_j,\tau)}{\Gamma(\gamma)}\int_0^\tau\frac{1}{(t_{k+1}-\eta)^{1-\gamma}}\mathrm{d}\eta+R_{11}=\\
&\frac{\kappa_1 b_k\tau^\gamma}{\Gamma(\gamma+1)}Lu(x_j,\tau)+R_{11}=\\
&\mu_1 b_k\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,\tau)+\frac{\kappa_1 b_k\tau^\gamma}{\Gamma(\gamma+1)}R_{12}+R_{11}=\\
&\mu_1 b_k\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,\tau)+R_1,
\end{aligned}$$

其中

$$R_{11}=\frac{\kappa_1}{\Gamma(\gamma)}\int_0^\tau\frac{Lu(x_j,\eta)-Lu(x_j,\tau)}{(t_{k+1}-\eta)^{1-\gamma}}\mathrm{d}\eta,$$

$$R_{12}=Lu(x_j,\tau)-\frac{\delta_x^2}{(1+\delta_x^2/12)h^2}u(x_j,\tau),$$

$$R_1=R_{11}+\frac{\kappa_1 b_k\tau^\gamma}{\Gamma(\gamma+1)}R_{12}.$$

由 Lagrange 中值定理可得

$$\begin{aligned}
|R_{11}|&=\left|\frac{\kappa_1}{\Gamma(\gamma)}\int_0^\tau\frac{\partial^3 u(x_j,\xi_1)}{\partial x^2\partial t}\frac{\eta-\tau}{(t_{k+1}-\eta)^{1-\gamma}}\mathrm{d}\eta\right|\leqslant\\
&\frac{C_1\tau\kappa_1}{\Gamma(\gamma)}\int_0^\tau\frac{\mathrm{d}\eta}{(t_{k+1}-\eta)^{1-\gamma}}\leqslant Cb_k\tau^{1+\gamma}.
\end{aligned}$$

另一方面由式(7)容易得到 $|R_{12}|\leqslant Ch^4$. 由此可以推得

$$|R_1|\leqslant Cb_k\tau^\gamma(\tau+h^4).$$

对于 I_2, 采用如下的近似：

$$\begin{aligned}
I_2&=\frac{\kappa_1}{\Gamma(\gamma)}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\frac{Lv(x_j,\eta)}{(t_k-\eta)^{1-\gamma}}\mathrm{d}\eta=\\
&\frac{\kappa_1}{\Gamma(\gamma)}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\frac{Lv(x_j,t_{i+1})}{(t_k-\eta)^{1-\gamma}}\mathrm{d}\eta+R_{21}=
\end{aligned}$$

$$\frac{\kappa_1\tau^\gamma}{\Gamma(\gamma+1)}\sum_{i=0}^{k-1}b_{k-i-1}Lv(x_j,t_{i+1})+R_{21}=$$
$$\mu_1\sum_{i=0}^{k-1}b_{k-i-1}\frac{\delta_x^2}{(1+\delta_x^2/12)}v(x_j,t_{i+1})+R_{22}+R_{21},$$

其中

$$R_{21}=\frac{\kappa_1}{\Gamma(\gamma)}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\frac{Lv(x_j,\eta)-Lv(x_j,t_{i+1})}{(t_k-\eta)^{1-\gamma}}\mathrm{d}\eta,$$
$$R_{22}=\frac{\kappa_1\tau^\gamma}{\Gamma(1+\gamma)}\sum_{i=0}^{k-1}b_{k-i-1}\left[Lv(x_j,t_{i+1})-\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}v(x_j,t_{i+1})\right].$$

再次应用 Lagrange 中值定理,有

$$Lv(x_j,\eta)-Lv(x_j,t_{i+1})=\frac{\partial^3 v(x_j,\xi_2)}{\partial x^2\partial t}(\eta-t_{i+1}),\qquad \eta\leqslant\xi_2\leqslant t_{i+1},$$
$$\frac{\partial^3 v(x_j,\xi_2)}{\partial x^2\partial t}=\frac{\partial^3 u(x_j,\xi_2+\tau)}{\partial x^2\partial t}-\frac{\partial^3 u(x_j,\xi_2)}{\partial x^2\partial t}=\frac{\partial^4 u(x_j,\xi_3)}{\partial x^2\partial t^2}\tau,\qquad \xi_2\leqslant\xi_3\leqslant\xi_2+\tau.$$

故而得到

$$|R_{21}|\leqslant C_2\tau^2\frac{\kappa_1}{\Gamma(\gamma)}\int_0^{t_k}\frac{\mathrm{d}\eta}{(t_k-\eta)^{1-\gamma}}\leqslant C\tau^2.$$

再次利用式(7),可导出:

$$Lv(x_j,t_{i+1})=\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}v(x_j,t_{i+1})+\frac{h^4}{240}\frac{\partial^4 v(\zeta_1,t_{i+1})}{\partial x^4}=$$
$$\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}v(x_j,t_{i+1})+\frac{h^4}{240}\left[\frac{\partial^4 u(\zeta_1,t_{i+2})}{\partial x^4}-\frac{\partial^4 u(\zeta_1,t_{i+1})}{\partial x^4}\right]=$$
$$\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}v(x_j,t_{i+1})+\frac{h^4\tau}{240}\frac{\partial^5 u(\zeta_1,\xi_4)}{\partial x^4\partial t}.$$

因此

$$|R_{22}|\leqslant Ch^4\tau^{1+\gamma}\frac{\kappa_1}{\Gamma(1+\gamma)}\sum_{i=0}^{k-1}b_{k-i-1}\leqslant Ch^4\tau.$$

对 I_3 和 I_4, 应用梯形积分公式可得

$$I_3=\kappa_2\int_{t_k}^{t_{k+1}}Lu(x_j,\eta)\mathrm{d}\eta=\kappa_2\frac{\tau}{2}(Lu(x_j,t_k)+Lu(x_j,t_{k+1}))+R_{31}=$$
$$\frac{\mu_2}{2}\left(\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,t_k)+\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,t_{k+1})\right)+R_{31}+R_{32}$$

和

$$I_4=\frac{\tau}{2}(f(x_j,t_{k+1})+f(x_j,t_k))+R_4, \tag{8}$$

其中, $|R_{31}|\leqslant C\tau^3$, $|R_4|\leqslant C\tau^3$, $|R_{32}|\leqslant C\tau h^4$. 事实上

$$R_{32}=\frac{\kappa_2\tau}{2}\left[Lu(x_j,t_k)-\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_k)+\right.$$
$$\left.Lu(x_j,t_{k+1})-\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_{k+1})\right],$$

应用式(7)可得其估计式 $|R_{32}|$.

综合上面的分析我们得到

$$
\begin{aligned}
&u(x_j,t_{k+1})=\\
&u(x_j,t_k)+\mu_1 b_k\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,\tau)+\mu_1\sum_{i=0}^{k-1}b_{k-i-1}\frac{\delta_x^2}{(1+\delta_x^2/12)}v(x_j,t_{i+1})+\\
&\frac{\mu_2}{2}\left(\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,t_k)+\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,t_{k+1})\right)+\\
&\frac{\tau}{2}\left(f(x_j,t_{k+1})+f(x_j,t_k)\right)+R_j^{k+1},\\
&\qquad j=1,2,\cdots,M-1;\ k=0,1,\cdots,N-1,
\end{aligned}\tag{9}
$$

其中, $|R_j^{k+1}|\leqslant C(b_k\tau^\gamma+\tau)(\tau+h^4)$. 利用引理 1,有

$$
|R_j^{k+1}|\leqslant Cb_k\tau^\gamma(\tau+h^4).\tag{10}
$$

容易看出式(9)可以改写为

$$
\begin{aligned}
&u(x_j,t_{k+1})=u(x_j,t_k)+\mu_1\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,t_{k+1})+\\
&\mu_1\sum_{i=0}^{k-1}(b_{k-i}-b_{k-i-1})\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,t_{i+1})+\\
&\frac{\mu_2}{2}\left(\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,t_k)+\frac{\delta_x^2}{(1+\delta_x^2/12)}u(x_j,t_{k+1})\right)+\\
&\frac{\tau}{2}\left(f(x_j,t_{k+1})+f(x_j,t_k)\right)+R_j^{k+1}.
\end{aligned}\tag{11}
$$

将式(11)的两端同乘以 $(1+\delta_x^2/12)$, 有

$$
\begin{aligned}
&\left(1+\frac{1}{12}\delta_x^2\right)u(x_j,t_{k+1})=\\
&\left(1+\frac{1}{12}\delta_x^2\right)u(x_j,t_k)+\mu_1\delta_x^2u(x_j,t_{k+1})+\\
&\mu_1\sum_{i=0}^{k-1}(b_{k-i}-b_{k-i-1})\delta_x^2u(x_j,t_{i+1})+\frac{\mu_2}{2}\left(\delta_x^2u(x_j,t_k)+\delta_x^2u(x_j,t_{k+1})\right)+\\
&\frac{\tau}{2}\left(1+\frac{1}{12}\delta_x^2\right)\left(f(x_j,t_{k+1})+f(x_j,t_k)\right)+\left(1+\frac{1}{12}\delta_x^2\right)R_j^{k+1}.
\end{aligned}\tag{12}
$$

让 u_j^k 表示问题(1) 在点(x_j,t_k) 处的数值解,并记

$$
\delta_x^2u_j^k=u_{j+1}^k-2u_j^k+u_{j-1}^k,\ f_j^k=f(x_j,t_k).
$$

通过上面的讨论,我们便得到求解问题(1) ~ (4)的数值计算格式:

$$
\begin{aligned}
&\left[1+\left(\frac{1}{12}-\left(\mu_1+\frac{\mu_2}{2}\right)\right)\delta_x^2\right]u_j^{k+1}=\\
&\left[1+\left(\frac{1}{12}+(b_1-b_0)\mu_1+\frac{\mu_2}{2}\right)\delta_x^2\right]u_j^k+\mu_1\sum_{i=1}^{k-1}(b_{k-i+1}-b_{k-i})\delta_x^2u_j^i+\\
&\frac{\tau}{2}\left(1+\frac{1}{12}\delta_x^2\right)(f_j^{k+1}+f_j^k),\qquad j=1,2,\cdots,M-1;\ k=0,1,\cdots,N-1,
\end{aligned}\tag{13}
$$

$$
u_0^k=\phi_1(k\tau),\ u_M^k=\phi_2(k\tau),\qquad k=1,2,\cdots,N,\tag{14}
$$

$$
u_j^0=\varphi(jh),\qquad j=0,1,\cdots,M.\tag{15}
$$

这里我们约定,当 $q < p$ 时,和式 $\sum_p^q$ 的值为 0.

现在分析格式(13)~(15)的唯一可解性,我们有如下结论:

定理 1 数值格式(13)~(15)存在唯一解.

证明 格式(13)~(15)可写成矩阵的形式:

$$\begin{cases} \boldsymbol{A}\boldsymbol{U}^{k+1} = \boldsymbol{B}\boldsymbol{U}^k + \sum_{i=1}^{k-1} \boldsymbol{B}_i \boldsymbol{U}^i + \boldsymbol{F}^{k+1}, & k \geqslant 0, \\ \boldsymbol{U}^0 = \boldsymbol{\Phi}, \end{cases} \tag{16}$$

其中

$$\boldsymbol{A} = \boldsymbol{W}\left(\frac{5}{6} + 2\mu_1 + \mu_2, \frac{1}{12} - \left(\mu_1 + \frac{\mu_2}{2}\right)\right), \ \boldsymbol{B}_i = \mu_1(b_{k-i+1} - b_{k-i})\ \boldsymbol{W}(-2, 1),$$

$$\boldsymbol{B} = \boldsymbol{W}\left(\frac{5}{6} - 2(b_1 - b_0)\mu_1 - \mu_2, \frac{1}{12} + (b_1 - b_0)\mu_1 + \frac{\mu_2}{2}\right),$$

$$\boldsymbol{F}^{k+1} = \begin{bmatrix} \left(\frac{1}{12} + \mu_1(b_1 - b_0) + \frac{\mu_2}{2}\right)u_0^k + \left(\mu_1 + \frac{\mu_2}{2} - \frac{1}{12}\right)u_0^{k+1} + \\ \quad \mu_1 \sum_{i=1}^{k-1}(b_{k-i+1} - b_{k-i})u_0^i + \\ \quad \frac{\tau}{24}(f_2^k + 10f_1^k + f_0^k + f_2^{k+1} + 10f_1^{k+1} + f_0^{k+1}), \\ \frac{\tau}{24}(f_3^k + 10f_2^k + f_1^k + f_3^{k+1} + 10f_2^{k+1} + f_1^{k+1}), \\ \vdots \\ \frac{\tau}{24}(f_{M-1}^k + 10f_{M-2}^k + f_{M-3}^k + f_{M-1}^{k+1} + 10f_{M-2}^{k+1} + f_{M-3}^{k+1}), \\ \left(\frac{1}{12} + \mu_1(b_1 - b_0) + \frac{\mu_2}{2}\right)u_M^k + \left(\mu_1 + \frac{\mu_2}{2} - \frac{1}{12}\right)u_M^{k+1} + \\ \quad \mu_1 \sum_{i=1}^{k-1}(b_{k-i+1} - b_{k-i})u_M^i + \\ \quad \frac{\tau}{24}(f_M^k + 10f_{M-1}^k + f_{M-2}^k + f_M^{k+1} + 10f_{M-1}^{k+1} + f_{M-2}^{k+1}) \end{bmatrix},$$

$$\boldsymbol{U}^k = (u_1^k, u_2^k, \cdots, u_{M-1}^k)^{\mathrm{T}}, \ \boldsymbol{\Phi} = (\varphi_1, \varphi_2, \cdots, \varphi_{M-1})^{\mathrm{T}}, \ \varphi_j = \varphi(jh).$$

显然系数矩阵 $\boldsymbol{A}$ 严格对角占优,从而它是可逆的[15]. 故格式(13)~(15)存在唯一解. □

2 IFDS 的稳定性分析

我们利用 Fourier 方法讨论格式(13)~(15)的稳定性.

令 z_j^k 为格式(13)~(15)的近似解,并记

$$\boldsymbol{\varepsilon}_j^k = u_j^k - z_j^k, \ \boldsymbol{\varepsilon}^k = (\boldsymbol{\varepsilon}_1^k, \boldsymbol{\varepsilon}_2^k, \cdots, \boldsymbol{\varepsilon}_{M-1}^k)^{\mathrm{T}}, \ \delta_x^2 \boldsymbol{\varepsilon}_j^k = \boldsymbol{\varepsilon}_{j+1}^k - 2\boldsymbol{\varepsilon}_j^k + \boldsymbol{\varepsilon}_{j-1}^k.$$

显然, $\boldsymbol{\varepsilon}_j^k$ 满足误差传播方程:

$$\left[1 + \left(\frac{1}{12} - \left(\mu_1 + \frac{\mu_2}{2}\right)\right)\delta_x^2\right]\boldsymbol{\varepsilon}_j^{k+1} =$$

$$\left[1 + \left(\frac{1}{12} + (b_1 - b_0)\mu_1 + \frac{\mu_2}{2}\right)\delta_x^2\right]\boldsymbol{\varepsilon}_j^k +$$

$$\mu_1 \sum_{i=0}^{k-1} \delta_x^2 \varepsilon_j^i (b_{k-i+1} - b_{k-i}), \qquad j = 0,1,\cdots,M-1;\ k = 0,1,\cdots,N-1. \tag{17}$$

对于 $k = 0,1,\cdots,\ N-1$, 我们引入阶梯函数:

$$\varepsilon^k(x) = \begin{cases} \varepsilon_j^k, & x_j - \dfrac{h}{2} < x \leqslant x_j + \dfrac{h}{2};\ j = 1,2,\cdots,M-1, \\ 0, & 0 \leqslant x \leqslant \dfrac{h}{2} \text{ 或 } L - \dfrac{h}{2} < x \leqslant L. \end{cases}$$

将 $\varepsilon^k(x)$ 展成 Fourier 级数形式:

$$\varepsilon^k(x) = \sum_{m=-\infty}^{+\infty} v^k(m) \mathrm{e}^{\mathrm{i}\sigma x}, \tag{18}$$

其中

$$v^k(m) = \frac{1}{L}\int_0^L \varepsilon^k(x)\mathrm{e}^{-\mathrm{i}\sigma x}\mathrm{d}x,\ \sigma = \frac{2m\pi}{L}.$$

定义离散范数 L_2: $\| \varepsilon^k \|_2 = \left(\sum_{j=1}^{M-1} h | \varepsilon_j^k|^2 \right)^{1/2}$. 则由 Parseval 等式可得

$$\sum_{m=-\infty}^{+\infty} | v^k(m) |^2 = \int_0^L | \varepsilon^k(x) |^2 \mathrm{d}x = \| \varepsilon^k \|_2^2. \tag{19}$$

对于具有连续变量的函数 $\varepsilon^k(x)$, 仍有

$$\left[1 + \left(\frac{1}{12} - \left(\mu_1 + \frac{\mu_2}{2}\right)\right)\delta_x^2\right]\varepsilon^{k+1}(x) =$$
$$\left[1 + \left(\frac{1}{12} + (b_1 - b_0)\mu_1 + \frac{\mu_2}{2}\right)\ \delta_x^2\right]\varepsilon^k(x) +$$
$$\mu_1 \sum_{i=0}^{k-1} \delta_x^2 \varepsilon^i(x)(b_{k-i+1} - b_{k-i}), \qquad k = 0,1,\cdots,N-1, \tag{20}$$

其中, $\delta_x^2 \varepsilon^k(x) = \varepsilon^k(x+h) - 2\varepsilon^k(x) + \varepsilon^k(x-h)$. 将式(18)代入到方程(20),并将方程(20)的两端同乘以 $\mathrm{e}^{-\mathrm{i}\sigma_1 x}$, 同时注意到:

$$\int_0^L \mathrm{e}^{\mathrm{i}\sigma x}\mathrm{e}^{-\mathrm{i}\sigma_1 x}\mathrm{d}x = \begin{cases} 0, & m \neq n, \\ L, & m = n, \end{cases} \qquad \sigma_1 = \frac{2n\pi}{L};\ n = 0,\ \pm 1,\ \pm 2,\cdots,$$

我们得到

$$\left[1 + \left(\mu_1 + \frac{\mu_2}{2} - \frac{1}{12}\right) r\right] v^{k+1}(m) =$$
$$\left[1 + \left((b_0 - b_1)\mu_1 - \frac{\mu_2}{2} - \frac{1}{12}\right) r\right] v^k(m) - r\mu_1 \sum_{i=0}^{k-1} v^i(m)(b_{k-i+1} - b_{k-i}), \tag{21}$$

其中, $r = 4\sin^2(\sigma h/2)$. 进一步地,式(21)可化为

$$v^{k+1}(m) = \frac{1 + ((b_0 - b_1)\mu_1 - \mu_2/2 - 1/12) r}{1 + (\mu_1 + \mu_2/2 - 1/12) r} v^k(m) +$$
$$\frac{r\mu_1}{1 + (\mu_1 + \mu_2/2 - 1/12) r} \sum_{i=0}^{k-1} v^i(m)(b_{k-i} - b_{k-i+1}). \tag{22}$$

由于 $\mu_1, \mu_2 \geqslant 0,\ 0 \leqslant r \leqslant 4$, 所以 $1 + (\mu_1 + \mu_2/2 - 1/12) r > 0$.

定理 2 若 $v^k(m)$ 满足方程(22),则必有 $| v^k(m) | \leqslant | v^0(m) |$.

证明 我们可用数学归纳法进行证明.

事实上,当 $k = 0$, 由式(22)有

$$v^1(m)=\frac{1+((b_0-b_1)\mu_1-\mu_2/2-1/12)r}{1+(\mu_1+\mu_2/2-1/12)r}v^0(m).$$

应用引理 1 可得 $0<b_0-b_1<1$. 则有 $|v^1(m)|\leqslant|v^0(m)|$.

假设 $|v^n(m)|\leqslant|v^0(m)|$ $(1\leqslant n\leqslant k)$. 仍由引理 1,有

$$\begin{aligned}|v^{k+1}(m)|\leqslant&\frac{1+((b_0-b_1)\mu_1+\mu_2/2-1/12)r}{1+(\mu_1+\mu_2/2-1/12)r}|v^k(m)|+\\&\frac{r\mu_1}{1+(\mu_1+\mu_2/2-1/12)r}\sum_{i=0}^{k-1}|v^i(m)|(b_{k-i}-b_{k-i+1})\leqslant\\&\left(\frac{1+((b_0-b_1)\mu_1+\mu_2/2-1/12)r}{1+(\mu_1+\mu_2/2-1/12)r}+\right.\\&\left.\frac{r\mu_1}{1+(\mu_1+\mu_2/2-1/12)r}\sum_{i=0}^{k-1}(b_{k-i}-b_{k-i+1})\right)|v^0(m)|=\\&\frac{1+((b_0-b_k)\mu_1+\mu_2/2-1/12)r}{1+(\mu_1+\mu_2/2-1/12)r}|v^0(m)|\leqslant|v^0(m)|.\end{aligned}$$

定理 2 证毕. □

定理 3 IFDS 式(13)~(15)是无条件稳定的.

证明 应用定理 2 到式(19),有

$$\|\varepsilon^k\|_2^2=\sum_{m=-\infty}^{+\infty}|v^k(m)|^2\leqslant\sum_{m=-\infty}^{+\infty}|v^0(m)|^2=\|\varepsilon^0\|_2^2,\qquad k=0,1,\cdots,N.$$

证毕. □

3 IFDS 的收敛性分析

为了讨论数值格式的收敛性,我们将用到离散的 Gronwall 引理[19-20].

引理 2(Gronwall 引理) 设 k_n 为一非负序列,ϕ_n 满足

$$\begin{cases}\phi_0\leqslant g_0,\\ \phi_n\leqslant g_0+\sum_{s=0}^{n-1}p_s+\sum_{s=0}^{n-1}k_s\phi_s,\qquad n\geqslant 1.\end{cases}$$

如果对于任给的 $n\geqslant 0$,都有 $g_0\geqslant 0$, $p_n\geqslant 0$, 则

$$\phi_n\leqslant\left(g_0+\sum_{s=0}^{n-1}p_s\right)\exp\left(\sum_{s=0}^{n-1}k_s\right),\qquad n\geqslant 1.$$

为简便计,我们引进一些记号.

对于 $j=0,1,2,\cdots,M$; $k=0,1,2,\cdots,N$, 我们定义

$$e_j^k=u(x_j,t_k)-u_j^k,\ \boldsymbol{u}^k=(u(x_1,t_k),\cdots,u(x_{M-1},t_k))^{\mathrm T},\ \boldsymbol{e}^k=(e_1^k,\cdots,e_{M-1}^k)^{\mathrm T},$$

$$\boldsymbol{R}^k=(R_1^k,\cdots,R_{M-1}^k)^{\mathrm T},\ \tilde{\boldsymbol{R}}^k=\left(\left(1+\frac{1}{12}\delta_x^2\right)R_1^k,\cdots,\left(1+\frac{1}{12}\delta_x^2\right)R_{M-1}^k\right)^{\mathrm T}.$$

对于 $\boldsymbol{\eta}=(\eta_1,\eta_2,\cdots,\eta_{M-1})^{\mathrm T}$, $\boldsymbol{v}=(v_1,v_2,\cdots,v_{M-1})^{\mathrm T}$, 我们定义内积:

$$(\boldsymbol{\eta},\boldsymbol{v})=\sum_{j=1}^{M-1}h\eta_j v_j.$$

符号 $\lambda_{\min}(\boldsymbol{A})$, $\lambda_{\max}(\boldsymbol{A})$ 和 $\lambda_j(\boldsymbol{A})$ 分别表示矩阵 $\boldsymbol{A}$ 的最小、最大和第 j 个特征值.

通过应用式(10),我们容易看出:

$$\| \tilde{\boldsymbol{R}}^{k+1} \|_2 = \left(\sum_{j=1}^{M-1} h \left| \left(1 + \frac{1}{12}\delta_x^2\right) R_j^{k+1} \right|^2 \right)^{1/2} \leqslant$$

$$C b_k \tau^{\gamma} (\tau + h^4) \sqrt{(M-1)h} \leqslant C b_k \tau^{\gamma} (\tau + h^4). \tag{23}$$

对于矢量 $\boldsymbol{u}^k$, 由式(12)我们有

$$\begin{cases} \boldsymbol{A}\boldsymbol{u}^{k+1} = \boldsymbol{B}\boldsymbol{u}^k + \sum_{i=1}^{k-1} \boldsymbol{B}_i \boldsymbol{u}^i + \boldsymbol{F}^{k+1} + \tilde{\boldsymbol{R}}^{k+1}, & k \geqslant 0, \\ \boldsymbol{u}^0 = \boldsymbol{\Phi}. \end{cases} \tag{24}$$

方程组(24)减去式(16),并注意到 $\boldsymbol{e}^0 = \boldsymbol{0}$, 可得

$$\boldsymbol{A}\boldsymbol{e}^{k+1} = \boldsymbol{B}\boldsymbol{e}^k + \sum_{i=1}^{k-1} \boldsymbol{B}_i \boldsymbol{e}^i + \tilde{\boldsymbol{R}}^{k+1}, \qquad k \geqslant 0. \tag{25}$$

定理 4 IFDS 格式(13)~(15)的收敛阶为 $O(\tau + h^4)$.

证明 方程(25)两边与 $\boldsymbol{e}^{k+1}$ 作内积, 有

$$(\boldsymbol{A}\boldsymbol{e}^{k+1}, \boldsymbol{e}^{k+1}) = (\boldsymbol{B}\boldsymbol{e}^k, \boldsymbol{e}^{k+1}) + \sum_{i=1}^{k-1} (\boldsymbol{B}_i \boldsymbol{e}^i, \boldsymbol{e}^{k+1}) + (\tilde{\boldsymbol{R}}^{k+1}, \boldsymbol{e}^{k+1}), \qquad k \geqslant 0. \tag{26}$$

由于 $\boldsymbol{A}, \boldsymbol{B}$ 和 $\boldsymbol{B}_i$ 是对称矩阵,故可应用 Rayleigh-Ritz 商定理[21]可得

$$\lambda_{\min} \leqslant \frac{(\boldsymbol{A}\boldsymbol{v}, \boldsymbol{v})}{(\boldsymbol{v}, \boldsymbol{v})} \leqslant \lambda_{\max}.$$

利用内积与 L_2 范数的关系不难得出:

$$\begin{cases} \lambda_{\min}(\boldsymbol{A}) \| \boldsymbol{e}^k \|_2 = \lambda_{\min}(\boldsymbol{A}) (\boldsymbol{e}^k, \boldsymbol{e}^k) \leqslant (\boldsymbol{A}\boldsymbol{e}^k, \boldsymbol{e}^k), \\ |(\boldsymbol{B}_i \boldsymbol{e}^i, \boldsymbol{e}^{k+1})| \leqslant \sqrt{(\boldsymbol{B}_i \boldsymbol{e}^i, \boldsymbol{e}^i)} \sqrt{(\boldsymbol{e}^{k+1}, \boldsymbol{e}^{k+1})} \leqslant \lambda_{\max}(\boldsymbol{B}_i) \| \boldsymbol{e}^i \|_2 \| \boldsymbol{e}^{k+1} \|_2, \\ |(\tilde{\boldsymbol{R}}^{k+1}, \boldsymbol{e}^{k+1})| \leqslant \| \tilde{\boldsymbol{R}}^{k+1} \|_2 \| \boldsymbol{e}^{k+1} \|_2. \end{cases}$$

从以上的分析,可以得到下列的不等式:

$$\| \boldsymbol{e}^{k+1} \|_2 \leqslant \frac{\lambda_{\max}(\boldsymbol{B})}{\lambda_{\min}(\boldsymbol{A})} \| \boldsymbol{e}^k \|_2 + \sum_{i=1}^{k-1} \frac{\lambda_{\max}(\boldsymbol{B}_i)}{\lambda_{\min}(\boldsymbol{A})} \| \boldsymbol{e}^i \|_2 + \frac{1}{\lambda_{\min}(\boldsymbol{A})} \| \tilde{\boldsymbol{R}}^{k+1} \|_2,$$

$$k \geqslant 0. \tag{27}$$

因为矩阵 $\boldsymbol{A}, \boldsymbol{B}$ 以及 $\boldsymbol{B}_i$ 都是三对角阵, 容易计算其特征值[19]如下:

$$\lambda_j(\boldsymbol{A}) = \frac{5}{6} + 2\left(\mu_1 + \frac{\mu_2}{2}\right) + 2\left(\frac{1}{12} - \left(\mu_1 + \frac{\mu_2}{2}\right)\right) \cos \frac{j\pi}{M} =$$

$$\frac{2}{3} + \frac{1}{3} \cos^2 \frac{j\pi}{2M} + 4\left(\mu_1 + \frac{\mu_2}{2}\right) \sin^2 \frac{j\pi}{2M},$$

$$\lambda_j(\boldsymbol{B}) = \frac{5}{6} - 2\left(\mu_1 (b_1 - b_0) + \frac{\mu_2}{2}\right) + 2\left[\frac{1}{12} + \mu_1 (b_1 - b_0) + \frac{\mu_2}{2}\right] \cos \frac{j\pi}{M} =$$

$$\frac{2}{3} + \frac{1}{3} \cos^2 \frac{j\pi}{2M} + 4\left[\mu_1 (b_0 - b_1) - \frac{\mu_2}{2}\right] \sin^2 \frac{j\pi}{2M} \leqslant \lambda_j(\boldsymbol{A}),$$

$$\lambda_j(\boldsymbol{B}_i) = \mu_1 (b_{k-i+1} - b_{k-i}) \left(-2 + 2\cos\frac{j\pi}{M}\right) = 4\mu_1 (b_{k-i} - b_{k-i+1}) \sin^2 \frac{j\pi}{2M},$$

其中 $j = 1, 2, \cdots, M-1$. 不等式(27)可化简为

$$\| \boldsymbol{e}^{k+1} \|_2 \leqslant \left(\frac{3}{2} + 6\mu_1 + 3\mu_2\right) \| \boldsymbol{e}^k \|_2 +$$

$$6\mu_1 \sum_{i=1}^{k-1} (b_{k-i} - b_{k-i+1}) \| \boldsymbol{e}^i \|_2 + \frac{1}{16} \| \tilde{\boldsymbol{R}}^{k+1} \|_2, \qquad k = 0, 1, 2, \cdots, N-1.$$

因此,应用引理 1 和引理 2 可得

$$\|\boldsymbol{e}^{k+1}\|_2 \leqslant C\sum_{i=1}^{k+1}\|\tilde{\boldsymbol{R}}^i\|_2\exp\left(\sum_{i=1}^{k}6\mu_1(b_{k-i}-b_{k-i+1})+\frac{3}{2}+6\mu_1+3\mu_2\right)\leqslant$$
$$C\sum_{i=1}^{k+1}\|\tilde{\boldsymbol{R}}^i\|_2\exp\left(12\mu_1+3\mu_2+\frac{3}{2}\right)\leqslant$$
$$C\sum_{i=1}^{k+1}\|\tilde{\boldsymbol{R}}^i\|_2\leqslant C\sum_{i=1}^{k+1}b_{i-1}\tau^{\gamma}(\tau+h^4)\leqslant C(\tau+h^4).$$

定理证毕. □

4 格式 IFDS 的改进(IIFDS)

对方程(1)的两端从 0 积到 t_{k+1}, 有

$$u(x_j,t_{k+1})=u(x_j,0)+\frac{\kappa_1}{\Gamma(\gamma)}\int_0^{t_{k+1}}\frac{\partial^2 u(x_j,\eta)}{\partial x^2}(t_{k+1}-\eta)^{\gamma-1}\mathrm{d}\eta+$$
$$\kappa_2\int_0^{t_{k+1}}\frac{\partial^2 u(x_j,t)}{\partial x^2}\mathrm{d}t+\int_0^{t_{k+1}}f(x_j,t)\mathrm{d}t=$$
$$u(x_j,0)+\frac{\kappa_1}{\Gamma(\gamma)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\frac{\partial^2 u(x_j,\eta)}{\partial x^2}(t_{k+1}-\eta)^{\gamma-1}\mathrm{d}\eta+$$
$$\kappa_2\int_0^{t_{k+1}}\frac{\partial^2 u(x_j,t)}{\partial x^2}\mathrm{d}t+\int_0^{t_{k+1}}f(x_j,t)\mathrm{d}t+I'+I''+I''',$$

其中

$$I'=\frac{\kappa_1}{\Gamma(\gamma)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\frac{\partial^2 u(x_j,\eta)}{\partial x^2}(t_{k+1}-\eta)^{\gamma-1}\mathrm{d}\eta, \tag{28}$$

$$I''=\kappa_2\int_0^{t_{k+1}}\frac{\partial^2 u(x_j,t)}{\partial x^2}\mathrm{d}t, \tag{29}$$

$$I'''=\int_0^{t_{k+1}}f(x_j,t)\mathrm{d}t. \tag{30}$$

由 Lagrange 插值公式,有

$$\frac{\partial^2 u(x_j,\eta)}{\partial x^2}=\frac{(t_{i+1}-\eta)}{\tau}\frac{\partial^2 u(x_j,t_i)}{\partial x^2}+\frac{(\eta-t_i)}{\tau}\frac{\partial^2 u(x_j,t_{i+1})}{\partial x^2}+$$
$$\frac{1}{2}\frac{\partial^4 u(x_j,\xi')}{\partial x^2\partial t^2}(\eta-t_i)(\eta-t_{i+1}).$$

再次利用式(7),可以得 I' 的近似:

$$I'=\frac{\kappa_1}{\Gamma(\gamma)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\left[\frac{(t_{i+1}-\eta)}{\tau}\left(\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_i)\right)+\right.$$
$$\left.\frac{(\eta-t_i)}{\tau}\left(\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_{i+1})\right)\right](t_{k+1}-\eta)^{\gamma-1}\mathrm{d}\eta+R_1'+R_2'+R_3',$$

其中

$$R_1'=\frac{\kappa_1}{\Gamma(\gamma)\tau}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\left(\frac{\partial^2 u(x_j,t_i)}{\partial x^2}-\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_i)\right)\times$$
$$(t_{i+1}-\eta)(t_{k+1}-\eta)^{\gamma-1}\mathrm{d}\eta,$$

$$R_2' = \frac{\kappa_1}{\Gamma(\gamma)\tau}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\left(\frac{\partial^2 u(x_j,t_{i+1})}{\partial x^2} - \frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_{i+1})\right)\times$$
$$(\eta - t_i)(t_{k+1}-\eta)^{\gamma-1}\mathrm{d}\eta,$$
$$R_3' = \frac{\kappa_1}{\Gamma(\gamma)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\frac{1}{2}\frac{\partial^4 u(x_j,\xi')}{\partial x^2\partial t^2}(\eta - t_i)(\eta - t_{i+1})(t_{k+1}-\eta)^{\gamma-1}\mathrm{d}\eta.$$

类似第 1 节的分析，我们容易得到余项 R_1',R_2',R_3' 的估计。事实上

$$|R_1'| \leqslant \frac{\kappa_1 Ch^4}{\Gamma(\gamma)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}(t_{k+1}-\eta)^{\gamma-1}\mathrm{d}\eta =$$
$$\frac{\kappa_1 Ch^4\tau^\gamma}{\Gamma(1+\gamma)}\sum_{l=0}^{k}(l+1)^\gamma - l^\gamma =$$
$$\frac{\kappa_1 Ch^4[(k+1)\tau]^\gamma}{\Gamma(1+\gamma)} \leqslant \frac{\kappa_1 T^\gamma}{\Gamma(1+\gamma)}Ch^4 \leqslant Ch^4.$$

类似地，有 $|R_2'| \leqslant Ch^4$ 及 $|R_3'| \leqslant C\tau^2$.

对于 I'', 可以采用如下的近似：

$$I'' = \kappa_2\int_0^{t_{k+1}}\frac{\partial^2 u(x_j,t)}{\partial x^2}\mathrm{d}t =$$
$$\kappa_2\sum_{i=0}^{k}\frac{\tau}{2}\left[\frac{\partial^2 u(x_j,t_i)}{\partial x^2} + \frac{\partial^2 u(x_j,t_{i+1})}{\partial x^2}\right] + O(\tau^3) =$$
$$\kappa_2\sum_{i=0}^{k}\frac{\tau}{2}\left[\frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_i) + \frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_{i+1})\right] + R_4' + O(\tau^3),$$

其中

$$R_4' = \kappa_2\sum_{i=0}^{k}\frac{\tau}{2}\left[\frac{\partial^2 u(x_j,t_i)}{\partial x^2} - \frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_i) + \right.$$
$$\left.\frac{\partial^2 u(x_j,t_{i+1})}{\partial x^2} - \frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_{i+1})\right].$$

仍应用式(7)，有

$$|R_4'| \leqslant \kappa_2\sum_{i=0}^{k}\frac{\tau}{2}\left[\left|\frac{\partial^2 u(x_j,t_i)}{\partial x^2} - \frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_i)\right| + \right.$$
$$\left.\left|\frac{\partial^2 u(x_j,t_{i+1})}{\partial x^2} - \frac{\delta_x^2}{h^2(1+\delta_x^2/12)}u(x_j,t_{i+1})\right|\right] \leqslant$$
$$\kappa_2 Ch^4(k+1)\tau \leqslant \kappa_2 Ch^4 T \leqslant Ch^4.$$

对于 I''', 使用下面的逼近式：

$$I''' = \sum_{l=0}^{k}\frac{\tau}{2}\left(f(x_j,t_{l+1}) + f(x_j,t_l)\right) + O(\tau^3).$$

对于 $l = 0,1,2,\cdots,N$, 我们定义

$$c_l = (l+1)^\gamma - \frac{1}{1+\gamma}[(l+1)^{1+\gamma} - l^{1+\gamma}],\ d_l = \frac{1}{1+\gamma}[(l+1)^{1+\gamma} - l^{1+\gamma}] - l^\gamma,$$

从上述的分析可以得到

$$u_j^{k+1} = u_j^0 + \mu_1\sum_{l=0}^{k}c_l\frac{\delta_x^2}{(1+\delta_x^2/12)}u_j^{k-l} + d_l\frac{\delta_x^2}{(1+\delta_x^2/12)}u_j^{k-l+1} +$$

$$\frac{\mu_2}{2}\sum_{l=0}^{k}\frac{\delta_x^2}{(1+\delta_x^2/12)}u_j^l+\frac{\delta_x^2}{(1+\delta_x^2/12)}u_j^{l+1}+\sum_{l=0}^{k}\frac{\tau}{2}(f_j^{l+1}+f_j^l). \tag{31}$$

上式两端同乘以 $(1+\delta_x^2/12)$，有

$$\left(1+\frac{1}{12}\delta_x^2\right)u_j^{k+1}=\left(1+\frac{1}{12}\delta_x^2\right)u_j^0+\mu_1\sum_{l=0}^{k}c_l\delta_x^2u_j^{k-l}+d_l\delta_x^2u_j^{k-l+1}+$$
$$\frac{\mu_2}{2}\sum_{l=0}^{k}\delta_x^2u_j^l+\delta_x^2u_j^{l+1}+\sum_{l=0}^{k}\frac{\tau}{2}\left(1+\frac{1}{12}\delta_x^2\right)(f_j^{l+1}+f_j^l). \tag{32}$$

由此我们得到改进的隐式差分格式(IIFDS)：

$$\left[1+\left(\frac{1}{12}-\mu_1d_0-\frac{\mu_2}{2}\right)\delta_x^2\right]u_j^{k+1}=$$
$$\left[1+\left(\frac{1}{12}+\mu_1c_k+\frac{\mu_2}{2}\right)\delta_x^2\right]u_j^0+\sum_{i=1}^{k}(\mu_1(c_{k-i}+d_{k-i+1})+\mu_2)\delta_x^2u_j^i+$$
$$\sum_{i=0}^{k}\frac{\tau}{2}\left(1+\frac{1}{12}\delta_x^2\right)(f_j^{i+1}+f_j^i).$$

该格式的截断误差为 $O(\tau^2+h^4)$.

5 数值算例

例 1 考虑下述问题[16]：

$$\begin{cases}\dfrac{\partial u}{\partial t}={}_0D_t^{1-\gamma}\left[\dfrac{\partial^2u}{\partial x^2}\right]+\dfrac{\partial^2u}{\partial x^2}+\mathrm{e}^x\left[(2+\gamma)t^{1+\gamma}-\dfrac{\Gamma(3+\gamma)}{\Gamma(2+2\gamma)}t^{1+2\gamma}-t^{2+\gamma}\right], \\ \qquad\qquad 0\leqslant x\leqslant 1;\ 0<t\leqslant 1, \\ u(1,t)=et^{2+\gamma},\ u(0,t)=t^{2+\gamma}, \qquad 0\leqslant t\leqslant 1, \\ u(x,0)=0, \qquad 0\leqslant x\leqslant 1.\end{cases} \tag{33}$$

该问题的真解为 $u(x,t)=\mathrm{e}^x\,t^{2+\gamma}$. 定义数值解与精确解的绝对误差定义为

$$E_\infty=\max_{1\leqslant j\leqslant M-1}\{|u_j^N-u(x_j,1)|\}.$$

表 1 列出了当 $\gamma=0.5$ 时，问题(33)的精确解与由格式 INAS[16]，IFDS 以及 IIFDS 所得到了数值解的绝对误差. 很显然，本文所提出的格式相对于 INAS 来说具有更高的精度. 进一步，还能从表 1 中证实 IFDS 具有误差阶 $O(\tau+h^4)$ 以及 IIFDS 的误差阶为 $O(\tau^2+h^4)$.

表 1 $\gamma=0.5$ 时的绝对误差

Table 1 The maximum error with $\gamma=0.5$

$\tau=h^2$	INAS[16]	IFDS	IIFDS	$\tau=h^4$	IFDS	IIFDS
1/16	1.102 7E-002	7.734 6E-003	8.700 0E-005	1/81	1.567 9E-003	5.737 5E-006
1/64	2.953 0E-003	2.095 5E-003	5.670 2E-006	1/256	5.415 0E-004	2.706 3E-006
1/256	7.621 2E-004	5.480 0E-004	3.635 3E-007	1/1 296	1.092 5E-004	5.945 0E-007
1/1 024	2.074 4E-004	1.397 3E-004	2.298 6E-008	1/4 096	3.482 3E-005	1.913 8E-007

例 2 考虑如下问题：

$$\begin{cases}\dfrac{\partial u}{\partial t}={}_0D_t^{1-\gamma}\left[\dfrac{\partial^2u}{\partial x^2}\right]+\dfrac{\partial^2u}{\partial x^2}+\left(2t+\dfrac{2\pi^2t^{1+\gamma}}{\Gamma(2+\gamma)}+\pi^2t^2\right)\cos(\pi x), \\ \qquad\qquad 0\leqslant x\leqslant 1;\ 0<t\leqslant 1, \\ u(1,t)=-t^2,\ u(0,t)=t^2, \qquad 0\leqslant t\leqslant 1, \\ u(x,0)=0, \qquad 0\leqslant x\leqslant 1.\end{cases} \tag{34}$$

该问题的真解为 $u(x,t)=\cos(\pi x)t^2$.

表 2 描述了当 $\gamma=0.6$ 时,问题(34)的精确解与由 IFDS 和 IIFDS 所得的数值解的绝对误差.表 3 列出了当 $\tau=1/4$ 并取不同的 γ 时问题(34)的精确解与由 IIFDS 所得的数值解的绝对误差.此数值例子的结果表明所作的理论分析与数值结果较为吻合.

表 2 IFDS 与 IIFDS 的绝对误差($\gamma=0.6$)

Table 2 The maximum error of the IFDS and IIFDS ($\gamma=0.6$)

	IFDS	IIFDS
$h=\tau=1/4$	2.232 2E-02	3.287 0E-04
$h^2=\tau=1/64$	1.659 1E-03	2.016 9E-05
$h=1/16,\ \tau=1/1\,024$	1.066 7E-04	1.254 8E-06
$h=\tau=1/8$	1.220 4E-02	2.016 9E-05
$h=1/16,\ \tau=1/256$	4.268 6E-04	1.254 8E-06
$h=1/32,\ \tau=1/1\,024$	1.096 6E-04	7.966 9E-08

表 3 IIFDS 时的绝对误差

Table 3 The maximum error of the IIFDS

γ	$h=1/10,\ \tau=1/4$	$h=1/100,\ \tau=1.4$	$h=1/1\,000,\ \tau=1/4$	$h=1/10\,000,\ \tau=1/4$
0.1	8.357 7E-06	8.382 0E-10	9.770 0E-14	5.400 1E-13
0.2	8.350 8E-06	8.374 9E-10	1.043 6E-13	1.632 6E-12
0.3	8.344 4E-06	8.368 3E-10	1.596 5E-13	3.543 1E-12
0.4	8.338 4E-06	8.362 2E-10	1.592 1E-13	1.331 7E-12
0.5	8.332 7E-06	8.356 4E-10	1.256 8E-13	3.348 0E-12
0.6	8.327 3E-06	8.350 8E-10	1.354 5E-13	1.355 6E-12
0.7	8.322 0E-06	8.345 5E-10	1.106 9E-13	1.181 8E-12
0.8	8.316 9E-06	8.340 3E-10	1.599 8E-13	1.220 2E-12
0.9	8.312 0E-06	8.335 2E-10	1.042 5E-13	9.469 1E-13

6 结　　论

本文给出了加热下分数阶广义二阶流体的 Stokes 第一问题的数值格式 IFDS 以及 IIFDS . 对所提出的两种格式进行了误差阶分析.对 IFDS 的稳定性与收敛性进行了严格论证.最后通过数值例子验证了所提出的格式的有效可靠性.

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High-Order Numerical Methods of the Fractional Order Stokes' First Problem for a Heated Generalized Second Grade Fluid

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Abstract: High-order implicit finite difference methods for solving the Stokes' first problem for a heated generalized second grade fluid with fractional derivative were studied. The stability, solvability and convergence of the numerical scheme were discussed via fourier analysis and matrix analysis method. An improved implicit scheme was also obtained. Finally, two numerical examples were presented to demonstrate the effectiveness of the mentioned schemes.

Key words: fractional order Stokes' first problem; implicit difference scheme; solvability stability; convergence