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# 电动力学电磁场边值问题的广义变分原理*

郑成博, 刘 彬, 王作君, 郑世科

(燕山大学 测试计量技术及仪器河北省重点实验室,河北 秦皇岛 066004)

(陈立群推荐)

**摘要:** 给出了线性各项异性电磁场边值问题的广义虚功原理表达式,运用钱伟长教授提出的方法建立了该问题的广义变分原理,可直接反映该问题的全部特征,即 4 个 Maxwell 方程、2 个场强-位势方程、2 个本构方程和 8 个边界条件. 继而导出了一族有先决条件的广义变分原理. 作为例证,导出了两个退化形式的广义变分原理,和已知的广义变分原理等价. 此外还导出了两个修正的广义变分原理,可为该问题提供杂交有限元模型. 建立的各广义变分原理可为电磁场边值问题的有限元应用提供更为完善的理论基础.

**关 键 词:** 广义变分原理; 电磁场; 电动力学; 边值问题; 有限元
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## 引 言

广义变分原理是建立杂交/混合有限元和 Ritz 法等变分近似解法的理论基础,已在连续介质力学领域取得了重大成功[1-2],因此一些学者也设法将广义变分原理的研究推广到电动力学和电磁弹性力学等领域. 钱伟长教授为广义变分原理在电磁场理论中的推广奠定了重要基础[2-3]. 文献[4]运用钱伟长教授提出的方法[1-3],分别建立了电磁场及压电问题的广义变分原理. 文献[5]建立了压电热弹性问题的广义变分原理. 文献[6]建立了磁热弹性问题的广义变分原理. 文献[7-8]建立了压磁电问题的广义变分原理. 文献[9]建立了电磁弹性静力学的 H-R 混合型广义变分原理. 文献[10]建立了耦合静态电磁场的弹性动力学的非传统 Hamilton 型广义变分原理. 然而,建立能够完全反映全部 4 个 Maxwell 方程的广义变分原理仍是一个难题.

本文给出了线性、各项异性材料中的电磁场边值问题的广义虚功原理和虚功原理的表达式,运用钱伟长教授提出的方法,建立了电动力学电磁场边值问题的广义变分原理,可直接反映该问题的全部特征, 即 4 个 Maxwell 方程、2 个场强-位势方程、2 个本构方程及 8 个边界条件. 继而导出了一组有先决条件的广义变分原理,给出了两个退化情况下的例证,并导出了可提供杂交有限元模型的修正的广义变分原理.

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**作者简介:** 郑成博(1979—),男,黑龙江齐齐哈尔人,讲师,博士生(E-mail:czheng@ysu.edu.cn);
王作君(1978—),女,山西太原人,博士生 (联系人. E-mail:scoutqhd@foxmail.com).

## 1 基本方程和边界条件

在线性、各项异性材料的三维正则区域 $V$ 中，广义的电磁场边值问题的基本方程和边界条件[11-12]包括：

Maxwell 方程

$$\partial_i D_i = \rho^{\mathrm{e}}, \tag{1}$$

$$e_{ijk}\partial_j D_k = -\varepsilon_{ij}(\partial_t B_j + J_j^{\mathrm{m}}), \tag{2}$$

$$\partial_i B_i = \rho^{\mathrm{m}}, \tag{3}$$

$$e_{ijk}\partial_j B_k = \mu_{ij}(\partial_t D_j + J_j^{\mathrm{e}}). \tag{4}$$

场强与位势的关系

$$E_i = -\partial_i \varphi^{\mathrm{e}} - \mu_{ij}\partial_t F_j^{\mathrm{e}} - e_{ijk}\partial_j F_k^{\mathrm{m}}, \tag{5}$$

$$H_i = -\partial_i \varphi^{\mathrm{m}} - \varepsilon_{ij}\partial_t F_j^{\mathrm{m}} + e_{ijk}\partial_j F_k^{\mathrm{e}}. \tag{6}$$

各项异性本构关系

$$D_i = \varepsilon_{ij} E_j, \tag{7}$$

$$B_i = \mu_{ij} H_j. \tag{8}$$

边界条件

$$n_i D_i = \bar{d}^{\perp}, \quad 在\ S_D^{\perp}\ 上, \tag{9}$$

$$\varphi^{\mathrm{e}} = \bar{\varphi}^{\mathrm{e}}, \quad 在\ S_{\varphi^{\mathrm{e}}}\ 上, \tag{10}$$

$$e_{ijk} n_j D_k = \bar{d}_i^{=}, \quad 在\ S_D^{=}\ 上, \tag{11}$$

$$F_i^{\mathrm{m}} = \bar{F}_i^{\mathrm{m}}, \quad 在\ S_{F^{\mathrm{m}}}\ 上, \tag{12}$$

$$n_i B_i = \bar{b}^{\perp}, \quad 在\ S_B^{\perp}\ 上, \tag{13}$$

$$\varphi^{\mathrm{m}} = \bar{\varphi}^{\mathrm{m}}, \quad 在\ S_{\varphi^{\mathrm{m}}}\ 上, \tag{14}$$

$$e_{ijk} n_j B_k = \bar{b}_i^{=}, \quad 在\ S_B^{=}\ 上, \tag{15}$$

$$F_i^{\mathrm{e}} = \bar{F}_i^{\mathrm{e}}, \quad 在\ S_{F^{\mathrm{e}}}\ 上, \tag{16}$$

式中，$D_i$ 为电位移，$B_i$ 为磁感应强度，$E_i$ 为电场强度，$H_i$ 为磁场强度，$\varphi^{\mathrm{e}}$ 为电标量势，$\varphi^{\mathrm{m}}$ 为磁标量势，$F_i^{\mathrm{e}}$ 为电矢量势，$F_i^{\mathrm{m}}$ 为磁矢量势，$\rho^{\mathrm{e}}$ 为电荷密度，$\rho^{\mathrm{m}}$ 为磁荷密度，$J_i^{\mathrm{e}}$ 为电流密度，$J_i^{\mathrm{m}}$ 为磁流密度，$\varepsilon_{ij}$ 为介电张量，$\mu_{ij}$ 为磁导率张量，$e_{ijk}$ 为置换符张量，$n_i$ 为边界上的单位外法向量，$\bar{d}^{\perp}$，$\bar{\varphi}^{\mathrm{e}}$，$\bar{d}_i^{=}$，$\bar{F}_i^{\mathrm{m}}$，$\bar{b}^{\perp}$，$\bar{\varphi}^{\mathrm{m}}$，$\bar{b}_i^{=}$ 和 $\bar{F}_i^{\mathrm{e}}$ 为各边界给定量，各边界的关系为

$$S_D^{\perp} \cup S_{\varphi^{\mathrm{e}}} = S_D^{=} \cup S_{F^{\mathrm{m}}} = S_B^{\perp} \cup S_{\varphi^{\mathrm{m}}} = S_B^{=} \cup S_{F^{\mathrm{e}}} = S \equiv \partial V,$$

$$S_D^{\perp} \cap S_{\varphi^{\mathrm{e}}} = S_D^{=} \cap S_{F^{\mathrm{m}}} = S_B^{\perp} \cap S_{\varphi^{\mathrm{m}}} = S_B^{=} \cap S_{F^{\mathrm{e}}} = \varnothing.$$

## 2 广义虚功原理和虚功原理

不难验证，对于互不相关的任意函数 $D_i$，$B_i$，$\varphi^{\mathrm{e}}$，$\varphi^{\mathrm{m}}$，$F_i^{\mathrm{e}}$ 和 $F_i^{\mathrm{m}}$，下式恒成立：

$$\int_{t_0}^{t_1}\mathrm{d}t\iiint_V\{-D_i(\partial_i\varphi^{\mathrm{e}} + \mu_{ij}\partial_t F_j^{\mathrm{e}} + e_{ijk}\partial_j F_k^{\mathrm{m}}) + B_i(\partial_i\varphi^{\mathrm{m}} + \varepsilon_{ij}\partial_t F_j^{\mathrm{m}} - e_{ijk}\partial_j F_k^{\mathrm{e}})\}\,\mathrm{d}V +$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iiint_V\{-\partial_i D_i\varphi^{\mathrm{e}} + (e_{ijk}\partial_j B_k - \mu_{ij}\partial_t D_j)F_i^{\mathrm{e}} +$$

$$\partial_i B_i \varphi^{m} + (\varepsilon_{ij}\partial_t B_j + e_{ijk}\partial_j D_k)F_i^{m}\}\mathrm{d}V +$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_S (D_i n_i \varphi^{e} - B_i n_i \varphi^{m} + D_i e_{ijk} n_j F_k^{m} + B_i e_{ijk} n_j F_k^{e})\mathrm{d}S = 0. \tag{17}$$

式(17)即为问题(1)~(16)的广义虚功原理表达式. 当 $D_i, B_i, \varphi^{e}, \varphi^{m}, F_i^{e}$ 和 $F_i^{m}$ 满足方程(1)~(6)时,式(17)变为

$$\int_{t_0}^{t_1}\mathrm{d}t\iiint_V (-\rho^{e}\varphi^{e} + \mu_{ij}J_j^{e}F_i^{e} + \rho^{m}\varphi^{m} - \varepsilon_{ij}J_j^{m}F_i^{m})\mathrm{d}V +$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_S (D_i n_i \varphi^{e} - B_i n_i \varphi^{m} + D_i e_{ijk} n_j F_k^{m} + B_i e_{ijk} n_j F_k^{e})\mathrm{d}S =$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iiint_V (-D_i E_i + B_i H_i)\mathrm{d}V. \tag{18}$$

式(18)即为问题(1)~(16)的虚功原理表达式.

# 3 广义变分原理

运用钱伟长教授提出的方法[1-3]构造广义 Lagrange 密度

$$L^{g} = \frac{\varepsilon_{ij}E_iE_j}{2} - \rho^{e}\varphi^{e} + \mu_{ij}J_j^{e}F_i^{e} - \lambda_i^{a}(E_i + \partial_i\varphi^{e} + \mu_{ij}\partial_t F_j^{e} + e_{ijk}\partial_j F_k^{m}) -$$

$$\frac{\mu_{ij}H_iH_j}{2} + \rho^{m}\varphi^{m} - \varepsilon_{ij}J_j^{m}F_i^{m} + \lambda_i^{b}(H_i + \partial_i\varphi^{m} + \varepsilon_{ij}\partial_t F_j^{m} - e_{ijk}\partial_j F_k^{e}). \tag{19}$$

式(19)中的 Lagrange 乘子可被识别为: $\lambda_i^{a} = D_i, \lambda_i^{b} = B_i$. 对 $L^{g}$ 积分并加上边界积分,可得问题(1)~(16)的广义作用量泛函

$$\pi^{g} = \int_{t_0}^{t_1}\mathrm{d}t\iiint_V \Big\{\frac{\varepsilon_{ij}E_iE_j}{2} - \rho^{e}\varphi^{e} + \mu_{ij}J_j^{e}F_i^{e} - D_i(E_i + \partial_i\varphi^{e} + \mu_{ij}\partial_t F_j^{e} + e_{ijk}\partial_j F_k^{m}) -$$

$$\frac{\mu_{ij}H_iH_j}{2} + \rho^{m}\varphi^{m} - \varepsilon_{ij}J_j^{m}F_i^{m} + B_i(H_i + \partial_i\varphi^{m} + \varepsilon_{ij}\partial_t F_j^{m} - e_{ijk}\partial_j F_k^{e})\Big\}\mathrm{d}V +$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{\perp}} \bar{d}^{\perp}\varphi^{e}\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{e}}} (\varphi^{e} - \bar{\varphi}^{e})n_iD_i\mathrm{d}S - \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{=}} \bar{d}_i^{=}F_i^{m}\mathrm{d}S +$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{m}}} e_{ijk}n_j(F_k^{m} - \bar{F}_k^{m})D_i\mathrm{d}S - \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^{\perp}} \bar{b}^{\perp}\varphi^{m}\mathrm{d}S -$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{m}}} (\varphi^{m} - \bar{\varphi}^{m})n_iB_i\mathrm{d}S - \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^{=}} \bar{b}_i^{=}F_i^{e}\mathrm{d}S +$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{e}}} e_{ijk}n_j(F_k^{e} - \bar{F}_k^{e})B_i\mathrm{d}S. \tag{20}$$

根据广义变分原理的对偶互补性质[13-14]和 $\pi^{g}$ 的表达式,可以构造出对偶的广义作用量泛函

$$\gamma^{g} = \int_{t_0}^{t_1}\mathrm{d}t\iiint_V \Big\{D_iE_i - \frac{\varepsilon_{ij}E_iE_j}{2} + (\rho^{e} - \partial_iD_i)\varphi^{e} - [\mu_{ij}(J_j^{e} + \partial_tD_j) - e_{ijk}\partial_jB_k]F_i^{e} -$$

$$B_iH_i + \frac{\mu_{ij}H_iH_j}{2} - (\rho^{m} - \partial_iB_i)\varphi^{m} + [\varepsilon_{ij}(J_j^{m} + \partial_tB_j) + e_{ijk}\partial_jD_k]F_i^{m}\Big\}\mathrm{d}V +$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{\perp}} (n_iD_i - \bar{d}^{\perp})\varphi^{e}\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{e}}} \bar{\varphi}^{e}n_iD_i\mathrm{d}S -$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{=}} (e_{ijk}n_jD_k - \bar{d}_i^{=})F_i^{m}\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{m}}} e_{ijk}n_j\bar{F}_k^{m}D_i\mathrm{d}S -$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^\perp}(n_iB_i-\bar{b}^\perp)\varphi^{\mathrm{m}}\mathrm{d}S-\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{m}}}}\bar{\varphi}^{\mathrm{m}}n_iB_i\mathrm{d}S-$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^=}(e_{ijk}n_jB_k-\bar{b}_i^=)F_i^{\mathrm{e}}\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{e}}}}e_{ijk}n_j\bar{F}_k^{\mathrm{e}}B_i\mathrm{d}S. \tag{21}$$

对于互不相关的变量 $D_i,B_i,E_i,H_i,\varphi^{\mathrm{e}},\varphi^{\mathrm{m}},F_i^{\mathrm{e}}$ 和 $F_i^{\mathrm{m}}$，均有 $\boldsymbol{\pi}^{\mathrm{g}}+\boldsymbol{\gamma}^{\mathrm{g}}=0$ 成立.

**定理1** 变分式 $\delta\boldsymbol{\pi}^{\mathrm{g}}=0$ 或 $\delta\boldsymbol{\gamma}^{\mathrm{g}}=0$ 成立，当且仅当 $D_i,B_i,E_i,H_i,\varphi^{\mathrm{e}},\varphi^{\mathrm{m}},F_i^{\mathrm{e}}$ 和 $F_i^{\mathrm{m}}$ 是问题(1)~(16)的解.

**证明** 对式(20)进行变分运算可得

$$\delta\boldsymbol{\pi}^{\mathrm{g}}=\int_{t_0}^{t_1}\mathrm{d}t\iiint_V\{\varepsilon_{ij}E_j\delta E_i-\rho^{\mathrm{e}}\delta\varphi^{\mathrm{e}}+\mu_{ij}J_j^{\mathrm{e}}\delta F_i^{\mathrm{e}}-(E_i+\partial_i\varphi^{\mathrm{e}}+$$
$$\mu_{ij}\partial_tF_j^{\mathrm{e}}+e_{ijk}\partial_jF_k^{\mathrm{m}})\delta D_i-D_i(\delta E_i+\partial_i\delta\varphi^{\mathrm{e}}+e_{ijk}\partial_j\delta F_k^{\mathrm{m}})-\mu_{ij}D_j\partial_t\delta F_i^{\mathrm{e}}-$$
$$\mu_{ij}H_j\delta H_i+\rho^{\mathrm{m}}\delta\varphi^{\mathrm{m}}-\varepsilon_{ij}J_j^{\mathrm{m}}\delta F_i^{\mathrm{m}}+(H_i+\partial_i\varphi^{\mathrm{m}}+\varepsilon_{ij}\partial_tF_j^{\mathrm{m}}-e_{ijk}\partial_jF_k^{\mathrm{e}})\delta B_i+$$
$$B_i(\delta H_i+\partial_i\delta\varphi^{\mathrm{m}}-e_{ijk}\partial_j\delta F_k^{\mathrm{e}})+\varepsilon_{ij}B_j\partial_t\delta F_i^{\mathrm{m}}\}\mathrm{d}V+$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^\perp}\bar{d}^\perp\delta\varphi^{\mathrm{e}}\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{e}}}}[(\varphi^{\mathrm{e}}-\bar{\varphi}^{\mathrm{e}})n_i\delta D_i+n_iD_i\delta\varphi^{\mathrm{e}}]\mathrm{d}S-$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^=}\bar{d}_i^=\delta F_i^{\mathrm{m}}\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{m}}}}[e_{ijk}n_j(F_k^{\mathrm{m}}-\bar{F}_k^{\mathrm{m}})\delta D_i+D_ie_{ijk}n_j\delta F_k^{\mathrm{m}}]\mathrm{d}S-$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^\perp}\bar{b}^\perp\delta\varphi^{\mathrm{m}}\mathrm{d}S-\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{m}}}}[(\varphi^{\mathrm{m}}-\bar{\varphi}^{\mathrm{m}})n_i\delta B_i+n_iB_i\delta\varphi^{\mathrm{m}}]\mathrm{d}S-$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^=}\bar{b}_i^=\delta F_i^{\mathrm{e}}\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{e}}}}[e_{ijk}n_j(F_k^{\mathrm{e}}-\bar{F}_k^{\mathrm{e}})\delta B_i+B_ie_{ijk}n_j\delta F_k^{\mathrm{e}}]\mathrm{d}S. \tag{22}$$

由 Gauss 定理和分部积分有

$$\iiint_V D_i\partial_i\delta\varphi^{\mathrm{e}}\mathrm{d}V=\iint_{S_D^\perp\cup S_{\varphi^{\mathrm{e}}}}n_iD_i\delta\varphi^{\mathrm{e}}\mathrm{d}S-\iiint_V\partial_iD_i\delta\varphi^{\mathrm{e}}\mathrm{d}V, \tag{23}$$

$$\iiint_V D_ie_{ijk}\partial_j\delta F_k^{\mathrm{m}}\mathrm{d}V=-\iint_{S_D^=\cup S_{F^{\mathrm{m}}}}e_{ijk}n_jD_k\delta F_i^{\mathrm{m}}\mathrm{d}S+\iiint_V e_{ijk}\partial_jD_k\delta F_i^{\mathrm{m}}\mathrm{d}V, \tag{24}$$

$$\iiint_V B_i\partial_i\delta\varphi^{\mathrm{m}}\mathrm{d}V=\iint_{S_B^\perp\cup S_{\varphi^{\mathrm{m}}}}n_iB_i\delta\varphi^{\mathrm{m}}\mathrm{d}S-\iiint_V\partial_iB_i\delta\varphi^{\mathrm{m}}\mathrm{d}V, \tag{25}$$

$$\iiint_V B_ie_{ijk}\partial_j\delta F_k^{\mathrm{e}}\mathrm{d}V=-\iint_{S_B^=\cup S_{F^{\mathrm{e}}}}e_{ijk}n_jB_k\delta F_i^{\mathrm{e}}\mathrm{d}S+\iiint_V e_{ijk}\partial_jB_k\delta F_i^{\mathrm{e}}\mathrm{d}V, \tag{26}$$

$$\mu_{ij}D_j\partial_t\delta F_i^{\mathrm{e}}=-\mu_{ij}\partial_tD_j\delta F_i^{\mathrm{e}}, \tag{27}$$

$$\varepsilon_{ij}B_j\partial_t\delta F_i^{\mathrm{m}}=-\varepsilon_{ij}\partial_tB_j\delta F_i^{\mathrm{m}}. \tag{28}$$

将式(23)~(28)代入式(22)，化简得

$$\delta\boldsymbol{\pi}^{\mathrm{g}}=\int_{t_0}^{t_1}\mathrm{d}t\iiint_V\{(\varepsilon_{ij}E_j-D_i)\delta E_i+(\partial_iD_i-\rho^{\mathrm{e}})\delta\varphi^{\mathrm{e}}+$$
$$[\mu_{ij}(J_j^{\mathrm{e}}+\partial_tD_j)-e_{ijk}\partial_jB_k]\delta F_i^{\mathrm{e}}-(E_i+\partial_i\varphi^{\mathrm{e}}+\mu_{ij}\partial_tF_j^{\mathrm{e}}+e_{ijk}\partial_jF_k^{\mathrm{m}})\delta D_i+$$
$$(B_i-\mu_{ij}H_j)\delta H_i+(\rho^{\mathrm{m}}-\partial_iB_i)\delta\varphi^{\mathrm{m}}-[\varepsilon_{ij}(J_j^{\mathrm{m}}+\partial_tB_j)+e_{ijk}\partial_jD_k]\delta F_i^{\mathrm{m}}+$$
$$(H_i+\partial_i\varphi^{\mathrm{m}}+\varepsilon_{ij}\partial_tF_j^{\mathrm{m}}-e_{ijk}\partial_jF_k^{\mathrm{e}})\delta B_i\}\mathrm{d}V+$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{m}}}}e_{ijk}n_j(F_k^{\mathrm{m}}-\bar{F}_k^{\mathrm{m}})\delta D_i\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^=}(e_{ijk}n_jD_k-\bar{d}_i^=)\delta F_i^{\mathrm{m}}\mathrm{d}S+$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{e}}}}(\varphi^{\mathrm{e}}-\bar{\varphi}^{\mathrm{e}})n_i\delta D_i\mathrm{d}S-\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^\perp}(n_iD_i-\bar{d}^\perp)\delta\varphi^{\mathrm{e}}\mathrm{d}S+$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^\perp}(n_iB_i-\bar{b}^\perp)\delta\varphi^{\mathrm{m}}\mathrm{d}S-\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{m}}}}(\varphi^{\mathrm{m}}-\bar{\varphi}^{\mathrm{m}})n_i\delta B_i\mathrm{d}S+$$

$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\bar{B}}^{=}}(e_{ijk}n_jB_k-\bar{b}_i^{=})\delta F_i^{\mathrm{e}}\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{e}}}}e_{ijk}n_j(F_k^{\mathrm{e}}-\bar{F}_k^{\mathrm{e}})\delta B_i\mathrm{d}S. \tag{29}$$

**充分性** 若 $D_i,B_i,E_i,H_i,\varphi^{\mathrm{e}},\varphi^{\mathrm{m}},F_i^{\mathrm{e}}$ 和 $F_i^{\mathrm{m}}$ 是问题(1) ~ (16) 的解,则式(29) 变为 $\delta\pi^{\mathrm{g}}=0$.

**必要性** 若 $\delta\pi^{\mathrm{g}}=0$,由于 $\delta D_i,\delta B_i,\delta E_i,\delta H_i,\delta\varphi^{\mathrm{e}},\delta\varphi^{\mathrm{m}},\delta F_i^{\mathrm{e}}$ 和 $\delta F_i^{\mathrm{m}}$ 的任意性,可由式(29)得出式(1) ~ (16),即 $D_i,B_i,E_i,H_i,\varphi^{\mathrm{e}},\varphi^{\mathrm{m}},F_i^{\mathrm{e}}$ 和 $F_i^{\mathrm{m}}$ 是问题(1) ~ (16)的解.

因过程类似,对 $\gamma^{\mathrm{g}}$ 的证明从略.

# 4 有先决条件的广义变分原理

对广义作用量泛函 $\pi^{\mathrm{g}}$ 和 $\gamma^{\mathrm{g}}$ 进行化简,可导出多种有先决条件的广义变分原理,其中以式(8)作为先决条件的 7 类变量的广义作用量泛函为

$$\pi_7^{\mathrm{a}}=\int_{t_0}^{t_1}\mathrm{d}t\iiint_V\Big\{\frac{\varepsilon_{ij}E_iE_j}{2}-\rho^{\mathrm{e}}\varphi^{\mathrm{e}}+\mu_{ij}J_j^{\mathrm{e}}F_i^{\mathrm{e}}-D_i(E_i+\partial_i\varphi^{\mathrm{e}}+\mu_{ij}\partial_tF_j^{\mathrm{e}}+e_{ijk}\partial_jF_k^{\mathrm{m}})+$$
$$(\partial_i\varphi^{\mathrm{m}}+\varepsilon_{il}\partial_tF_l^{\mathrm{m}}-e_{ilm}\partial_lF_m^{\mathrm{e}})\mu_{ij}H_j+\frac{\mu_{ij}H_iH_j}{2}+\rho^{\mathrm{m}}\varphi^{\mathrm{m}}-\varepsilon_{ij}J_j^{\mathrm{m}}F_i^{\mathrm{m}}\Big\}\mathrm{d}V+I_\pi^{\mathrm{B}}, \tag{30}$$

$$\gamma_7^{\mathrm{a}}=\int_{t_0}^{t_1}\mathrm{d}t\iiint_V\Big\{D_iE_i-\frac{\varepsilon_{ij}E_iE_j}{2}+(\rho^{\mathrm{e}}-\partial_iD_i)\varphi^{\mathrm{e}}-$$
$$\mu_{ij}(J_j^{\mathrm{e}}+\partial_tD_j-e_{jkn}\partial_kH_n)F_i^{\mathrm{e}}-\frac{\mu_{ij}H_iH_j}{2}-(\rho^{\mathrm{m}}-\mu_{ij}\partial_iH_j)\varphi^{\mathrm{m}}+$$
$$[\varepsilon_{ij}(J_j^{\mathrm{m}}+\mu_{jk}\partial_tH_k)+e_{ijk}\partial_jD_k]F_i^{\mathrm{m}}\Big\}\mathrm{d}V+I_\gamma^{\mathrm{B}}, \tag{31}$$

式中

$$I_\pi^{\mathrm{B}}=\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{\perp}}\bar{d}^{\perp}\varphi^{\mathrm{e}}\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{e}}}}(\varphi^{\mathrm{e}}-\bar{\varphi}^{\mathrm{e}})n_iD_i\mathrm{d}S-\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{=}}\bar{d}_i^{=}F_i^{\mathrm{m}}\mathrm{d}S+$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{m}}}}e_{ijk}n_j(F_k^{\mathrm{m}}-\bar{F}_k^{\mathrm{m}})D_i\mathrm{d}S-\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^{\perp}}\bar{b}^{\perp}\varphi^{\mathrm{m}}\mathrm{d}S-$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{m}}}}(\varphi^{\mathrm{m}}-\bar{\varphi}^{\mathrm{m}})n_iB_i\mathrm{d}S-\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^{=}}\bar{b}_i^{=}F_i^{\mathrm{e}}\mathrm{d}S+$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{e}}}}e_{ijk}n_j(F_k^{\mathrm{e}}-\bar{F}_k^{\mathrm{e}})B_i\mathrm{d}S, \tag{32}$$

$$I_\gamma^{\mathrm{B}}=\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{\perp}}(n_iD_i-\bar{d}^{\perp})\varphi^{\mathrm{e}}\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{e}}}}\bar{\varphi}^{\mathrm{e}}n_iD_i\mathrm{d}S-$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{=}}(e_{ijk}n_jD_k-\bar{d}_i^{=})F_i^{\mathrm{m}}\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{m}}}}e_{ijk}n_j\bar{F}_k^{\mathrm{m}}D_i\mathrm{d}S-$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^{\perp}}(n_iB_i-\bar{b}^{\perp})\varphi^{\mathrm{m}}\mathrm{d}S-\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{m}}}}\bar{\varphi}^{\mathrm{m}}n_iB_i\mathrm{d}S-$$
$$\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^{=}}(e_{ijk}n_jB_k-\bar{b}_i^{=})F_i^{\mathrm{e}}\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{e}}}}e_{ijk}n_j\bar{F}_k^{\mathrm{e}}B_i\mathrm{d}S. \tag{33}$$

以式(7)作为先决条件的 7 类变量的广义作用量泛函为

$$\pi_7^{\mathrm{b}}=\int_{t_0}^{t_1}\mathrm{d}t\iiint_V\Big\{-(\partial_i\varphi^{\mathrm{e}}+\mu_{il}\partial_tF_l^{\mathrm{e}}+e_{ilm}\partial_lF_m^{\mathrm{m}})\varepsilon_{ij}E_j-\frac{\varepsilon_{ij}E_iE_j}{2}-\rho^{\mathrm{e}}\varphi^{\mathrm{e}}+\mu_{ij}J_j^{\mathrm{e}}F_i^{\mathrm{e}}-$$
$$\frac{\mu_{ij}H_iH_j}{2}+\rho^{\mathrm{m}}\varphi^{\mathrm{m}}-\varepsilon_{ij}J_j^{\mathrm{m}}F_i^{\mathrm{m}}+B_i(H_i+\partial_i\varphi^{\mathrm{m}}+\varepsilon_{ij}\partial_tF_j^{\mathrm{m}}-e_{ijk}\partial_jF_k^{\mathrm{e}})\Big\}\mathrm{d}V+I_\pi^{\mathrm{B}}, \tag{34}$$

$$\gamma_7^{\mathrm{b}} = \int_{t_0}^{t_1}\mathrm{d}t\iiint_V\left\{\frac{\varepsilon_{ij}E_iE_j}{2} + (\rho^{\mathrm{e}} - \varepsilon_{ij}\partial_iE_j)\varphi^{\mathrm{e}} - \mu_{ij}(J_j^{\mathrm{e}} + \varepsilon_{jk}\partial_tE_k - e_{jkn}\partial_kH_n)F_i^{\mathrm{e}} - B_iH_i + \frac{\mu_{ij}H_iH_j}{2} - (\rho^{\mathrm{m}} - \partial_iB_i)\varphi^{\mathrm{m}} + \varepsilon_{ij}(J_j^{\mathrm{m}} + \partial_tB_j + e_{jkn}\partial_kE_n)F_i^{\mathrm{m}}\right\}\mathrm{d}V + I_\gamma^{\mathrm{B}}. \tag{35}$$

以式(7)和式(8)作为先决条件的6类变量的广义作用量泛函为

$$\pi_6 = \int_{t_0}^{t_1}\mathrm{d}t\iiint_V\left\{-(\partial_i\varphi^{\mathrm{e}} + \mu_{il}\partial_tF_l^{\mathrm{e}} + e_{ilm}\partial_lF_m^{\mathrm{m}})\varepsilon_{ij}E_j - \frac{\varepsilon_{ij}E_iE_j}{2} - \rho^{\mathrm{e}}\varphi^{\mathrm{e}} + \mu_{ij}J_j^{\mathrm{e}}F_i^{\mathrm{e}} + (\partial_i\varphi^{\mathrm{m}} + \varepsilon_{il}\partial_tF_l^{\mathrm{m}} - e_{ilm}\partial_lF_m^{\mathrm{e}})\mu_{ij}H_j + \frac{\mu_{ij}H_iH_j}{2} + \rho^{\mathrm{m}}\varphi^{\mathrm{m}} - \varepsilon_{ij}J_j^{\mathrm{m}}F_i^{\mathrm{m}}\right\}\mathrm{d}V + I_\pi^{\mathrm{B}}, \tag{36}$$

$$\gamma_6 = \int_{t_0}^{t_1}\mathrm{d}t\iiint_V\left\{\frac{\varepsilon_{ij}E_iE_j}{2} + (\rho^{\mathrm{e}} - \varepsilon_{ij}\partial_iE_j)\varphi^{\mathrm{e}} - \mu_{ij}(J_j^{\mathrm{e}} + \varepsilon_{jk}\partial_tE_k - e_{jkn}\partial_kH_n)F_i^{\mathrm{e}} - \frac{\mu_{ij}H_iH_j}{2} - (\rho^{\mathrm{m}} - \mu_{ij}\partial_iH_j)\varphi^{\mathrm{m}} + \varepsilon_{ij}(J_j^{\mathrm{m}} + \mu_{jk}\partial_tH_k + e_{jkn}\partial_kE_n)F_i^{\mathrm{m}}\right\}\mathrm{d}V + I_\gamma^{\mathrm{B}}. \tag{37}$$

以式(5)~(8)作为先决条件的4类变量的广义作用量泛函为

$$\pi_4 = \int_{t_0}^{t_1}\mathrm{d}t\iiint_V\left\{\frac{(\partial_i\varphi^{\mathrm{e}} + \mu_{il}\partial_tF_l^{\mathrm{e}} + e_{ilm}\partial_lF_m^{\mathrm{m}})\varepsilon_{ij}(\partial_j\varphi^{\mathrm{e}} + \mu_{jk}\partial_tF_k^{\mathrm{e}} + e_{jkn}\partial_kF_n^{\mathrm{m}})}{2} - \frac{(\partial_i\varphi^{\mathrm{m}} + \varepsilon_{il}\partial_tF_l^{\mathrm{m}} - e_{ilm}\partial_lF_m^{\mathrm{e}})\mu_{ij}(\partial_j\varphi^{\mathrm{m}} + \varepsilon_{jk}\partial_tF_k^{\mathrm{m}} - e_{jkn}\partial_kF_n^{\mathrm{e}})}{2} - \rho^{\mathrm{e}}\varphi^{\mathrm{e}} + \mu_{ij}J_j^{\mathrm{e}}F_i^{\mathrm{e}} + \rho^{\mathrm{m}}\varphi^{\mathrm{m}} - \varepsilon_{ij}J_j^{\mathrm{m}}F_i^{\mathrm{m}}\right\}\mathrm{d}V + I_\pi^{\mathrm{B}}. \tag{38}$$

这些不同的广义变分原理将导致多样的有限元模型.

## 5 退化形式的例证

当电磁场内不含有磁源 $\rho^{\mathrm{m}}$ 和 $J_i^{\mathrm{m}}$ 时,广义作用量泛函 $\pi^{\mathrm{g}}$ 和 $\Gamma^{\mathrm{g}}$ 分别退化为以下形式:

$$\pi^{\mathrm{d}} = \int_{t_0}^{t_1}\mathrm{d}t\iiint_V\left\{\frac{\varepsilon_{ij}E_iE_j}{2} - \rho^{\mathrm{e}}\varphi^{\mathrm{e}} + \mu_{ij}J_j^{\mathrm{e}}F_i^{\mathrm{e}} - D_i(E_i + \partial_i\varphi^{\mathrm{e}} + \mu_{ij}\partial_tF_j^{\mathrm{e}}) - \frac{\mu_{ij}H_iH_j}{2} + B_i(H_i - e_{ijk}\partial_jF_k^{\mathrm{e}})\right\}\mathrm{d}V + \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{\perp}}\bar{d}^{\perp}\varphi^{\mathrm{e}}\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{e}}}}(\varphi^{\mathrm{e}} - \bar{\varphi}^{\mathrm{e}})n_iD_i\mathrm{d}S - \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^{=}}\bar{b}_i^{=}F_i^{\mathrm{e}}\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{e}}}}e_{ijk}n_j(F_k^{\mathrm{e}} - \bar{F}_k^{\mathrm{e}})B_i\mathrm{d}S, \tag{39}$$

$$\gamma^{\mathrm{d}} = \int_{t_0}^{t_1}\mathrm{d}t\iiint_V\left\{D_iE_i - \frac{\varepsilon_{ij}E_iE_j}{2} + (\rho^{\mathrm{e}} - \partial_iD_i)\varphi^{\mathrm{e}} - B_iH_i + \frac{\mu_{ij}H_iH_j}{2} - [\mu_{ij}(J_j^{\mathrm{e}} + \partial_tD_j) - e_{ijk}\partial_jB_k]F_i^{\mathrm{e}}\right\}\mathrm{d}V + \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_D^{\perp}}(n_iD_i - \bar{d}^{\perp})\varphi^{\mathrm{e}}\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{\varphi^{\mathrm{e}}}}\bar{\varphi}^{\mathrm{e}}n_iD_i\mathrm{d}S - \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_B^{=}}(e_{ijk}n_jB_k - \bar{b}_i^{=})F_i^{\mathrm{e}}\mathrm{d}S + \int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{F^{\mathrm{e}}}}e_{ijk}n_j\bar{F}_k^{\mathrm{e}}B_i\mathrm{d}S. \tag{40}$$

对式(39)和式(40)进行变分运算,并令 $A_i = \mu_{ij}F_j^{\mathrm{e}}$,易证 $\pi^{\mathrm{d}}$ 和 $\gamma^{\mathrm{d}}$ 分别与文献[4]中的泛函 $\pi_{\mathrm{em1}}$ 和 $\gamma_{\mathrm{em1}}$ 等价,这间接地验证了广义作用量 $\pi^{\mathrm{g}}$ 和 $\gamma^{\mathrm{g}}$ 的正确性.

## 6 修正的广义变分原理

基于泛函 $\pi^{\mathrm{g}}$ 和 $\gamma^{\mathrm{g}}$ 的表达式可得修正的广义作用量泛函:

$$\begin{aligned}\pi_{\mathrm{mod}}^{\mathrm{g}} = &\sum_{(a)}\int_{t_0}^{t_1}\mathrm{d}t\iiint_{V(a)}\Big\{\frac{\varepsilon_{ij}^{(a)}E_i^{(a)}E_j^{(a)}}{2} - D_i^{(a)}(E_i^{(a)} + \partial_i\varphi^{\mathrm{e},(a)} + \mu_{ij}\partial_t F_j^{\mathrm{e},(a)} + e_{ijk}\partial_j F_k^{\mathrm{m},(a)}) - \\ &\rho^{\mathrm{e}}\varphi^{\mathrm{e},(a)} + \mu_{ij}^{(a)}J_j^{\mathrm{e}}F_i^{\mathrm{e},(a)} - \frac{\mu_{ij}^{(a)}H_i^{(a)}H_j^{(a)}}{2} + \rho^{\mathrm{m}}\varphi^{\mathrm{m},(a)} - \varepsilon_{ij}^{(a)}J_j^{\mathrm{m}}F_i^{\mathrm{m},(a)} + \\ &B_i^{(a)}(H_i^{(a)} + \partial_i\varphi^{\mathrm{m},(a)} + \varepsilon_{ij}^{(a)}\partial_t F_j^{\mathrm{m},(a)} - e_{ijk}\partial_j F_k^{\mathrm{e},(a)})\Big\}\mathrm{d}V + \\ &\sum_{(a,b)}\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{ab}}(\varphi^{\mathrm{e},(a)} - \varphi^{\mathrm{e},(b)})n_i^{(a)}D_i^{(a)}\mathrm{d}S + \\ &\sum_{(a,b)}\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{ab}}e_{ijk}n_j^{(a)}(F_k^{\mathrm{m},(a)} - F_k^{\mathrm{m},(b)})D_i^{(a)}\mathrm{d}S - \\ &\sum_{(a,b)}\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{ab}}(\varphi^{\mathrm{m},(a)} - \varphi^{\mathrm{m},(b)})n_i^{(a)}B_i^{(a)}\mathrm{d}S + \\ &\sum_{(a,b)}\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{ab}}e_{ijk}n_j^{(a)}(F_k^{\mathrm{e},(a)} - F_k^{\mathrm{e},(b)})B_i^{(a)}\mathrm{d}S + I_\pi^{\mathrm{B}},\end{aligned} \tag{41}$$

$$\begin{aligned}\gamma_{\mathrm{mod}}^{\mathrm{g}} = &\sum_{(a)}\int_{t_0}^{t_1}\mathrm{d}t\iiint_{V}\Big\{D_i^{(a)}E_i^{(a)} - \frac{\varepsilon_{ij}^{(a)}E_i^{(a)}E_j^{(a)}}{2} + (\rho^{\mathrm{e}} - \partial_i D_i^{(a)})\varphi^{\mathrm{e},(a)} - \\ &[\mu_{ij}^{(a)}(J_j^{\mathrm{e}} + \partial_t D_j^{(a)}) - e_{ijk}\partial_j B_k^{(a)}]F_i^{\mathrm{e},(a)} - B_i^{(a)}H_i^{(a)} + \frac{\mu_{ij}^{(a)}H_i^{(a)}H_j^{(a)}}{2} - \\ &(\rho^{\mathrm{m}} - \partial_i B_i^{(a)})\varphi^{\mathrm{m},(a)} + [\varepsilon_{ij}^{(a)}(J_j^{\mathrm{m}} + \partial_t B_j^{(a)}) + e_{ijk}\partial_j D_k^{(a)}]F_i^{\mathrm{m},(a)}\Big\}\mathrm{d}V + \\ &\sum_{(a,b)}\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{ab}}(n_i^{(a)}D_i^{(a)} + n_i^{(b)}D_i^{(b)})\varphi^{\mathrm{e},(a)}\mathrm{d}S - \\ &\sum_{(a,b)}\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{ab}}(e_{ijk}n_j^{(a)}D_k^{(a)} + e_{ijk}n_j^{(b)}D_k^{(b)})F_i^{\mathrm{m},(a)}\mathrm{d}S - \\ &\sum_{(a,b)}\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{ab}}(n_i^{(a)}B_i^{(a)} + n_i^{(b)}B_i^{(b)})\varphi^{\mathrm{m},(a)}\mathrm{d}S - \\ &\sum_{(a,b)}\int_{t_0}^{t_1}\mathrm{d}t\iint_{S_{ab}}(e_{ijk}n_j^{(a)}B_k^{(a)} + e_{ijk}n_j^{(b)}B_k^{(b)})F_i^{\mathrm{e},(a)}\mathrm{d}S + I_\gamma^{\mathrm{B}}.\end{aligned} \tag{42}$$

$\pi_{\mathrm{mod}}^{\mathrm{g}}$ 和 $\gamma_{\mathrm{mod}}^{\mathrm{g}}$ 分别提供了位势杂交和通量杂交的混合元模型. 对于有先决条件的广义变分原理，也可导出各种相应的有限元模型.

# 7 结　　论

本文建立的广义变分原理可以完全反映线性各项异性电磁场边值问题的全部特征,在它们的基础上相继导出了各种有先决条件的广义变分原理、退化的广义变分原理和修正的广义变分原理. 以上这些变分原理可为建立相应的有限元应用提供更为完善的理论基础.

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# Generalized Variational Principles for Boundary Value Problem of Electromagnetic Field in Electrodynamics

ZHENG Cheng-bo, LIU Bin, WANG Zuo-jun, ZHENG Shi-ke

(*Key Lab of Measurement Technology and Instrumentation of Hebei Province*, *Yanshan University*, *Qinhuangdao*, *Hebei* 066004, *P. R. China*)

**Abstract**: The expression of the generalized principle of virtual work for the boundary value problem of linear and anisotropic electromagnetic field was given. Using Prof. W. Z. Chien's method, a pair of generalized variational principles (GVPs) were established, which could directly lead to all four Maxwell equations, two intensity-potential equations, two constitutive equations and eight boundary conditions. A family of constrained variational principles was deduced sequentially. As the additional verifications, two degenerated forms were obtained, which were equivalent to two known variational principles. Two modified GVPs were given to provide the hybrid finite element models for the present problem. A more complete theoretical foundation for the finite element applications was provided for the discussed problem.

**Key words**: generalized variational principle; electromagnetic field; electrodynamics; boundary value problem; finite element method