文章编号:1000-0887(2011)04-0455-15

多孔压电线性理论中的唯一性定理、互易定理和特征值问题*

A·K·瓦西世, V·古泊塔

(古鲁格舍德拉大学 数学系,古鲁格舍德拉 136119,印度)

摘要: 假定弹性场和电场为正定,在多孔压电线性理论中建立起唯一性定理和互易定理。在准静态电场近似下,证明多孔压电材料线性理论中的一般性定理。利用弹性场的正定性,唯一性定理得到证明。在与多孔压电体自由振动相联系的特征值问题的研究中,给出了简明的公式。文中还研究了有关算子的某些特性。以简明公式为基础,利用变分法和算子法,研究了由于小扰动产生的频移问题。还给出了特殊情况下的扰动分析。

关 键 词: 特征值问题; 压电性; 多孔性; 唯一性定理; 互易定理

中图分类号: O343.7 **文献标志码:** A

DOI:10.3879/j.issn.1000-0887.2011.04.009

引 言

单一材料中的压电材料,由于耦合了电学和力学特征的特色,多年来成为重要的研究对象。压电材料广泛地应用于机电系统,例如压电传感器、微观结构"飞虫"(micromechanics flying insects)、光束偏转器、谐振器和滤波器。这些智能结构的构件还有着其他的重要作用,振动和噪声的抑制、主变形的控制及健康的监测。经典论文[1-3]按耦合场问题作为压电影响描述的基础。Mindlin[4]在板的厚度坐标项中,应用扩展的机械位移和电势,导出了压电晶体平板的二维运动方程。Paul 和 Natarajan[5]对结晶极 6 mm 有限长压电实心圆柱体,边界无牵引力和电势时,研究其挠性振动。文献[6]用带复参数的 Bessel 函数,得到了频率方程的精确解。Ding 等[7]将压电圆板的自由振动,看作横观各向同性压电体三维耦合方程的一般解。Sabu[8]在几何中面的侧面固支假设下,研究了压电材料壳特征值问题的极限状态。

由于微电力学系统、小型电源和其他设备,如压电马达的现代化发展,激起了对线性压电材料及其应用理论基础新的兴趣。文献[9]对电位移场,引入适当的内部约束和假定,推断三维压电体为一维压电细直杆模型。Lioubimova 和 Schiavone[10-11]对线性压电介质中的一般平面应变状态,进行了稳定状态振动的研究,应用积分方程法,证明了相应边值问题解的存在性及其唯一性定理。

* 收稿日期: 2010-09-20; 修订日期: 2011-01-19
基金项目: 印度大学教育资助委员会资助项目(No. f. No. 8(42)/2010(MRP/NRCB))
作者简介: A. K. Vashishth(联系人. E-mail:anil_vashishth@yahoo.co.in).
本文原文为英文,吴承平 译,张禄坤 校.

频率的控制和选择、精确的定时和同步,使得压电谐振器在商业和军用电子中有着广泛的应用. 最初是因为 Mindlin[4] 对谐振器的应用,激起了压电板自由振动问题和特征值问题分析的兴趣. Yang 和 Batra[12] 系统地分析了与有限压电体自由振动相联系的特征值问题,用变分法和摄动法,研究了与压电体有联系的频移问题. Yang[13] 在电弹性体中,研究自由偏转场上叠加自由小幅振动时的特征值问题. 最近,Guo[14] 以静力学理论为基础,得到了加筋压电体弹性和介电常数表达式,文中有一个线弹性体、线压电弹性体基本理论文章的详细清单. Iesan[15] 使用 Laplace 变换,对准静力学和动力学压电体,也提出了互易定理和唯一性定理. Yang 和 Batra[16] 利用 Nother 定理,连同 4 变量准静力学压电体公式一起,得到了一类线压电材料守恒定律. 有些作者[17-19] 尝试在压电微极热电线性理论中提出一般定理. Ciarletta 和 Scalia[20] 利用 Green 和 Laws 提出的熵产出不等式,发展了带空洞材料的热压电体的线性理论,还导出了带空洞压电材料线性理论的互易性关系和唯一性定理.

多孔压电材料(PPM)应用广泛,如低频水诊器、水下传感和激励的应用[21-23] 等. 由于这些材料具有高静水力学的灵敏值和低声速,所以声学阻抗低,水耦合性强. 一些文献[24-27]报道了多孔压电材料表征、合成和性质的实验研究. 有些作者[28-30] 就多孔性对多孔压电材料弹性、压电性和介电性性能的影响,进行理论模型的研究开发. 最近,Veshishth 和 Gupta[31] 利用 Biot 理论和电焓密度函数,导出了 PPM 的本构方程,还在高频情况下得到了厚剪切修正因子. Veshishth 和 Gupta[32] 在横观各向同性压电多孔材料中,开展了波传播的解析研究. 并以钛酸钡($BaTiO_3$)这一特殊模型,就频率、多孔性、传播方向和压电性,对介质中的相速度和波传播时的衰减质量因子进行了数值研究. 多孔材料理论似乎是描述压电体性质的理想工具,因为大多数此类材料是颗粒结构的.

文献搜索显示,许多已做的和正在做的工作,都与压电材料的振动和特征值问题有关,但是同样的问题,对多孔压电材料的研究还显不够. 本文的目的是,为多孔压电材料线弹性理论建立基本定理. 这里,采用简明的公式,并介绍相应的场向量及算子. 第 1 节假设弹性场正定时,提出三维多孔压电体的唯一性定理. 第 2 节用 Laplace 变换,并结合运动方程中初始条件,证明了多孔压电体线性理论中的互易关系. 在弹性场正定的假设下,交替证明了唯一性定理. 鉴于谐振器频率稳定性分析中特征值问题的重要性,在第 3 节中对与多孔压电材料相联系的特征值问题,给出了系统的分析. 并对所引入算子的一些性质给出了证明.

1　解的唯一性

考虑一个多孔的压电体,体积为 V,边界为 S. 各向异性多孔压电体的本构方程为[31]:

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} + m_{ij}\varepsilon^{*} - e_{kij}E_k - \zeta_{kij}E_k^{*}, \tag{1a}$$

$$\sigma^{*} = m_{ij}\varepsilon_{ij} + R\varepsilon^{*} - \tilde{\zeta}_i E_i - e_i^{*}E_i^{*}, \tag{1b}$$

$$D_i = e_{ijk}\varepsilon_{jk} + \tilde{\zeta}_i\varepsilon^{*} + \xi_{ij}E_j + A_{ij}E_j^{*}, \tag{1c}$$

$$D_i^{*} = \zeta_{ijk}\varepsilon_{jk} + e_i^{*}\varepsilon^{*} + A_{ij}E_j + \xi_{ij}^{*}E_j^{*} \qquad (i,\ j = 1,2,3), \tag{1d}$$

其中,($\sigma_{ij},\varepsilon_{ij},E_i,D_i$) 和($\sigma^{*},\varepsilon^{*},E_i^{*},D_i^{*}$) 分别为多孔聚合物固相和液相的(应力、应变、电场强度和电位移) 分量;c_{ijkl} 为弹性刚度常数. 弹性常数 R 测量施加在流体上的压力,将流体的单位体积推向多孔的基体. (e_{kij},ξ_{ij}),(e_i^{*},ξ_{ij}^{*}) 分别为固相和液相的压电常数和介电常数;参数($m_{ij};\zeta_{kij},\tilde{\zeta}_i;A_{ij}$) 看作多孔聚合物两相之间弹性、压电性、介电性的耦合. 应变张量($\varepsilon_{ij},\varepsilon^{*}$) 和电场向量($E_i,E_i^{*}$) 分别与机械位移($u_i,u_i^{*}$) 和电势($\phi,\phi^{*}$) 有关,即

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\ \varepsilon^{*}=u_{i,i}^{*},\ E_{i}=-\phi_{,i},\ E_{i}^{*}=-\phi_{,i}^{*}. \tag{2}$$

不计体力和面电荷密度时的运动方程为

$$\sigma_{ij,i}=\rho_{11}\ddot{u}_{j}+\rho_{12}\ddot{u}_{j}^{*}, \tag{3a}$$

$$\sigma_{,i}^{*}=\rho_{12}\ddot{u}_{i}+\rho_{22}\ddot{u}_{i}^{*}, \tag{3b}$$

$$D_{i,i}=0, \tag{3c}$$

$$D_{i,i}^{*}=0. \tag{3d}$$

曲面 S 上的边界条件为

$$\sigma_{ij}n_{i}=t_{j}, \tag{4a}$$

$$\sigma^{*}n_{i}=t_{i}^{*}, \tag{4b}$$

$$D_{i}n_{i}=c, \tag{4c}$$

$$D_{i}^{*}n_{i}=c^{*}, \tag{4d}$$

其中，n_i 为表面 S 上单位外法向量 $\boldsymbol{n}$ 的分量. 这里，动力学系数（$\rho_{11},\rho_{12},\rho_{22}$）是依赖于孔隙度 f、多孔聚合物密度 ρ、孔隙流体密度 ρ_{f} 及其惯性的耦合参数. t_j,t_j^* 分别为多孔体积材料的固相和液相的面牵引力. c,c^* 分别为固相和液相的面电荷密度. 多孔压电体[31]的电焓密度函数 W 和动能密度 K 为

$$W=\frac{1}{2}[\sigma_{ij}\varepsilon_{ij}+\sigma^{*}\varepsilon^{*}-E_{i}D_{i}-E_{i}^{*}D_{i}^{*}], \tag{5a}$$

$$K=\frac{1}{2}[\rho_{11}\dot{u}_{i}\dot{u}_{i}+2\rho_{12}\dot{u}_{i}\dot{u}_{i}^{*}+\rho_{22}\dot{u}_{i}^{*}\dot{u}_{i}^{*}]. \tag{5b}$$

设 $\{u_i',u_i^{*\prime},\sigma_{ij}',\sigma^{*\prime},\phi',\phi^{*\prime},E_i',E_i^{*\prime}\}$ 和 $\{u_i'',u_i^{*\prime\prime},\sigma_{ij}'',\sigma^{*\prime\prime},\phi'',\phi^{*\prime\prime},E_i'',E_i^{*\prime\prime}\}$ 为问题(3)的两个解集，如果可能的话，满足边界条件(4).

定义

$$u_{i}=u_{i}'-u_{i}'',\ u_{i}^{*}=u_{i}^{*\prime}-u_{i}^{*\prime\prime},\ \sigma_{ij}=\sigma_{ij}'-\sigma_{ij}'',\ \sigma^{*}=\sigma^{*\prime}-\sigma^{*\prime\prime},$$
$$\phi=\phi'-\phi'',\ \phi^{*}=\phi^{*\prime}-\phi^{*\prime\prime},\ E_{i}=E_{i}'-E_{i}'',\ E_{i}^{*}=E_{i}^{*\prime}-E_{i}^{*\prime\prime}.$$

$\{u_i,u_i^*,\sigma_{ij},\sigma^*,\phi,\phi^*,E_i,E_i^*\}$ 也是问题(3)的解，满足边界条件

$$\sigma_{ij}n_{j}=0,\ \sigma^{*}n_{i}=0,\ D_{i}n_{i}=0,\ D_{i}^{*}n_{i}=0. \tag{6}$$

方程(3)意味着

$$\int_{V}[(\sigma_{ij,i}-\rho_{11}\ddot{u}_{j}-\rho_{12}\ddot{u}_{j}^{*})\dot{u}_{j}+(\sigma_{,j}^{*}-\rho_{12}\ddot{u}_{j}-\rho_{22}\ddot{u}_{j}^{*})\dot{u}_{j}^{*}-$$
$$\dot{D}_{j,j}\phi-\dot{D}_{j,j}^{*}\phi^{*}]\mathrm{d}V=0. \tag{7}$$

利用方程(5)和(6)，由方程(7)导得

$$\int_{V}(\sigma_{ij}\dot{u}_{j,i}+\sigma^{*}\dot{u}_{i,i}^{*}-\dot{D}_{i}\phi_{,i}-\dot{D}_{i}^{*}\phi_{,i}^{*})\mathrm{d}V+\int_{V}\frac{\mathrm{d}K}{\mathrm{d}t}\mathrm{d}V=0.$$

上述方程中利用电焓密度函数的定义，得到

$$\int_{V}\left[\frac{\partial W}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij}+\frac{\partial W}{\partial\varepsilon^{*}}\dot{\varepsilon}^{*}+\frac{\partial W}{\partial E_{i}}\dot{E}_{i}+\frac{\partial W}{\partial E_{i}^{*}}\dot{E}_{i}^{*}+\frac{\mathrm{d}}{\mathrm{d}t}(D_{i}E_{i}+D_{i}^{*}E_{i}^{*})\right]\mathrm{d}V+\int_{V}\frac{\mathrm{d}K}{\mathrm{d}t}\mathrm{d}V=0$$
$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}t}\int_{V}(U+K)\mathrm{d}V=0,$$

其中

$$U=\frac{1}{2}(\sigma_{ij}\varepsilon_{ij}+\sigma^*\varepsilon^*+D_iE_i+D_i^*E_i^*)$$

$$\Rightarrow U+K=\text{const}.$$

根据定义, U 和 K 为正定, 且初始值为0. 这就意味着, 若解是唯一的, 只可能有 $U=0$ 且 $K=0$.

2　一 般 定 理

考虑一物体,时间 $t=0$ 时,有体积为 V,边界面为 S. 设 $\bar{V}$ 表示 V 是闭合的. 多孔压电体的运动方程和本构方程与上节相同.

假设

(i) $c_{ijkl}, e_{kij}, m_{ij}, \zeta_{kij}, \xi_{ij}, A_{ij}$ 和 ξ_{ij}^* 在 $\bar{V}$ 上连续可微,且满足对称关系:

$$c_{ijkl}=c_{jikl},\ c_{ijkl}=c_{ijlk},\ c_{ijkl}=c_{klij}, e_{kij}=e_{kji},$$

$$\zeta_{kij}=\zeta_{kji},\ m_{ij}=m_{ji},\ A_{ij}=A_{ji},\ \xi_{ij}=\xi_{ji},\ \xi_{ij}^*=\xi_{ji}^*.$$

(ii) $\rho_{11}, \rho_{12}, \rho_{22}$ 在 $\bar{V}$ 上连续.

牵引力向量和电位移的法向分量为

$$t_i=\sigma_{ij}n_j,\ t_i^*=\sigma^*n_i,\ d=D_in_i,\ d^*=D_i^*n_i. \tag{8}$$

设 $S_u, S_{u^*}, S_T, S_{T^*}, S_\phi, S_{\phi^*}$, S_D 和 S_{D^*} 为 S 的一部分,即

$$\bar{S}_u\cup\bar{S}_{u^*}\cup S_T\cup S_{T^*}=S, \tag{9}$$

$$\bar{S}_\phi\cup\bar{S}_{\phi^*}\cup S_D\cup S_{D^*}=S, \tag{10}$$

其中, $S_u, S_{u^*}; S_T, S_{T^*}; S_\phi, S_{\phi^*}$ 和 S_D, S_{D^*} 两两互不相交.

从几何上来说,面 $S_u, S_T, S_\phi, S_D; S_{u^*}, S_{T^*}, S_{\phi^*}, S_{D^*}$ 都是面 S 的区域,分别对应于多孔聚合物的固相和液相,这些面上的机械位移、牵引力向量、电势和电位移是指定的.

考虑如下边界条件:

$$u_i=\tilde{u}_i,\quad \text{在 } \bar{S}_u\times I \text{ 上},\qquad \phi=\tilde{\phi},\quad \text{在 } \bar{S}_\phi\times I \text{ 上}, \tag{11a}$$

$$u_i^*=\tilde{u}_i^*,\quad \text{在 } \bar{S}_{u^*}\times I \text{ 上},\quad \text{或}\quad \phi^*=\tilde{\phi}^*,\quad \text{在 } \bar{S}_{\phi^*}\times I \text{ 上}, \tag{11b}$$

$$\sigma_{ij}n_j=\tilde{t}_i,\quad \text{在 } S_T\times I \text{ 上},\qquad D_in_i=\tilde{d},\quad \text{在 } S_D\times I \text{ 上}, \tag{11c}$$

$$\sigma^*n_i=\tilde{t}_i^*,\quad \text{在 } S_{T^*}\times I \text{ 上},\qquad D_i^*n_i=\tilde{d}^*,\quad \text{在 } S_{D^*}\times I \text{ 上}, \tag{11d}$$

其中, $I=[0,\infty)$ 和 $\tilde{u}, \tilde{u}^*, \tilde{\phi}, \tilde{\phi}^*, \tilde{t}, \tilde{t}^*, \tilde{d}$ 和 $\tilde{d}^*$ 为已知.

进一步假设

$$\begin{cases} u_i(x,0)=v_i(x),\ u_i^*(x,0)=V_i(x), \\ \dot{u}_i(x,0)=\hat{v}_i(x),\ \dot{u}_i^*(x,0)=\hat{V}_i(x), \end{cases}\quad x\in\bar{V}, \tag{12}$$

其中 $v_i, V_i, \hat{v}_i, \hat{V}_i$ 为给定函数.

考虑两组外部数据:

$$L^{(\alpha)}=\{\tilde{u}_i^{(\alpha)}, \tilde{u}_i^{*(\alpha)}, \tilde{t}_i^{(\alpha)}, \tilde{t}_i^{*(\alpha)}, \tilde{\phi}^{(\alpha)}, \tilde{\phi}^{*(\alpha)}, \tilde{d}^{(\alpha)}, \tilde{d}^{*(\alpha)}, v_i^{(\alpha)}, V_i^{(\alpha)}, \hat{v}_i^{(\alpha)}, \hat{V}_i^{(\alpha)}\},\quad \alpha=1,2.$$

设

$$A^{(\alpha)}=\{u^{(\alpha)},\ u^{*(\alpha)},\ E^{(\alpha)},\ E^{*(\alpha)},\ \sigma_{ij}^{(\alpha)},\ \sigma^{*(\alpha)},\ D^{(\alpha)},\ D^{*(\alpha)},\ \phi^{(\alpha)},\ \phi^{*(\alpha)}\}$$

为对应于 $L^{(\alpha)}$ 的解.

定义

$$t_i^{(\alpha)} = \sigma_{ij}^{(\alpha)} n_j,\ t_i^{*(\alpha)} = \sigma^{*(\alpha)} n_i,\ d^{(\alpha)} = D_i^{(\alpha)} n_i,\ d^{*(\alpha)} = D_i^{*(\alpha)} n_i. \tag{13}$$

定理 2.1 设 $c_{ijkl}, e_{kij}, m_{ij}, \zeta_{kij}, \xi_{ij}, A_{ij}$ 和 ξ_{ij}^* 满足对称关系. 令

$$\begin{aligned} E_{\alpha\beta}(r,s) = &\int_S [t_i^{(\alpha)}(x,r) u_i^{(\beta)}(x,s) + t_i^{*(\alpha)}(x,r) u_i^{*(\beta)}(x,s) + \\ &d^{(\alpha)}(x,r) \phi^{(\beta)}(x,s) + d^{*(\alpha)}(x,r) \phi^{*(\beta)}(x,s)] \mathrm{d}S - \\ &\int_V [\rho_{11} \ddot{u}_i^{(\alpha)}(x,r) u_i^{(\beta)}(x,s) + \rho_{22} \ddot{u}_i^{*(\alpha)}(x,r) u_i^{*(\beta)}(x,s) + \\ &\rho_{12}(\ddot{u}_i^{(\alpha)} u_i^{*(\beta)} + u_i^{(\beta)} \ddot{u}_i^{*(\alpha)})] \mathrm{d}V, \end{aligned} \tag{14}$$

其中, $r,s \in I$, 则 $E_{\alpha\beta}(r,s) = E_{\beta\alpha}(s,r)$.

证明 考虑

$$\begin{aligned} I_{\alpha\beta}(r,s) = &\sigma_{ij}^{(\alpha)}(r) u_{i,j}^{(\beta)}(s) + \sigma^{*(\alpha)}(r) u_{i,i}^{*(\alpha)}(s) - \\ &D_i^{(\alpha)}(r) E_i^{(\beta)}(s) - D_i^{*(\alpha)}(r) E_i^{*(\beta)}(s). \end{aligned} \tag{15}$$

利用方程(1)和对称关系,有

$$I_{\alpha\beta}(r,s) = I_{\beta\alpha}(s,r). \tag{16}$$

将方程(3)代入方程(15),得到

$$\begin{aligned} I_{\alpha\beta}(r,s) = &[\sigma_{ij}^{(\alpha)}(r) u_i^{(\beta)}(s) + \sigma^{*(\alpha)}(r) u_j^{*(\beta)}(s) + \\ &D_j^{(\alpha)}(r) \phi^{(\beta)}(s) + D_j^{*(\alpha)}(r) \phi^{*(\beta)}(s)]_{,j} - \\ &[\rho_{11} \ddot{u}_i^{(\alpha)}(r) u_i^{(\beta)}(s) + \rho_{12} \ddot{u}_i^{*(\alpha)}(r) u_i^{(\beta)}(s) + \\ &\rho_{12} u_i^{*(\beta)}(s) \ddot{u}_i^{(\alpha)}(r) + \rho_{22} \ddot{u}_i^{*(\alpha)}(r) u_i^{*(\beta)}(s)]. \end{aligned} \tag{17}$$

在体积 V 上,对方程(17)两边积分,得到

$$\int_V I_{\alpha\beta}(r,s) \mathrm{d}V = E_{\alpha\beta}(r,s). \tag{18}$$

方程(16)和(18)意味着

$$E_{\alpha\beta}(r,s) = E_{\beta\alpha}(s,r). \tag{19}$$

定理 2.2 设弹性、压电和介电系数满足对称关系. 并设 $A^{(\alpha)}$ 为对应于外部数据系统 $L^{(\alpha)}(\alpha = 1,2)$ 的解,则

$$\begin{aligned} &\int_S [\gamma * (t_i^{(1)} * u_i^{(2)} + t_i^{*(1)} * u_i^{*(2)} + d^{(1)} * \phi^{(2)} + d^{*(1)} * \phi^{*(2)})] \mathrm{d}S + \\ &\int_V [\rho_{11} f_i^{(1)} * u_i^{(2)} + \rho_{12}(f_i^{(1)} * u_i^{*(2)} + F_i^{(1)} * u_i^{(2)}) + \rho_{22} F_i^{(1)} * u_i^{*(2)}] \mathrm{d}V = \\ &\int_S [\gamma * (t_i^{(2)} * u_i^{(1)} + t_i^{*(2)} * u_i^{*(1)} + d^{(2)} * \phi^{(1)} + d^{*(2)} * \phi^{*(1)})] \mathrm{d}a + \\ &\int_V [\rho_{11} f_i^{(2)} * u_i^{(1)} + \rho_{12}(f_i^{(2)} * u_i^{*(1)} + F_i^{(2)} * u_i^{(1)}) + \rho_{22} F_i^{(2)} * u_i^{*(1)}] \mathrm{d}V, \end{aligned} \tag{20}$$

其中

$$f_i^{(\alpha)} = v_i^{(\alpha)} + t\hat{v}_i^{(\alpha)},\ F_i^{(\alpha)} = V_i^{(\alpha)} + t\hat{V}_i^{(\alpha)},\ \gamma(t) = t, \qquad t \in I$$

和

$$[u * v](x,t) = \int_0^t u(x,t-s) v(x,s) \mathrm{d}s. \tag{21}$$

证明 利用定理 2.1,有

$$E_{12}(\tau, t-\tau) = E_{21}(t-\tau, \tau). \tag{22}$$

由方程(14)、(21)、(22),得到

$$\int_S [t_i^{(1)} * u_i^{(2)} + t_i^{*(1)} * u_i^{*(2)} + d^{(1)} * \phi^{(2)} + d^{*(1)} * \phi^{*(2)}] \mathrm{d}S - \int_V [\rho_{11} \ddot{u}_i^{(1)} * u_i^{(2)} + \rho_{12} (\ddot{u}_i^{(1)} * u_i^{*(2)} + \ddot{u}_i^{*(1)} * u_i^{(2)}) + \rho_{22} \ddot{u}_i^{*(1)} * u_i^{*(2)}] \mathrm{d}V = \int_S [t_i^{(2)} * u_i^{(1)} + t_i^{*(2)} * u_i^{*(1)} + d^{(2)} * \phi^{(1)} + d^{*(2)} * \phi^{*(1)}] \mathrm{d}S - \int_V [\rho_{11} \ddot{u}_i^{(2)} * u_i^{(1)} + \rho_{12} (\ddot{u}_i^{(2)} * u_i^{*(1)} + \ddot{u}_i^{*(2)} * u_i^{(1)}) + \rho_{22} \ddot{u}_i^{*(2)} * u_i^{*(1)}] \mathrm{d}V. \quad (23)$$

方程(23)的两边上对 γ 求卷积,就得到所要的结果.

定理 2.3 设弹性、压电和介电系数满足对称关系.并设

$$G(r,s) = \int_S [t_i(x,r) u_i(x,s) + t_i^*(x,r) u_i^*(x,s) + d(x,r) \phi(x,s) + d^*(x,r) \phi^*(x,s)] \mathrm{d}S, \quad (24)$$

其中 $r,s \in I$, 则

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V [\rho_{11} (u^{(1)})^2 + \rho_{12} u^{(1)} u^{*(1)} + \rho_{22} (u^{*(1)})^2] \mathrm{d}V = \int_0^t [G(t+s, t-s) - G(t-s, t+s)] \mathrm{d}s + \int_V \rho_{11} [u_i^{(1)}(0) \dot{u}_i^{(1)}(2t) + u_i^{(1)}(2t) \dot{u}_i^{(1)}(0)] \mathrm{d}V + \int_V \rho_{12} [u_i^{*(1)}(0) \dot{u}_i^{(1)}(2t) + u_i^{*(1)}(2t) \dot{u}_i^{(1)}(0) + u_i^{(1)}(0) \dot{u}_i^{*(1)}(2t) + u_i^{(1)}(2t) \dot{u}_i^{*(1)}(0)] \mathrm{d}V + \int_V \rho_{22} [u_i^{*(1)}(0) \dot{u}_i^{*(1)}(2t) + u_i^{*(1)}(2t) \dot{u}_i^{*(1)}(0)] \mathrm{d}V. \quad (25)$$

证明 利用定理 2.1,有

$$\int_0^t E_{11}(t+s, t-s) \mathrm{d}s = \int_0^t E_{11}(t-s, t+s) \mathrm{d}s. \quad (26)$$

再由方程(14)、(24)、(26),得到

$$\int_0^t [G(t+s, t-s) - G(t-s, t+s)] \mathrm{d}s = \int_V \left[\rho_{11} \int_0^t (\ddot{u}_i^{(1)}(t+s) u_i^{(1)}(t-s) - \ddot{u}_i^{(1)}(t-s) u_i^{(1)}(t+s)) \mathrm{d}s\right] \mathrm{d}V + \int_V \left[\rho_{12} \int_0^t (\ddot{u}_i^{(1)}(t+s) u_i^{*(1)}(t-s) + u_i^{(1)}(t-s) \ddot{u}_i^{*(1)}(t+s) - \ddot{u}_i^{(1)}(t-s) u_i^{*(1)}(t+s) - u_i^{(1)}(t+s)) \ddot{u}_i^{*(1)}(t-s) \mathrm{d}s\right] \mathrm{d}V + \int_V \left[\rho_{22} \int_0^t (\ddot{u}_i^{*(1)}(t+s) u_i^{*(1)}(t-s) - \ddot{u}_i^{*(1)}(t-s) u_i^{*(1)}(t+s)) \mathrm{d}s\right] \mathrm{d}V. \quad (27)$$

简化后,即得到所需的结果.

定理 2.4(唯一性定理) 设

1) 弹性、压电和介电系数满足对称关系;

2) $\xi_{ij}, A_{ij}, \xi_{ij}^*$ 为正定.

再假定 ($u_i^{(\alpha)}$, $u_i^{*(\alpha)}$, $\phi^{(\alpha)}$, $\phi^{*(\alpha)}$; $\alpha = 1,2$) 为混合边值问题的任意两组解,设

$$u_i^{(0)} = u_i^{(1)} - u_i^{(2)},\ u_i^{*(0)} = u_i^{*(1)} - u_i^{*(2)},$$
$$\phi^{(0)} = \phi^{(1)} - \phi^{(2)},\ \phi^{*(0)} = \phi^{*(1)} - \phi^{*(2)}.$$

则

$$u_i^{(0)} = 0;\ u_i^{*(0)} = 0;\ \phi^{(0)} = 0,\ \phi^{*(0)} = 0, \qquad 在\ V \times I\ 上.$$

证明 因为 $\{u_i^{(1)}, u_i^{*(1)}, \phi^{(1)}, \phi^{*(1)}\}$ 和 $\{u_i^{(2)}, u_i^{*(2)}, \phi^{(2)}, \phi^{*(2)}\}$ 是方程(3)在边界条件(11)和(12)时的解,所以 $\{u_i^{(0)}, u_i^{*(0)}, \phi^{(0)}, \phi^{*(0)}\}$ 也是方程(3)满足如下边界条件的解:

$$u_i^{(0)} = 0,\quad 在\ \bar{S}_u \times I\ 上, \qquad \phi^{(0)} = 0,\quad 在\ \bar{S}_\phi \times I\ 上, \tag{28a}$$
$$u_i^{*(0)} = 0,\quad 在\ \bar{S}_{u^*} \times I\ 上, \quad 或 \quad \phi^{*(0)} = 0,\quad 在\ \bar{S}_{\phi^*} \times I\ 上, \tag{28b}$$
$$\sigma_{ij}^{(0)} n_j = 0,\quad 在\ S_T \times I\ 上, \qquad D_i^{(0)} n_i = 0,\quad 在\ S_D \times I\ 上, \tag{28c}$$
$$\sigma^{*(0)} n_i = 0,\quad 在\ S_{T^*} \times I\ 上, \qquad D_i^{*(0)} n_i = 0,\quad 在\ S_{D^*} \times I\ 上 \tag{28d}$$

和

$$u_i^{(0)}(x,0) = 0,\ u_i^{*(0)}(x,0) = 0,\ \dot{u}_i^{(0)}(x,0) = 0,\ \dot{u}_i^{*(0)}(x,0) = 0, \qquad x \in \bar{V}. \tag{29}$$

根据定理 2.3 以及上述边界条件,得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V [\rho_{11}(u_i^{(0)})^2 + \rho_{12} u_i^{(0)} u_i^{*(0)} + \rho_{22}(u_i^{*(0)})^2]\mathrm{d}V = 0$$
$$\Rightarrow \int_V [\rho_{11}(u_i^{(0)})^2 + \rho_{12} u_i^{(0)} u_i^{*(0)} + \rho_{22}(u_i^{*(0)})^2]\mathrm{d}V = \mathrm{const}. \tag{30}$$

因为 $u_i^{(0)}$ 和 $u_i^{*(0)}$ 初始为 0,则 const = 0,因此 u_i 和 u_i^* 是唯一的. 利用 $\xi_{ij}, A_{ij}, \xi_{ij}^*$ 的正定性,就可以得到所需要的结果.

3 三维多孔压电体的自由振动

考虑一个多孔压电体的自由振动,其边界条件为

$$\sigma_{ij} n_j = 0,\quad 在\ S_T\ 上, \qquad \sigma^* n_j = 0,\quad 在\ S_{T^*}\ 上, \tag{31a}$$
$$D_i n_i = 0,\quad 在\ S_D\ 上, \quad 或 \quad D_i^* n_i = 0,\quad 在\ S_{D^*}\ 上, \tag{31b}$$
$$u_i = 0,\quad 在\ S_u\ 上, \qquad u_i^* = 0,\quad 在\ S_{u^*}\ 上, \tag{31c}$$
$$\phi = 0,\quad 在\ S_\phi\ 上, \qquad \phi^* = 0,\quad 在\ S_{\phi^*}\ 上. \tag{31d}$$

定义如下:

$$\boldsymbol{U} = \{u_i, u_i^*, \phi, \phi^*\},\ \boldsymbol{V} = \{v_i, v_i^*, \psi, \psi^*\}, \tag{32a}$$
$$\begin{cases} A\boldsymbol{U} = \{-\sigma_{ij,j}, -\sigma_{,i}^*, -D_{i,i}, -D_{i,i}^*\}, \\ B\boldsymbol{U} = \{\rho_{11} u_i + \rho_{12} u_i^*, \rho_{12} u_i + \rho_{22} u_i^*, 0, 0\}, \end{cases} \tag{32b}$$

则由方程(3)和(32)导得特征值问题

$$A\boldsymbol{U} = \lambda B\boldsymbol{U}, \tag{33}$$

其中 $\lambda = \omega^2$, ω 为角频率.

下面,定义两个集合:

$F = \{\boldsymbol{U}/\boldsymbol{U}$ 满足边界条件(31)$\}$

及

$$F^* = \{\boldsymbol{U} \in F/\boldsymbol{U}\ 有实分量, D_{i,i} = 0, D_{i,i}^* = 0\}. \tag{34}$$

F 和 F^* 为 Hilbert 空间,其内积定义如下:

$$\langle \boldsymbol{U},\boldsymbol{V}\rangle = \int_V (u_i v_i + u_i^* v_i^* + \phi\psi + \phi^*\psi^*)\mathrm{d}V. \tag{35}$$

利用上面的公式,方程(1)和(3)定义的问题,可以被归结为已处理过的特征值问题,即:寻找 λ,存在一个非平凡解 $\boldsymbol{U}\in F$,使得 $A\boldsymbol{U}=\lambda B\boldsymbol{U}$.

定理 3.1 设 $c_{ijkl}, m_{ij}, R, \xi_{ij}, \xi_{ij}^*, A_{ij}$ 正定,则定义在函数空间 F 上的算子 A 和 B 是自伴随算子. 若算子 A 和 B 定义在空间 F^* 上,则它们是非负的.

证明 设 $\boldsymbol{U},\boldsymbol{V}\in F$,

$$\langle A\boldsymbol{U},\boldsymbol{V}\rangle = \int_V (-\sigma_{ij,j}v_i - \sigma_{,i}v_i^* - D_{i,i}\psi - D_{i,i}^*\psi^*)\mathrm{d}V. \tag{36}$$

则由方程(36)连同方程(1)和(31)一起,给出

$$\begin{aligned}\langle A\boldsymbol{U},\boldsymbol{V}\rangle = &\int_V [(c_{ijkl}u_{k,l} + m_{ij}u_{k,k}^* + e_{kij}\phi_{,k} + \zeta_{kij}\phi_{,k}^*)v_{i,j} + \\ &(m_{kl}u_{k,l} + Ru_{k,k}^* + \tilde{\zeta}_k\phi_{,k} + e_k^*\phi_{,k}^*)v_{i,i}^* + \\ &(e_{ikl}u_{k,l} + \tilde{\zeta}_i u_{k,k}^* - \xi_{il}\phi_{,l} - A_{il}\phi_{,l}^*)\psi_{,i} + \\ &(\zeta_{ikl}u_{k,l} + e_i^* u_{k,k}^* - A_{il}\phi_{,l} - \xi_{il}^*\phi_{,l}^*)\psi_{,i}^*]\mathrm{d}V = \\ &\int_V [\sigma_{kl}(V)u_{k,l} + \sigma^*(V)u_{k,k}^* + D_k(V)\phi_{,k} + D_k^*(V)\phi_{,k}^*]\mathrm{d}V = \\ &\int_V [-\sigma_{kl,l}(V)u_k - \sigma_{,k}^*(V)u_k^* - D_{k,k}(V)\phi - D_{k,k}^*(V)\phi^*]\mathrm{d}V = \langle \boldsymbol{U},A\boldsymbol{V}\rangle.\end{aligned}$$

这就证明了算子 A 是自伴随算子. 同时

$$\langle B\boldsymbol{U},\boldsymbol{V}\rangle = \int_V [(\rho_{11}u_i + \rho_{12}u_i^*)v_i + (\rho_{12}u_i + \rho_{22}u_i^*)v_i^*]\mathrm{d}V = \langle \boldsymbol{U},B\boldsymbol{V}\rangle.$$

这就意味着算子 B 是自伴随算子.

下面,证明算子 A,B 是非负的.

设 $\boldsymbol{U}\in F^*$,

$$\langle A\boldsymbol{U},\boldsymbol{U}\rangle = \int_V (-\sigma_{ij,j}u_i - \sigma_{,i}^*u_i^* - D_{i,i}\phi - D_{i,i}^*\phi^*)\mathrm{d}V. \tag{37}$$

在方程(37)中利用方程(1)和(31),得到

$$\begin{aligned}\langle A\boldsymbol{U},\boldsymbol{U}\rangle = &\int_V [c_{ijkl}u_{i,j}u_{k,l} + 2m_{ij}u_{i,j}u_{k,k}^* + 2e_{kij}u_{i,j}\phi_{,k} + 2\zeta_{kij}u_{i,j}\phi_{,k}^* + Ru_{i,i}^*u_{k,k}^* + \\ &2\tilde{\zeta}_k u_{i,i}^*\phi_{,k} + 2e_k^* u_{i,i}^*\phi_{,k}^* - \xi_{il}\phi_{,i}\phi_{,l} - \xi_{il}^*\phi_{,i}^*\phi_{,l}^* - 2A_{il}\phi_{,i}\phi_{,l}^*]\mathrm{d}V.\end{aligned}$$

所以

$$\begin{aligned}\langle A\boldsymbol{U},\boldsymbol{U}\rangle = &\int_V (c_{ijkl}u_{i,j}u_{k,l} + 2m_{ij}u_{i,j}u_{k,k}^* + Ru_{i,i}^*u_{k,k}^* + \\ &2A_{il}\phi_{,i}\phi_{,l}^* + \xi_{il}\phi_{,i}\phi_{,l} + \xi_{il}^*\phi_{,i}^*\phi_{,l}^*)\mathrm{d}V \Rightarrow \langle A\boldsymbol{U},\boldsymbol{U}\rangle \geqslant 0.\end{aligned}$$

又有

$$\langle B\boldsymbol{U},\boldsymbol{U}\rangle = \int_V (\rho_{11}u_i^2 + 2\rho_{12}u_iu_i^* + \rho_{22}u_i^{*2})\mathrm{d}V \geqslant 0.$$

这就意味着算子 A,B 是非负的.

定理 3.2 设算子 A,B 为实的,则联合边值问题(33)的特征值是实的. 特别地,与 $\boldsymbol{U}\in F^*$ 相应的特征值是正的.

证明 设 λ 为问题(33)的特征值,$\boldsymbol{U}$ 为相应的特征向量,则

$$A\boldsymbol{U} = \lambda B\boldsymbol{U}. \tag{38}$$

对方程(38)两边取复共轭,并利用真实算子,得到

$$A\bar{\boldsymbol{U}} = \bar{\lambda} B\bar{\boldsymbol{U}}. \tag{39}$$

方程(38)、(39)两边分别对 $\boldsymbol{U}$ 和 $\bar{U}$ 取内积,然后相减,得到

$$\langle A\boldsymbol{U},\bar{\boldsymbol{U}}\rangle - \langle A\bar{\boldsymbol{U}},\boldsymbol{U}\rangle = \lambda\langle B\boldsymbol{U},\bar{\boldsymbol{U}}\rangle - \bar{\lambda}\langle B\bar{\boldsymbol{U}},\boldsymbol{U}\rangle.$$

应用定理 3.1,得到

$$(\lambda - \bar{\lambda})\langle B\boldsymbol{U},\bar{\boldsymbol{U}}\rangle = 0. \tag{40}$$

因为

$$\langle B\boldsymbol{U},\bar{\boldsymbol{U}}\rangle > 0 \Rightarrow \lambda = \bar{\lambda}.$$

因此,特征值是实的.进而由于特征值问题的线性性,以及真实算子,说明存在实特征向量.

若特征向量为 $\boldsymbol{U} \in F^*$, 则

$$A\boldsymbol{U} = \lambda B\boldsymbol{U} \Rightarrow \langle A\boldsymbol{U},\boldsymbol{U}\rangle = \lambda\langle B\boldsymbol{U},\boldsymbol{U}\rangle.$$

因为算子 A,B 在 F^* 上是非负的,因此特征值 λ 是正的.

定理 3.3 对应于不同特征值的特征向量是正交的.

证明 设 $\boldsymbol{U}^{(1)},\boldsymbol{U}^{(2)}$ 是对应于不同特征值 λ_1,λ_2 的特征向量.有

$$A\boldsymbol{U}^{(1)} = \lambda_1 B\boldsymbol{U}^{(1)}, \tag{41}$$

$$A\boldsymbol{U}^{(2)} = \lambda_2 B\boldsymbol{U}^{(2)}. \tag{42}$$

方程(41)和(42)两边分别对 $\boldsymbol{U}^{(1)},\boldsymbol{U}^{(2)}$ 取内积,然后相减,得到

$$(\lambda_1 - \lambda_2)\langle B\boldsymbol{U}^{(1)},\boldsymbol{U}^{(2)}\rangle = 0$$

$$\Rightarrow \int_V [\rho_{11} u_i^{(1)} u_i^{(2)} + \rho_{12}(u_i^{(1)} u_i^{*(2)} + u_i^{(2)} u_i^{*(1)}) + \rho_{22} u_i^{*(1)} u_i^{*(2)}]\mathrm{d}V = 0.$$

根据 Tiersten 的文献[3],不同特征值相对应的特征向量正交性得证.

3.1 变分原理

在函数空间 F 上定义一个泛函(Reyleigh 商):

$$I(\boldsymbol{U}) = \frac{\langle A\boldsymbol{U},\boldsymbol{U}\rangle}{\langle B\boldsymbol{U},\boldsymbol{U}\rangle}. \tag{43}$$

则

$$\delta I = \frac{\langle B\boldsymbol{U},\boldsymbol{U}\rangle\delta\langle A\boldsymbol{U},\boldsymbol{U}\rangle - \langle A\boldsymbol{U},\boldsymbol{U}\rangle\delta\langle B\boldsymbol{U},\boldsymbol{U}\rangle}{\langle B\boldsymbol{U},\boldsymbol{U}\rangle^2}.$$

由方程(43)定义的泛函,稳定的必要条件是

$$\begin{aligned}
&\delta I = 0 \\
&\quad\Rightarrow \langle B\boldsymbol{U},\boldsymbol{U}\rangle\delta\langle A\boldsymbol{U},\boldsymbol{U}\rangle - \langle A\boldsymbol{U},\boldsymbol{U}\rangle\delta\langle B\boldsymbol{U},\boldsymbol{U}\rangle = 0 \\
&\quad\Rightarrow \langle B\boldsymbol{U},\boldsymbol{U}\rangle[\langle A\boldsymbol{U}, \delta\boldsymbol{U}\rangle + \langle A\,\delta\boldsymbol{U}, \boldsymbol{U}\rangle] - \\
&\quad\langle A\boldsymbol{U},\boldsymbol{U}\rangle[\langle B\boldsymbol{U}, \delta\boldsymbol{U}\rangle + \langle B\,\delta\boldsymbol{U}, \boldsymbol{U}\rangle] = 0 \\
&\quad\Rightarrow \langle B\boldsymbol{U},\boldsymbol{U}\rangle\langle A\boldsymbol{U},\delta\boldsymbol{U}\rangle + \langle B\boldsymbol{U}, \boldsymbol{U}\rangle\langle\delta\boldsymbol{U}, A\boldsymbol{U}\rangle - \\
&\quad\langle A\boldsymbol{U},\boldsymbol{U}\rangle\langle B\boldsymbol{U},\delta\boldsymbol{U}\rangle + \langle A\boldsymbol{U}, \boldsymbol{U}\rangle\langle\delta\boldsymbol{U}, B\boldsymbol{U}\rangle = 0 \\
&\quad\Rightarrow 2\langle B\boldsymbol{U},\boldsymbol{U}\rangle[\langle A\boldsymbol{U} - IB\boldsymbol{U}, \delta\boldsymbol{U}\rangle] = 0.
\end{aligned}$$

这就意味着

$$\langle A\boldsymbol{U} - IB\boldsymbol{U},\delta\boldsymbol{U}\rangle = 0.$$

因为 $\delta\boldsymbol{U}$ 是任意的,于是有

$$AU = IBU.$$

因此,若 U 满足运动方程,且相应的稳定值为 ω^2,则由方程(43)定义的泛函是稳定的. 函数空间 F 上定义的 Rayleigh 商可以写为

$$I(U) = \int_V [c_{ijkl}u_{i,j}u_{k,l} + 2m_{ij}u_{i,j}u_{k,k}^* + 2e_{kij}u_{i,j}\phi_{,k} + 2\zeta_{kij}u_{i,j}\phi_{,k}^* + Ru_{i,i}^*u_{k,k}^* + 2\tilde{\zeta}_k u_{i,i}^*\phi_{,k} + 2e_k^* u_{i,i}^*\phi_{,k}^* - \xi_{il}\phi_{,i}\phi_{,l} - \xi_{il}^*\phi_{,i}^*\phi_{,l}^* - 2A_{il}\phi_{,i}\phi_{,l}^*]\mathrm{d}V \Big/ \left(\int_V(\rho_{11}u_i^2 + 2\rho_{12}u_iu_i^* + \rho_{22}u_i^{*2})\mathrm{d}V\right). \tag{44}$$

又,在函数空间上 F^* 上,方程(44)变为

$$I(U) = \int_V [c_{ijkl}u_{i,j}u_{k,l} + 2m_{ij}u_{i,j}u_{k,k}^* + Ru_{i,i}^*u_{k,k}^* + \xi_{il}\phi_{,i}\phi_{,l} + \xi_{il}^*\phi_{,i}^*\phi_{,l}^* + 2A_{il}\phi_{,i}\phi_{,l}^*]\mathrm{d}V \Big/ \left(\int_V(\rho_{11}u_i^2 + 2\rho_{12}u_iu_i^* + \rho_{22}u_i^{*2})\mathrm{d}V\right). \tag{45}$$

3.2　频移的摄动计算

系统的物理和几何性质微小变化,引起特征值问题的微小改变,这就是大家知道的频移. 由于小扰动产生的频移,是一个有着具大现实意义的问题. 这里要讨论的扰动问题,是指初始变形状态周围的扰动. 这一类摄动问题的频移,可以用变分法和算子法得到.

3.2.1　变分法

设由于摄动, λ, A, B 分别产生微小的变化 $\Delta\lambda$, ΔA, ΔB. 与这些变化相应的特征值问题成为

$$(A + \Delta A)(U + \Delta U) = (\lambda + \Delta\lambda)(B + \Delta B)(U + \Delta U)$$

$$\Rightarrow \lambda + \Delta\lambda = \frac{\langle (A + \Delta A)(U + \Delta U),\ U + \Delta U\rangle}{\langle (B + \Delta B)(U + \Delta U), U + \Delta U\rangle} \approx$$

$$\frac{\langle AU,\ U\rangle + \langle AU,\ \Delta U\rangle + \langle (\Delta A)U + A(\Delta U),\ U\rangle}{\langle BU,\ U\rangle + \langle BU,\ \Delta U\rangle + \langle (\Delta B)U + B(\Delta U),\ U\rangle} \approx$$

$$\frac{\langle AU, U\rangle}{\langle BU, U\rangle}\left(1 + \frac{2\langle AU,\ \Delta U\rangle + \langle (\Delta A)U,\ U\rangle}{\langle AU, U\rangle}\right) \times$$

$$\left(1 - \frac{2\langle BU,\ \Delta U\rangle + \langle (\Delta B)U,\ U\rangle}{\langle BU, U\rangle}\right) =$$

$$\lambda + \frac{\langle (\Delta A)U, U\rangle - \lambda\langle (\delta B)U,\ U\rangle}{\langle BU, U\rangle} \Rightarrow$$

$$\Delta\lambda = \frac{\langle (\Delta A)U, U\rangle - \lambda\langle (\delta B)U,\ U\rangle}{\langle BU, U\rangle}. \tag{46}$$

3.2.2　算子法

由方程(46)提取一阶项,得到

$$A(\Delta U) + (\Delta A)U = (\Delta\lambda)BU + \lambda(\Delta B)U + \lambda B(\Delta U). \tag{47}$$

在方程(47)的两边,对 U 取内积,得到

$$\langle A(\Delta U),\ U\rangle + \langle (\Delta A)U,\ U\rangle = \Delta\lambda\langle BU,\ U\rangle + \lambda\langle (\Delta B)U,\ U\rangle + \lambda\langle B(\Delta U),\ U\rangle.$$

利用算子 A, B 的自伴性以及 $AU = \lambda BU$, 有

$$\Delta\lambda = \frac{\langle (\Delta A)U, U\rangle - \lambda\langle (\Delta B)U, U\rangle}{\langle BU, U\rangle}.$$

3.3 特殊情况

对于有限的偏离场,前面诸小节的分析依然有效.实际应用中,最经常碰到的是小偏离场情况.本小节中,这种特殊情况下的摄动分析,表现为自然状态周围的摄动.与该摄动问题相应的特征值问题为

$$A^{(p)}\boldsymbol{U}^{(p)}=\lambda^{(p)}B\boldsymbol{U}^{(p)},\qquad \text{在 } V \text{ 中}. \tag{48}$$

边界条件为

$$\begin{cases}\boldsymbol{\sigma}_{ij}^{(p)}(\boldsymbol{U}^{(p)})n_j=0, & \text{在 } S_T \text{ 上},\\ \boldsymbol{\sigma}^{*(p)}(\boldsymbol{U}^{(p)})n_j=0, & \text{在 } S_{T^*} \text{ 上},\end{cases} \tag{49a}$$

$$\begin{cases}D_i^{(p)}(\boldsymbol{U}^{(p)})n_i=0, & \text{在 } S_D \text{ 上},\\ D_i^{*(p)}(\boldsymbol{U}^{(p)})n_i=0, & \text{在 } S_{D^*} \text{ 上},\end{cases} \tag{49b}$$

$$\begin{cases}u_i^{(p)}=0, & \text{在 } S_u \text{ 上},\\ u_i^{*(p)}=0, & \text{在 } S_{u^*} \text{ 上},\end{cases} \tag{49c}$$

$$\begin{cases}\phi^{(p)}=0, & \text{在 } S_\phi \text{ 上},\\ \phi^{*(p)}=0, & \text{在 } S_{\phi^*} \text{ 上},\end{cases} \tag{49d}$$

其中

$$\begin{cases}A^{(p)}=A^{(0)}+\epsilon A^{(1)},\ \boldsymbol{\sigma}_{ij}^{(p)}=\boldsymbol{\sigma}_{ij}^{(0)}+\epsilon\boldsymbol{\sigma}_{ij}^{(1)},\ \boldsymbol{\sigma}^{*(p)}=\boldsymbol{\sigma}^{*(0)}+\epsilon\boldsymbol{\sigma}^{*(1)},\\ D_i^{(p)}=D_i^{(0)}+\epsilon D_i^{(1)},\ D_i^{*(p)}=D_i^{*(0)}+\epsilon D_i^{*(1)},\\ \lambda^{(p)}=\lambda^{(0)}+\epsilon\lambda^{(1)},\ \boldsymbol{U}^{(p)}=\boldsymbol{U}^{(0)}+\epsilon\boldsymbol{U}^{(1)}.\end{cases} \tag{50}$$

根据方程(48)和(50),得到一阶特征值问题:

$$A^{(0)}\boldsymbol{U}^{(1)}=\lambda^{(0)}B\boldsymbol{U}^{(1)}+\lambda^{(1)}B\boldsymbol{U}^{(0)}-A^{(1)}\boldsymbol{U}^{(0)},\qquad \text{在 } V \text{ 上}. \tag{51}$$

边界条件为

$$\begin{cases}(\boldsymbol{\sigma}_{ij}^{(0)}(\boldsymbol{U}^{(1)})+\boldsymbol{\sigma}_{ij}^{(1)}(\boldsymbol{U}^{(0)}))n_j=0, & \text{在 } S_T \text{ 上},\\ (\boldsymbol{\sigma}^{*(0)}(\boldsymbol{U}^{(1)})+\boldsymbol{\sigma}^{*(1)}(\boldsymbol{U}^{(0)}))n_j=0, & \text{在 } S_{T^*} \text{ 上},\end{cases} \tag{52a}$$

$$\begin{cases}(D_i^{(0)}(\boldsymbol{U}^{(1)})+D_i^{(1)}(\boldsymbol{U}^{(0)}))n_i=0, & \text{在 } S_D \text{ 上},\\ (D_i^{*(0)}(\boldsymbol{U}^{(1)})+D_i^{*(1)}(\boldsymbol{U}^{(0)}))n_i=0, & \text{在 } S_{D^*} \text{ 上},\end{cases} \tag{52b}$$

$$\begin{cases}u_i=0, & \text{在 } S_u \text{ 上},\\ u_i^*=0, & \text{在 } S_{u^*} \text{ 上},\end{cases} \tag{52c}$$

$$\begin{cases}\phi_i=0, & \text{在 } S_\phi \text{ 上},\\ \phi_i^*=0, & \text{在 } S_{\phi^*} \text{ 上},\end{cases} \tag{52d}$$

其中,$\lambda^{(0)}$ 和 $\boldsymbol{U}^{(0)}$ 为线性压电体无偏离场振动时的特征值解.

在方程(51)的两边,对 $\boldsymbol{U}^{(0)}$ 取内积,得到

$$\langle A^{(0)}\boldsymbol{U}^{(1)},\boldsymbol{U}^{(0)}\rangle=$$
$$\lambda^{(0)}\langle B\boldsymbol{U}^{(1)},\boldsymbol{U}^{(0)}\rangle+\lambda^{(1)}\langle B\boldsymbol{U}^{(0)},\boldsymbol{U}^{(1)}\rangle-\langle A^{(1)}\boldsymbol{U}^{(0)},\boldsymbol{U}^{(0)}\rangle. \tag{53}$$

根据内积的定义和 Gauss 散度定理,得到

$$\langle A^{(0)}\boldsymbol{U}^{(0)},\boldsymbol{U}^{(1)}\rangle=$$
$$-\int_S[\boldsymbol{\sigma}_{ij}^{(0)}(\boldsymbol{U}^{(0)})n_ju_i^{(1)}+\boldsymbol{\sigma}^{*(0)}(\boldsymbol{U}^{(0)})n_ju_j^{*(1)}+$$

$$D_i^{(0)}(\boldsymbol{U}^{(0)})n_i\phi^{(1)} + D_i^{*(0)}(\boldsymbol{U}^{(0)})n_i\phi^{*(1)}]\mathrm{d}S +$$
$$\int_S[\boldsymbol{\sigma}_{ij}^{(0)}(\boldsymbol{U}^{(1)})n_ju_i^{(0)} + \boldsymbol{\sigma}^{*(0)}(\boldsymbol{U}^{(1)})n_ju_j^{*(0)} +$$
$$D_i^{(0)}(\boldsymbol{U}^{(1)})n_i\phi^{(0)} + D_i^{*(0)}(\boldsymbol{U}^{(1)})n_i\phi^{*(0)}]\mathrm{d}S +$$
$$\langle \boldsymbol{U}^{(0)}, A^{(0)}\boldsymbol{U}^{(1)}\rangle. \tag{54}$$

利用边界条件,方程(54)简化为

$$\langle A^{(0)}\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(1)}\rangle = -\int_{S_T}\boldsymbol{\sigma}_{ij}^{(1)}(\boldsymbol{U}^{(0)})n_ju_i^{(0)}\mathrm{d}S -$$
$$\int_{S_{T^*}}\boldsymbol{\sigma}^{*(1)}(\boldsymbol{U}^{(0)})n_ju_i^{*(0)}\mathrm{d}S - \int_{S_D}D_i^{(1)}(\boldsymbol{U}^{(0)})n_j\phi^{(0)}\mathrm{d}S -$$
$$\int_{S_{D^*}}D_i^{*(1)}(\boldsymbol{U}^{(0)})n_j\phi^{*(0)}\mathrm{d}S + \langle \boldsymbol{U}^{(0)}, A^{(0)}\boldsymbol{U}^{(1)}\rangle. \tag{55}$$

应用方程(53)和(55),有

$$-\int_{S_T}\boldsymbol{\sigma}_{ij}^{(1)}(\boldsymbol{U}^{(0)})n_ju_i^{(0)}\mathrm{d}S - \int_{S_{T^*}}\boldsymbol{\sigma}^{*(1)}(\boldsymbol{U}^{(0)})n_ju_i^{*(0)}\mathrm{d}S -$$
$$\int_{S_D}D_i^{(1)}(\boldsymbol{U}^{(0)})n_j\phi^{(0)}\mathrm{d}S - \int_{S_{D^*}}D_i^{*(1)}(\boldsymbol{U}^{(0)})n_j\phi^{*(0)}\mathrm{d}S + \langle \boldsymbol{U}^{(0)}, A^{(0)}\boldsymbol{U}^{(1)}\rangle =$$
$$\lambda^{(0)}\langle B\,\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(0)}\rangle + \lambda^{(1)}\langle B\,\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(1)}\rangle - \langle A^{(1)}\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(0)}\rangle \Rightarrow$$
$$\langle A^{(0)}\boldsymbol{U}^{(0)} - \lambda^{(0)}B\,\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(1)}\rangle - \int_{S_T}\boldsymbol{\sigma}_{ij}^{(1)}(\boldsymbol{U}^{(0)})n_ju_i^{(0)}\mathrm{d}S -$$
$$\int_{S_{T^*}}\boldsymbol{\sigma}^{*(1)}(\boldsymbol{U}^{(0)})n_ju_i^{*(0)}\mathrm{d}S - \int_{S_D}D_i^{(1)}(\boldsymbol{U}^{(0)})n_j\phi^{(0)}\mathrm{d}S -$$
$$\int_{S_{D^*}}D_i^{*(1)}(\boldsymbol{U}^{(0)})n_j\phi^{*(0)}\mathrm{d}S =$$
$$\lambda^{(1)}\langle B\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(0)}\rangle - \langle A^{(1)}\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(0)}\rangle.$$

因为

$$A^{(0)}\boldsymbol{U}^{(0)} = \lambda^{(0)}B\boldsymbol{U}^{(0)},$$

所以

$$\lambda^{(1)} = \Big[\langle A^{(1)}\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(0)}\rangle - \int_{S_T}\boldsymbol{\sigma}_{ij}^{(1)}(\boldsymbol{U}^{(0)})n_ju_i^{(0)}\mathrm{d}S -$$
$$\int_{S_{T^*}}\boldsymbol{\sigma}^{*(1)}(\boldsymbol{U}^{(0)})n_ju_i^{*(0)}\mathrm{d}S - \int_{S_D}D_i^{(1)}(\boldsymbol{U}^{(0)})n_j\phi^{(0)}\mathrm{d}S -$$
$$\int_{S_{D^*}}D_i^{*(1)}(\boldsymbol{U}^{(0)})n_j\phi^{*(0)}\mathrm{d}S\Big]\Big/\langle B\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(0)}\rangle.$$

现在

$$\lambda^{(\mathrm{p})} = (\omega^{(\mathrm{p})})^{(2)} \approx (\omega^{(\mathrm{p})})^{(0)} + 2\,\epsilon\,\omega^{(0)}\omega^{(1)},$$

又有

$$\lambda^{(\mathrm{p})} = \lambda^{(0)} + \epsilon\,\lambda^{(1)},$$

所以

$$\frac{\epsilon\omega^{(1)}}{\omega^{(0)}} \approx \frac{\epsilon\lambda^{(1)}}{2(\omega^{(0)})^{(2)}} =$$
$$\frac{\epsilon}{2(\omega^{(0)})^{(2)}\langle B\,\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(0)}\rangle}\Big[\langle A^{(1)}\boldsymbol{U}^{(0)}, \boldsymbol{U}^{(0)}\rangle -$$

$$\int_{S_T}\sigma_{ij}^{(1)}(\boldsymbol{U}^{(0)})n_j u_i^{(0)}\mathrm{d}S-\int_{S_{T^*}}\sigma^{*(1)}(\boldsymbol{U}^{(0)})n_j u_i^{*(0)}\mathrm{d}S-$$
$$\int_{S_D}D_i^{(1)}(\boldsymbol{U}^{(0)})n_j\phi^{(0)}\mathrm{d}S-\int_{S_{D^*}}D_i^{*(1)}(\boldsymbol{U}^{(0)})n_j\phi^{*(0)}\mathrm{d}S\Big].$$

采取与上节相同的步骤,有

$$\frac{\omega^{(\mathrm{p})}-\omega^{(0)}}{\omega^{(0)}}\approx\frac{\epsilon}{2(\omega^{(0)})^{(2)}}\Big[\int_V[c_{ijkl}^{(1)}u_{i,j}^{(0)}u_{k,l}^{(0)}+$$
$$2m_{ij}^{(1)}u_{i,j}^{(0)}u_{k,k}^{*(0)}+2e_{kij}^{(1)}u_{i,j}^{(0)}\phi_{,k}^{(0)}+2\zeta_{kij}^{(1)}u_{i,j}^{(0)}\phi_{,k}^{*(0)}+R^{(1)}u_{i,i}^{*(0)}u_{k,k}^{*(0)}+$$
$$2\tilde{\zeta}_k^{(1)}u_{i,i}^{*(0)}\phi_{,k}^{(0)}+2e_k^{*(1)}u_{i,i}^{*(0)}\phi_{,k}^{*(0)}-\xi_{il}^{(1)}\phi_{,i}^{(0)}\phi_{,l}^{(0)}-\xi_{il}^{*(1)}\phi_{,i}^{*(0)}\phi_{,l}^{*(0)}-$$
$$2A_{il}^{(1)}\phi_{,i}^{(0)}\phi_{,l}^{*(0)}]\mathrm{d}V\Big]\Big/\Big[\int_V(\rho_{11}(u_i^{(0)})^2+2\rho_{12}u_i^{(0)}u_i^{*(0)}+\rho_{22}(u_i^{*(0)})^2)\mathrm{d}V\Big].$$

上述方程给出了自然状态周围摄动时的一阶频移,可以用来计算体声波和面声波的频移.

4 结　　论

本文证明了多孔压电线性理论的一般定理,以简明的形式研究边值问题.在适当的 Hilbert 空间上,引入自伴随和非负算子,特征值为实数.在简明公式的基础上,给出了特征值问题的变分公式和摄动分析,还给出了特殊情况下的摄动分析.

参考文献:

[1] Cady W G. *Piezoelectricity*[M]. New York: McGraw Hill Book Company, 1946:1-405.

[2] Mason W P. *Piezoelectric Crystals and Their Application to Ultrasonics*[M]. Princeton, New York: D Van Nostrand Company, Inc, 1950: 1-502.

[3] Tiersten H F. *Linear Piezoelectric Plate Vibrations*[M]. New York:Plenum Press, 1969:1-212.

[4] Mindlin R D. High frequency vibrations of piezoelectric crystal plates[J]. *International Journal of Solids and Structures*,1972,**8**(7):895-906.

[5] Paul H S, Natarajan K. Flexural vibration in a finite piezoelectric circular cylinder of crystal class 6 mm[J]. *International Journal of Engineering Science*,1994,**32**(8):1303-1314.

[6] DING Hao-jiang, GUO Yi-mu, YANG Qing-da, CHEN Wei-qiu. Free vibration of piezoelectric cylindrical shells[J]. *Acta Mechanica Solida Sinica*,1997,**10**(1):48-55.

[7] DING Hao-jiang, XU Rong-qiao, CHEN Wei-qui. Exact solutions for free vibration of transversely isotropic piezoelectric circular plates[J]. *Acta Mechanica Sinica*(*English Series*), 2000, **16**(2): 141-147.

[8] Sabu N. Vibrations of thin piezoelectric flexural shells: two dimensional approximation[J]. *Journal of Elasticity*, 2002, **68**(1/3): 145-165.

[9] Davi F. Dynamics of linear piezoelectric rods[J]. *Journal of Elasticity*, 1997,**46**(3): 181-198.

[10] Lioubimova E, Schiavone P. Steady state vibrations of an unbounded linear piezoelectric medium[J]. *Z Angew Math Phys*, 2006, **57**(5): 862-874.

[11] Lioubimova E, Schiavone P. Steady state vibrations for the state of generalized plane strain in a linear piezoelectric medium[J]. *International Journal of Engineering Science*, 2006, **44**(8/9): 471-483.

[12] Yang J S, Batra R C. Free vibrations of a piezoelectric body[J]. *Journal of Elasticity*, 1994,

34(3): 239-254.

[13] Yang J. Free vibrations of an electrostatic body under biasing fields[J]. *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control*, 2005, **52**: 358-364.

[14] Guo S. An eigen expression for piezoelectrically stiffened elastic and dielectric constants based on the static theory[J]. *Acta Mechanica*, 2010, **210**(3/4): 345-350.

[15] Ieşsan D. Reciprocity, uniqueness and minimum principles in the linear theory of piezoelectricity[J]. *International Journal of Engineering Science*, 1990, **28**(11): 1139-1149.

[16] Yang J S, Batra R C. Conservation laws in linear piezoelectricity[J]. *Engineering Fracture Mechanics*, 1995, **51**(6): 1041-1047.

[17] Cr ǎciun I A. Uniqueness theorem in the linear theory of piezoelectric micropolar thermoelasticity[J]. *International Journal of Engineering Science*, 1995, **33**(7): 1027-1036.

[18] Iesan D, Quintanilla R. Some theorems in the theory of microstretch thermopiezoelectricity [J]. *International Journal of Engineering Science*, 2007, **45**(1): 1-16.

[19] Karamany A S E. Uniqueness theorem and Hamilton's principle in linear micropolar thermopiezoelctric/piezomagnetic continuum with two relaxation times[J]. *Meccanica*, 2009, **44**(1): 47-59.

[20] Ciarletta M, Scalia A. Thermodynamic theory for porous piezoelectric materials[J]. *Meccanica*, 1993, **28**(4): 303-308.

[21] Hashimoto K Y, Yamaguchi M. Elastic, piezoelectric and dielectric properties of composite materials[J]. *Proceeding IEEE Ultrasonics Symposium*, 1986, **2**: 697-702. 10. 1109/ULTSYM. 1986. 198824

[22] Arai T, Ayusawa K, Sato H, Miyata T, Kawamura K, Kobayashi K. Properties of hydrophone with porous piezoelectric ceramics[J]. *Japanese Journal of Applied Physics*, 1991, **30**: 2253-2255.

[23] Mizumura K, Kurihara Y, Ohashi H, Kumamoto S, Okuno K. Porous piezoelectric ceramic transducer[J]. *Japanese Journal of Applied Physics*, 1991, **30**: 2271-2273.

[24] Hayashi T, Sugihara K, Okazaki K. Processing of porous 3-3 PZT ceramics using capsule-free O2 H P[J]. *Japanese Journal of Applied Physics*, 1991, **30**: 2243-2246.

[25] Xia Z, Ma S, Qui X, Wu Y, Wang F. Influence of porosity on the stability of charge and piezoelectricity for porous polytetrafluoroethylene film electrets[J]. *Journal of Electrostatics*, 2003, **59**(1): 57-69.

[26] Piazza D, Stoleriu L, Mitoseriu L, Stancu A, Galassi C. Characterization of porous PZT ceramics by first order reversal curves (FORC) diagrams[J]. *Journal of the European Ceramic Society*, 2006, **26**(14): 2959- 2962.

[27] Wang Q, Chen Q, Zhu J, Huang C, Darvell B W, Chen Z. Effects of pore shape and porosity on the properties of porous LNKN ceramics as bone substitute[J]. *Materials Chemistry and Physics*, 2008, **109**(2/3): 488-491.

[28] Banno H. Effects of porosity on dielectric, elastic and electromechanical properties of Pb(Zr, Ti)O_3 ceramics with open pores: a theoretical approach[J]. *Japanese Journal of Applied Physics*, 1993, **32**(9S): 4214-4217.

[29] Lacour O, Lagier M, Sornette D. Effect of dynamic fluid compressibility and permeability on porous piezoelectric ceramics[J]. *The Journal of the Acoustic Society of America*, 1994, **96**(6): 3548-3557.

[30] Gomez T E, Montero F. Highly coupled dielectric behavior of porous ceramics embedding a

polymer[J]. *Applied Physics Letters*, 1996, **68**(2): 263-265.

[31] Vashishth A K, Gupta V. Vibration of porous piezoelectric plates[J]. *Journal of Sound and Vibration*, 2009, **325**: 781-797.

[32] Vashishth A K, Gupta V. Wave propagation in transversely isotropic porous piezoelectric materials[J]. *International Journal of Solids and Structures*, 2009, **46**(20): 3620-3632.

Uniqueness Theorem, Theorem of Reciprocity and Eigen Value Problems in the Linear Theory of Porous Piezoelectricity

Anil K. Vashishth, Vishakha Gupta

(*Department of Mathematics*, *Kurukshetra University*, *Kurukshetra* 136119, *India*)

Abstract: The uniqueness theorem and theorem of reciprocity in the linearized theory of porous piezoelectricity were established with the assumption of positive definiteness of elastic and electric field. General theorems in the linear theory of porous piezoelectric materials were proved for the quasi-static electric field approximation. The uniqueness theorem was also proved without using positive definiteness of elastic field. An eigen value problem, associated with free vibrations of porous piezoelectric body, was studied employing abstract formulation. Some properties of involved operators were also studied. The problem of frequency shift due to small disturbances, based on an abstract formulation, was studied using variational and operator approach. A perturbation analysis of a special case is also given.

Key words: eigen value problem; piezoelectric; porous; uniqueness theorem; reciprocal theorem