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二维流面上的流动问题的速度图方法*

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摘要: 对于一些特殊的流动,尤其是平面上的位势流动,速度图方法有其显著的优点. 对于理想流体来说,流面总是存在的,在流面上,流动的速度向量总是在其切空间里. 通过引入流函数和势函数,采用张量分析作为工具,给出了二维曲流面上位势流动的速度图方法,得到了流函数满足的速度图方程,为一些特殊的流动问题提供了一类分析方法. 并且,对于得到的二维速度图方程,得到了相应的特征方程和特征根,从而可以对方程的类型进行分类. 最后,给出了一些特殊流动的实例.

关 键 词: 速度图方法; 位势流; 流面; 流函数; 位势函数

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引 言

众所周知,传统的速度图方法在空气动力学研究中发挥了重要的作用. 然而,该方法仅仅被应用在二维的平面无粘流动问题中,可见文献[1-3]. 在本文中,我们使用流面的概念:一个 R^3 中的二维的流形,如果流动的速度向量总是在其切空间里,称这个流形为流动的流面,可以证明这种流面总是存在的,见文献[4-6]. 在流面上,是否也可以建立相应的速度图方法呢? 这是本文的主要动机和目的. 通过本文,我们给出了二维流形上的速度图方法,并且得到了速度图方程的特征方程和特征根,从而可以对方程进行分类. 对于二维平面流动,这些结果和经典结论是一致的.

本文的组织结构如下:在第 1 节中我们基于二维流形引入三维空间的半测地坐标系,并得到该坐标系下的可压缩粘性流动的 Navier-Stokes 方程. 在第 2 节中,基于前面的 Navier-Stokes 方程的表示,我们引入流函数,通过速度图变换得到了流函数满足的二阶方程,并给出了该方程对应的特征方程和特征根,从而可以对速度图方程进行分类. 最后,我们在第 3 节中给出了一些特殊流动下的结果.

1 半测地坐标系下的 Navier-Stokes 方程

在本节中,我们将给出基于二维流面的半测地坐标系下的关于 Navier-Stokes 方程的一些

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结果,主要结论来自于文献[4-7].

1.1 基于二维流面的半测地坐标系

记 S 是一个二维流形,而 P 是三维空间中 S 附近的一点,O 是原点. 我们知道 P 点可以表示为

$$\boldsymbol{R} = \overrightarrow{OP} = \boldsymbol{r} + \xi\boldsymbol{n},\ 以及\ \boldsymbol{r} = \overrightarrow{OP_o},\ \overrightarrow{P_oP} = \xi\boldsymbol{n},$$

其中,P_o 是点 P 在 S 上的投影,$\boldsymbol{n}$ 是 $\overrightarrow{P_oP}$ 方向上的单位向量. 当 (x^1, x^2) 是 S 上的一个 Gauss 曲线坐标系时,点 P 可以唯一地表示为 (x^1, x^2, ξ). 在文献[8]中,Il' in 证明了当 $\boldsymbol{r}: R^2 \supset S \to R^3$ 是一个 C^3 映照时,存在从 S 上的 $\boldsymbol{r}$ 到其邻域 Ω_ϵ 的正则延拓 $\boldsymbol{R}$:

$$\boldsymbol{R}(x,\xi) = \boldsymbol{r}(x) + \xi\boldsymbol{n}(x), \qquad \forall (x,\xi) \in \Omega^\varepsilon,$$

其中 $\boldsymbol{R}$ 是从 Ω^ε 到 $\boldsymbol{R}(\Omega^\varepsilon)$ 上的 C^1 映照,而且 3 个向量 $\boldsymbol{e}_\alpha = \partial\boldsymbol{R}/\partial x^\alpha, \alpha = 1,2, \boldsymbol{e}_3 = \partial\boldsymbol{R}/\partial\xi$ 在 Ω^ε 内是处处线性无关的. 因此 (x,ξ) 可以看做是 S 附近一点 P 在 R^3 中的坐标表示,被称为 S 邻域的半测地坐标.

在本文中,拉丁字母 $i,j,k,\cdots$ 等在 $\{1,2,3\}$ 中取值,而希腊字母 $\alpha,\beta,\gamma,\cdots$ 等在 $\{1,2\}$ 中取值. 另外,对重复的指标使用 Einstein 求和约定.

流形 S 上的第一、二和三基本型 $\{a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}\}$ 定义为

$$\begin{cases} a_{\alpha\beta} = \boldsymbol{r}_\alpha\cdot\boldsymbol{r}_\beta,\ b_{\alpha\beta} = \boldsymbol{n}\cdot\boldsymbol{r}_{\alpha\beta} = -(\boldsymbol{n}_\alpha\cdot\boldsymbol{r}_\beta + \boldsymbol{r}_\alpha\cdot\boldsymbol{n}_\beta)/2, \\ c_{\alpha\beta} = \boldsymbol{n}_\alpha\cdot\boldsymbol{n}_\beta = a^{\lambda\sigma}b_{\alpha\lambda}b_{\beta\sigma},\ a^{\alpha\beta}a_{\beta\gamma} = \delta^\alpha_\gamma, \\ \hat{b}^{\alpha\beta}b_{\beta\lambda} = \delta^\alpha_\lambda,\ \hat{c}^{\alpha\beta}c_{\beta\lambda} = \delta^\alpha_\lambda,\ b^{\alpha\beta} = a^{\alpha\lambda}a^{\beta\sigma}b_{\lambda\sigma},\ c^{\alpha\beta} = a^{\alpha\lambda}a^{\beta\sigma}c_{\lambda\sigma}, \end{cases} \tag{1}$$

其中 $\hat{b}^{\alpha\beta}, \hat{c}^{\alpha\beta}$ 分别为 $a_{\alpha\beta}, b_{\alpha\beta}$ 的逆矩阵元素. S 的平均曲率 H 和 Gauss 曲率 K 分别为

$$H = \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta},\ K = \frac{\det(b_{\alpha\beta})}{\det(a_{\alpha\beta})}.$$

很容易验证下面两个常用的关系式成立[5]:

$$Ka_{\alpha\beta} - 2Hb_{\alpha\beta} + c_{\alpha\beta} = 0,\ a^{\alpha\beta} - 2H\hat{b}^{\alpha\beta} + K\hat{c}^{\alpha\beta} = 0. \tag{2}$$

同时我们还需要用到 S 上的行列式张量 $\varepsilon^{\alpha\beta}, \varepsilon_{\alpha\beta}$, 定义为

$$\varepsilon_{\alpha\beta} = \begin{cases} \sqrt{a}, \\ -\sqrt{a}, \\ 0, \end{cases} \quad \varepsilon^{\alpha\beta} = \begin{cases} 1/\sqrt{a}, & (\alpha,\beta)\ 是(1,2)\ 的偶排列, \\ -1/\sqrt{a}, & (\alpha,\beta)\ 是(1,2)\ 的奇排列, \\ 0, & 其它情形. \end{cases} \tag{3}$$

此外,我们还需要下面的引理[5]:

引理 1 在半测地坐标系下,R^3 上的度量张量、Christoffel 符号、协变导数、散度和旋度可以用 S 上的 $\{a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}\}$ 分别表示为

$$\begin{cases} g_{\alpha\beta} = a_{\alpha\beta} - 2\xi b_{\alpha\beta} + \xi^2 c_{\alpha\beta},\ g_{3\alpha} = g_{\alpha3} = 0,\ g_{33} = 1, \\ g^{\alpha\beta} = \theta^{-2}G^{\alpha\beta},\ g^{3\alpha} = g^{\alpha3} = 0,\ g^{33} = 1, \\ g = \det(g_{ij}) = \theta^2 a,\ a = \det(a_{\alpha\beta}), \end{cases} \tag{4}$$

$$\begin{cases} \Gamma^\gamma_{\alpha\beta} = \overset{*}{\Gamma}{}^\gamma_{\alpha\beta} + \theta^{-1}R^\gamma_{\alpha\beta},\ \Gamma^\alpha_{\beta3} = \theta^{-1}I^\alpha_\beta,\ \Gamma^3_{\alpha\beta} = J_{\alpha\beta}, \\ \Gamma^3_{\beta3} = \Gamma^3_{3\beta} = \Gamma^\alpha_{33} = \Gamma^3_{33} = 0, \end{cases} \tag{5}$$

$$\begin{cases}\nabla_\alpha u^\beta = \overset{*}{\nabla}_\alpha u^\beta + \theta^{-1}(I_\alpha^\beta u^3 + R_{\alpha\lambda}^\beta u^\lambda),\ \nabla_\alpha u^3 = \dfrac{\partial u^3}{\partial x^\alpha} + J_{\alpha\beta}u^\beta, \\ \nabla_3 u^3 = \dfrac{\partial u^3}{\partial \xi},\ \nabla_3 u^\beta = \dfrac{\partial u^\beta}{\partial \xi} + \theta^{-1}I_\alpha^\beta u^\alpha, \\ \mathrm{div}\boldsymbol{u} = \overset{*}{\mathrm{div}}\boldsymbol{u} + \dfrac{\partial u^3}{\partial \xi} + \theta^{-1}[2(-H + K\xi)u^3 + \\ \qquad u^\alpha(-2\overset{*}{\nabla}_\alpha H\xi + \overset{*}{\nabla}_\alpha K\xi^2)], \\ \mathrm{rot}(\boldsymbol{u}) = (\mathrm{rot}(\boldsymbol{u}))^\alpha \boldsymbol{e}_\alpha + (\mathrm{rot}(\boldsymbol{u}))^3\boldsymbol{n}, \\ (\mathrm{rot}(\boldsymbol{u}))^\alpha = \theta^{-1}\varepsilon^{\alpha\beta}\left(\overset{*}{\nabla}_\beta u^3 - g_{\beta\lambda}\dfrac{\partial u^\lambda}{\partial \xi}\right), \\ (\mathrm{rot}(\boldsymbol{u}))^3 = \varepsilon^{\beta\sigma}g_{\beta\lambda}[\theta^{-1}\overset{*}{\nabla}_\sigma u^\lambda + \theta^{-2}R_{\sigma\nu}^\lambda u^\nu],\end{cases} \tag{6}$$

其中，$\overset{*}{\Gamma}{}_{\alpha\beta}^{\gamma}$，$\overset{*}{\Delta}$，$\overset{*}{\mathrm{div}}$，$\overset{*}{\nabla}_\alpha$ 分别表示 S 上的 Christoffel 符号、Laplace 算子、散度算子和协变导数，以及

$$\theta = 1 - 2H\xi + K\xi^2,\ G^{\alpha\beta} = a^{\alpha\beta} - 2K\xi\hat{b}^{\alpha\beta} + \xi^2K^2\hat{c}^{\alpha\beta},$$

$$I_\beta^\alpha = -b_\beta^\alpha + K\xi\delta_\beta^\alpha,\ J_{\alpha\beta} = b_{\alpha\beta} - \xi c_{\alpha\beta},\ R_{\beta\sigma}^\alpha = (2H\xi^2 - \xi)\overset{*}{\nabla}_\sigma b_\beta^\alpha - b_\mu^\alpha\overset{*}{\nabla}_\sigma b_\beta^\mu\xi^2.$$

特别地，当 S 是一个流面时，$u^3 = 0$，此时，在流面 S（对应于 $\xi = 0$）上有

$$\begin{cases} g_{\alpha\beta} = a_{\alpha\beta},\ g_{\alpha 3} = g_{3\alpha} = 0,\ g_{33} = 1,\ g = a, \\ g^{\alpha\beta} = a^{\alpha\beta},\ g^{\alpha 3} = g^{3\alpha} = 0,\ g^{33} = 1, \\ \Gamma_{\alpha\beta}^\gamma = \overset{*}{\Gamma}{}_{\alpha\beta}^\gamma,\ \Gamma_{\beta 3}^\alpha = -b_\beta^\alpha,\ \Gamma_{\alpha\beta}^3 = b_{\alpha\beta},\ \Gamma_{i3}^3 = \Gamma_{33}^i = 0,\end{cases} \tag{7}$$

$$\begin{cases}\nabla_\alpha u^\beta = \overset{*}{\nabla}_\alpha u^\beta,\ \nabla_3 u^\beta = \dfrac{\partial u^\beta}{\partial \xi} - b_\alpha^\beta u^\alpha,\ \nabla_\alpha u^3 = \dfrac{\partial u^3}{\partial x^\alpha} + b_{\alpha\beta}u^\beta,\ \nabla_3 u^3 = \dfrac{\partial u^3}{\partial \xi}, \\ \mathrm{div}\boldsymbol{u} = \overset{*}{\mathrm{div}}\boldsymbol{u} + \dfrac{\partial u^3}{\partial \xi},\ (\mathrm{rot}(\boldsymbol{u}))^\alpha = -\varepsilon^{\alpha\beta}a_{\beta\lambda}\dfrac{\partial u^\lambda}{\partial \xi},\ (\mathrm{rot}(\boldsymbol{u}))^3 = \varepsilon^{\beta\sigma}a_{\beta\lambda}\overset{*}{\nabla}_\sigma u^\lambda,\end{cases} \tag{8}$$

$$\begin{cases}\Delta u^\alpha = \overset{*}{\Delta}u^\alpha + \dfrac{\partial^2 u^\alpha}{\partial \xi^2} - 2(b_\beta^\alpha + H\delta_\beta^\alpha)\dfrac{\partial u^\beta}{\partial \xi} + Ku^\alpha, \\ \Delta u^3 = \dfrac{\partial^2 u^\alpha}{\partial \xi^2} - 2H\dfrac{\partial u^3}{\partial \xi} + 2b_\lambda^\nu\overset{*}{\nabla}_\nu u^\lambda + 2u^\beta\overset{*}{\nabla}_\beta H, \\ \Delta T = \overset{*}{\Delta}T + \dfrac{\partial^2 T}{\partial \xi^2} - H\dfrac{\partial T}{\partial \xi},\end{cases} \tag{9}$$

这里我们用到了结论

$$\overset{*}{\Delta}u^3 \equiv a^{\alpha\beta}\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^3 = 0,\ \overset{*}{\nabla}_\nu u^3 = 0,\qquad \text{在 } S \text{ 上}, \tag{10}$$

这是由于在流面 S 上 $u^3 = 0$.

另外我们还会用到流面 S 上的旋度算子

$$\overset{*}{\mathrm{rot}}(\boldsymbol{u}) = (\mathrm{rot}(\boldsymbol{u}))^3|_S = \varepsilon^{\beta\sigma}a_{\beta\lambda}\overset{*}{\nabla}_\sigma u^\lambda = \varepsilon_{\alpha\beta}\overset{*}{\nabla}{}^\alpha u^\beta = \varepsilon^{\alpha\beta}\overset{*}{\nabla}_\alpha u_\beta. \tag{11}$$

1.2 半测地坐标系下的 Navier-Stokes 方程

我们考虑定常的可压缩粘性流动问题. 当采用以角速度大小 ω 绕 z 轴旋转的非惯性坐标系时，控制方程可以表示为

$$
\begin{cases}
\operatorname{div}(\rho \boldsymbol{u}) = 0, & \text{连续性方程,} \\
-\mu \Delta \boldsymbol{u} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\left(p - \dfrac{\mu}{3}\operatorname{div}\boldsymbol{u}\right) + 2\rho\boldsymbol{\omega}\times\boldsymbol{u} = & \\
\qquad \rho\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{R}) + \boldsymbol{f}, & \text{动量方程,} \\
\operatorname{div}(\rho E\boldsymbol{u}) + p\operatorname{div}\boldsymbol{u} - \operatorname{div}(\kappa \operatorname{grad} T) - \Phi = 0, & \text{能量方程,} \\
p = p(\rho, T), & \text{状态方程,}
\end{cases}
\tag{12}
$$

其中，$\boldsymbol{u},\rho,p,E = C_vT, C_v, T, \kappa, \nu = \mu/\rho, 2\rho\omega\times\boldsymbol{u}, \boldsymbol{F} = \boldsymbol{f}/\rho + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{R})$ 分别为流体相对旋转坐标系的相对速度、密度、压力、单位体积下的内能、比热容、温度、热传导系数、动力粘性常数、Coriolis 力、单位质量下的外力和离心力，$\boldsymbol{\omega}$ 为角速度向量，应力张量 $\boldsymbol{\sigma}$ 和耗散函数 Φ 分别定义为

$$
\boldsymbol{\sigma}^{ij} = g^{ij}\left(-p - \frac{2}{3}\mu\operatorname{div}\boldsymbol{u}\right) + 2\mu \mathrm{e}^{ij}(\boldsymbol{u}),\ \mathrm{e}^{ij}(\boldsymbol{u}) = \frac{1}{2}(\nabla^i u^j + \nabla^j u^i), \tag{13}
$$

$$
\Phi = 2\mu \mathrm{e}^{ij}(\boldsymbol{u})\mathrm{e}_{ij}(\boldsymbol{u}) - \frac{2}{3}\mu(\operatorname{div}\boldsymbol{u})^2. \tag{14}
$$

有些情形下，我们会用如下的熵方程代替前面的能量方程

$$
\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{|\boldsymbol{u}|^2 T}\left(\frac{\kappa}{\rho}\Delta T + \frac{\Phi}{\rho}\right), \tag{15}
$$

其中 $s = R\lg(T^{\gamma/(\gamma-1)}/p)$ 表示单位质量的熵，$|\boldsymbol{u}|^2 = g_{ij}u^iu^j$.

由引理 1 我们可知

定理 1 在流面 $S(\xi = 0, u^3 = 0)$ 上，Navier-Stokes 方程(12)和(15)可以表示为

$$
-\mu\left(\overset{*}{\Delta}u^\alpha + \frac{\partial^2 u^\alpha}{\partial\xi^2} - 2(H\delta^\alpha_\lambda + b^\alpha_\lambda)\frac{\partial u^\lambda}{\partial\xi} + Ku^\alpha\right) + \rho u^\beta \overset{*}{\nabla}_\beta u^\alpha +
$$
$$
a^{\alpha\beta}\overset{*}{\nabla}_\beta\left(p - \frac{\mu}{3}\overset{*}{\operatorname{div}}\boldsymbol{u}\right) - 2\rho a^{\alpha\beta}\varepsilon_{\beta\lambda}u^\lambda\omega^3 = F^\alpha, \tag{16a}
$$

$$
-\mu\left(\frac{\partial^2 u^3}{\partial\xi^2} - 2H\frac{\partial u^3}{\partial\xi}\right) + \frac{\partial}{\partial\xi}\left(p - \frac{\mu}{3}\operatorname{div}\boldsymbol{u}\right)\Bigg|_{\xi=0} - 2\rho\varepsilon_{\alpha\beta}\omega^\alpha u^\beta -
$$
$$
2\mu\left(\frac{4}{3}u^\alpha\overset{*}{\nabla}_\beta H + b^\alpha_\beta\overset{*}{\nabla}_\alpha u^\beta\right) + \rho b_{\alpha\beta}u^\alpha u^\beta = F^3, \tag{16b}
$$

$$
\overset{*}{\operatorname{div}}(\rho\boldsymbol{u}) + \frac{\partial(\rho\boldsymbol{u})^3}{\partial\xi} = 0, \tag{16c}
$$

$$
(u\cdot\overset{*}{\nabla})s = \frac{1}{|u|^2 T\rho}\left(\kappa\left(\overset{*}{\Delta}T + \frac{\partial^2 T}{\partial\xi^2} - 2H\frac{\partial T}{\partial\xi}\right) + \Phi_0\right), \tag{16d}
$$

其中 ω^3 为角速度向量在半测地坐标系下的第 3 分量，

$$
\Phi_0 = 2\mu a^{\alpha\lambda}a^{\beta\sigma}\overset{*}{e}_{\alpha\beta}\overset{*}{e}_{\lambda\sigma} + \frac{1}{2}\mu a_{\alpha\beta}\frac{\partial u^\alpha}{\partial\xi}\frac{\partial u^\beta}{\partial\xi} - \frac{2}{3}\mu\left(\overset{*}{\operatorname{div}}\boldsymbol{u} + \frac{\partial u^3}{\partial\xi}\right)^2.
$$

关于曲面上算子的交换性质，我们有下面的结论(参见文献[5])。

引理 2

$$
\overset{*}{\operatorname{rot}}(\overset{*}{\Delta}\boldsymbol{u} + K\boldsymbol{u}) = \overset{*}{\Delta}\overset{*}{\operatorname{rot}}\,\boldsymbol{u};\ \overset{*}{\operatorname{rot}}((\boldsymbol{u}\cdot\overset{*}{\nabla})\boldsymbol{u}) = (\boldsymbol{u}\cdot\overset{*}{\nabla})\overset{*}{\operatorname{rot}}\,\boldsymbol{u} + \varepsilon_{\alpha\beta}\overset{*}{\nabla}^\alpha u^\lambda\overset{*}{\nabla}_\lambda u^\beta. \tag{17}
$$

利用在 S 上 $\boldsymbol{\sigma}^{33} = -\tau H$，并定义函数 b 为

$$
u^\alpha\frac{\partial \ln b}{\partial x^\alpha} = \frac{\partial u^3}{\partial x^3},\qquad \text{在 } S \text{ 上}. \tag{18}
$$

注1 b 的物理意义是相邻流面之间的相对厚度[4-5].

最终我们可得

定理2 流面 S 上的 Navier-Stokes 方程(16)可以表示为

$$-\mu\overset{*}{\Delta}\overset{*}{\mathrm{rot}}\boldsymbol{u}+\rho(\boldsymbol{u}\cdot\overset{*}{\nabla})\overset{*}{\mathrm{rot}}\,\boldsymbol{u}+\rho\varepsilon_{\alpha\beta}\overset{*}{\nabla}{}^{\alpha}u^{\lambda}\overset{*}{\nabla}_{\lambda}u^{\beta}+\varepsilon_{\alpha\beta}(\boldsymbol{u}\cdot\overset{*}{\nabla})u^{\beta}\overset{*}{\nabla}{}^{\alpha}\rho+ \\ 2\overset{*}{\nabla}_{\lambda}(\rho\omega^{3}u^{\lambda})=\overset{*}{\mathrm{rot}}\left(F-\mu\frac{\partial^{2}u^{\alpha}}{\partial\xi^{2}}+2(H\delta_{\lambda}^{\alpha}+b_{\lambda}^{\alpha})\frac{\partial u^{\lambda}}{\partial\xi}\right), \tag{19a}$$

$$\mu\left(\frac{\partial^{2}u^{3}}{\partial\xi^{2}}+\left(2H+\frac{\partial\ln\rho}{\partial\xi}\right)\frac{\partial u^{3}}{\partial\xi}\right)=F^{3}-\tau\frac{\partial H}{\partial\xi}-\rho b_{\alpha\beta}u^{\alpha}u^{\beta}+2\rho\varepsilon_{\alpha\beta}\omega^{\alpha}u^{\beta}+ \\ 2\mu(u^{\beta}\overset{*}{\nabla}_{\beta}H+b_{\beta}^{\alpha}\overset{*}{\nabla}_{\alpha}u^{\beta})-\frac{\partial u^{\alpha}}{\partial\xi}\overset{*}{\nabla}_{\alpha}\ln\rho-u^{\alpha}\overset{*}{\nabla}_{\alpha}\frac{\partial\ln\rho}{\partial\xi}, \tag{19b}$$

$$\overset{*}{\mathrm{div}}(b\rho\boldsymbol{u})=0, \tag{19c}$$

$$-\kappa\left(\overset{*}{\Delta}T+\frac{\partial^{2}T}{\partial\xi^{2}}-2H\frac{\partial T}{\partial\xi}\right)+C_{v}\rho u^{\alpha}\overset{*}{\nabla}_{\alpha}T+p\left(\overset{*}{\mathrm{div}}\boldsymbol{u}+\frac{\partial u^{3}}{\partial\xi}\right)- \\ 2\mu\overset{*}{\mathrm{e}}{}^{\alpha\beta}(\boldsymbol{u})\overset{*}{\mathrm{e}}_{\alpha\beta}(\boldsymbol{u})-\frac{1}{2}\mu a^{\alpha\beta}\frac{\partial u^{\alpha}}{\partial\xi}\frac{\partial u^{\beta}}{\partial\xi}+\frac{2}{3}\mu\left(\overset{*}{\mathrm{div}}\boldsymbol{u}+\frac{\partial u^{3}}{\partial\xi}\right)^{2}=0. \tag{19d}$$

2 速度图方法和流函数满足的速度图方程

在本节中对于位势流动,我们定义流面 S 上的流函数 $\boldsymbol{\Psi}$ 和位势函数 $\boldsymbol{\Phi}$, 并得出了流函数满足的方程.

2.1 流面上的流函数和位势函数

由(19c)式,我们可以定义满足以下关系的流函数

$$b\rho u^{\beta}=\varepsilon^{\beta\alpha}\overset{*}{\nabla}_{\alpha}\Psi,\quad\overset{*}{\nabla}_{\alpha}\Psi=b\rho\varepsilon_{\beta\alpha}u^{\beta}, \tag{20}$$

其中 $\varepsilon_{\beta\alpha}$ 为 S 上的行列式张量,定义见(3)式.

另外,如果流动是位势流动,可以定义位势函数 $\boldsymbol{\Phi}$:

$$\boldsymbol{u}=\mathrm{grad}\Phi \text{ 或 } u_{i}=\frac{\partial\Phi}{\partial x^{i}},\quad u^{i}=g^{ij}\frac{\partial\Phi}{\partial x^{j}}.$$

由于 S 是流面,可知在 S 上,$u^{3}=u_{3}=0$, 因此

$$u_{\alpha}=\frac{\partial\Phi}{\partial x^{\alpha}},\quad u^{\alpha}=a^{\alpha\beta}u_{\beta}=a^{\alpha\beta}\frac{\partial\Phi}{\partial x^{\beta}}. \tag{21}$$

S 上的速度向量、流函数和位势函数有下面的关系式:

$$u^{\alpha}=a^{\alpha\beta}\nabla_{\beta}\Phi,\quad\nabla_{\alpha}\Phi=a_{\alpha\beta}u^{\beta};\ u^{\alpha}=\frac{1}{b\rho}\varepsilon^{\alpha\beta}\nabla_{\beta}\Psi,\quad\nabla_{\alpha}\Psi=b\rho\varepsilon_{\beta\alpha}u^{\beta}. \tag{22}$$

2.2 流函数满足的速度图方程

首先引入速度图变量 (V,Θ),其中 V 是速度向量 $\boldsymbol{u}$ 的模,Θ 是速度向量 $\boldsymbol{u}$ 与流面上第一基向量 $\boldsymbol{e}_1$ 的夹角,即

$$V=\sqrt{a_{\alpha\beta}u^{\alpha}u^{\beta}},\quad V\cos\Theta=\sqrt{a_{11}}\,u^{1}.$$

改变换的行列式为 $\hat{J}=\partial(V,\Theta)/\partial(x^{1},x^{2})$.

我们先给出本文的主要定理如下:

定理3 设速度图变换 Jacobi 行列式 $\hat{J}$ 非奇异,那么流面上流函数 Ψ 满足速度图方程:

$$A_{\Theta\Theta}\frac{\partial^2\Psi}{\partial V^2}+A_{VV}\frac{\partial^2\Psi}{\partial\Theta^2}-(A_{V\Theta}+A_{\Theta V})\frac{\partial^2\Psi}{\partial V\partial\Theta}+$$
$$\left(\frac{A_{\Theta\Theta}}{\partial V}-\frac{\partial A_{V\Theta}}{\partial\Theta}\right)\frac{\partial\Psi}{\partial V}+\left(\frac{A_{VV}}{\partial\Theta}-\frac{\partial A_{\Theta V}}{\partial V}\right)\frac{\partial\Psi}{\partial\Theta}=0, \tag{23}$$

其系数在后面的(33)式中定义,即

$$A_{\xi\eta}=\frac{1}{Q\sqrt{a}}\varepsilon_{\alpha\beta}\frac{\partial}{\partial\xi}(D_\rho u^\beta)\frac{\partial}{\partial\eta}(D_1\varepsilon_{\gamma}^{\cdot\alpha}u^\gamma),\qquad \xi,\eta=V\text{ 或 }\Theta,$$

以及　$D=b\rho\sqrt{a}V^2,\ D_1=\dfrac{\sqrt{a}}{D},\ D_\rho=D_1b\rho=\dfrac{1}{V^2}.$

为证明定理3,我们需要用到下面的几个引理.

引理3　流面 S 上的流函数和位势函数(Ψ,Φ) 在 Gauss 坐标系 $x=(x^1,x^2)$ 下满足方程

$$\frac{\partial x^\sigma}{\partial u^\beta}=\frac{1}{b\rho V^2}(b\rho\delta_\alpha^\sigma\Phi_\beta+\varepsilon_\alpha^{\cdot\sigma}\Psi_\beta)u^\alpha. \tag{24}$$

证明　由(22)式有

$$\begin{cases}\mathbf{d}\Phi=\nabla_\alpha\Phi\,\mathbf{d}x^\alpha=a_{\alpha\beta}u^\beta\mathbf{d}x^\alpha,\\ \mathbf{d}\Psi=\nabla_\alpha\Psi\,\mathbf{d}x^\alpha=b\rho\varepsilon_{\beta\alpha}u^\beta\mathbf{d}x^\alpha.\end{cases} \tag{25}$$

该方程组的系数矩阵对应的行列式为

$$D=\begin{vmatrix}a_{1\beta}u^\beta & a_{2\beta}u^\beta\\ b\rho\varepsilon_{\beta1}u^\beta & b\rho\varepsilon_{\beta2}u^\beta\end{vmatrix}=b\rho\varepsilon_{12}a_{\alpha\beta}u^\alpha u^\beta=b\rho\sqrt{a}V^2,$$

其中 $V^2=a_{\alpha\beta}u^\alpha u^\beta=|\boldsymbol{u}|^2$, 从而有

$$\mathbf{d}x^1=\frac{1}{D}\begin{vmatrix}\mathbf{d}\Phi & a_{2\beta}u^\beta\\ \mathbf{d}\Psi & b\rho\varepsilon_{\beta2}u^\beta\end{vmatrix}=\frac{1}{D}(b\rho\varepsilon_{\beta2}u^\beta\mathbf{d}\Phi-a_{2\beta}u^\beta\mathbf{d}\Psi), \tag{26a}$$

$$\mathbf{d}x^2=\frac{1}{D}\begin{vmatrix}a_{1\beta}u^\beta & \mathbf{d}\Phi\\ b\rho\varepsilon_{\beta1}u^\beta & \mathbf{d}\Psi\end{vmatrix}=\frac{1}{D}(a_{1\beta}u^\beta\mathbf{d}\Psi-b\rho\varepsilon_{\beta1}u^\beta\mathbf{d}\Phi). \tag{26b}$$

另外将 Φ 和 Ψ 看作 u^1 和 u^2 的函数,并定义

$$\Phi_\alpha=\frac{\partial\Phi}{\partial u^\alpha},\ \Psi_\alpha=\frac{\partial\Psi}{\partial u^\alpha},$$

我们有

$$\begin{cases}\mathbf{d}x^1=-\dfrac{1}{D}(b\rho\varepsilon_{2\alpha}\Phi_\beta+a_{2\alpha}\Psi_\beta)u^\alpha\mathbf{d}u^\beta,\\ \mathbf{d}x^2=\dfrac{1}{D}(b\rho\varepsilon_{1\alpha}\Phi_\beta+a_{1\alpha}\Psi_\beta)u^\alpha\mathbf{d}u^\beta.\end{cases} \tag{27}$$

利用 $\varepsilon_{\alpha\beta}$ 和 $\varepsilon^{\alpha\beta}$, 则有

$$\mathbf{d}x^\sigma=(\sqrt{a}/D)(b\rho\delta_\alpha^\sigma\Phi_\beta+a_{\lambda\alpha}\varepsilon^{\lambda\sigma}\Psi_\beta)u^\alpha\mathbf{d}u^\beta.$$

定义循环张量 $\varepsilon_\alpha^{\cdot\sigma}=a_{\alpha\lambda}\varepsilon^{\lambda\sigma}$, 最终可得

$$\frac{\partial x^\sigma}{\partial u^\beta}=\frac{1}{b\rho V^2}(b\rho\delta_\alpha^\sigma\Phi_\beta+\varepsilon_\alpha^{\cdot\sigma}\Psi_\beta)u^\alpha.$$

证毕.　□

引理4　由(20)式所定义的流函数 Ψ 满足方程

$$\frac{\partial}{\partial u^{\alpha}}\left(m^{\alpha\beta}\frac{\partial}{\partial u^{\beta}}\Psi\right)=0, \tag{28}$$

其中

$$m^{\alpha\beta}=\frac{\sqrt{a}}{D_2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\varepsilon_{\nu\mu}\frac{\partial}{\partial u^{\lambda}}(D_\rho u^{\nu})\frac{\partial}{\partial u^{\sigma}}(D_1\varepsilon_{\gamma}^{\cdot\nu}u^{\gamma}).$$

证明 从引理 3 的结果和二阶导数的可交换性

$$\frac{\partial^2 x^{\sigma}}{\partial u^{\beta}\partial u^{\lambda}}=\frac{\partial^2 x^{\sigma}}{\partial u^{\lambda}\partial u^{\beta}},$$

可得

$$\frac{\partial}{\partial u^{\lambda}}(D_1 b\rho\delta_{\alpha}^{\sigma}u^{\alpha}\Phi_{\beta}+D_1\varepsilon_{\alpha}^{\cdot\sigma}u^{\alpha}\Psi_{\beta})=\frac{\partial}{\partial u^{\beta}}(D_1 b\rho\delta_{\alpha}^{\sigma}u^{\alpha}\Phi_{\lambda}+D_1\varepsilon_{\alpha}^{\cdot\sigma}u^{\alpha}\Psi_{\lambda}),$$

即

$$\frac{\partial}{\partial u^{\lambda}}(D_1 b\rho\delta_{\alpha}^{\sigma}u^{\alpha})\Phi_{\beta}-\frac{\partial}{\partial u^{\beta}}(D_1 b\rho\delta_{\alpha}^{\sigma}u^{\alpha})\Phi_{\lambda}=$$

$$\frac{\partial}{\partial u^{\beta}}(D_1\varepsilon_{\alpha}^{\cdot\sigma}u^{\alpha})\Psi_{\lambda}-\frac{\partial}{\partial u^{\lambda}}(D_1\varepsilon_{\alpha}^{\cdot\sigma}u^{\alpha})\Psi_{\beta}$$

或

$$\frac{\partial}{\partial u^{2}}(D_1 b\rho u^{\sigma})\Phi_1-\frac{\partial}{\partial u^{1}}(D_1 b\rho u^{\sigma})\Phi_2=\frac{\partial}{\partial u^{1}}(D_1\varepsilon_{\alpha}^{\cdot\sigma}u^{\alpha})\Psi_2-\frac{\partial}{\partial u^{2}}(D_1\varepsilon_{\alpha}^{\cdot\sigma}u^{\alpha})\Psi_1. \tag{29}$$

记 $D_\rho=D_1 b\rho=1/V^2$ 并令

$$D_2=\begin{vmatrix}\frac{\partial}{\partial u^{2}}(D_\rho u^1) & -\frac{\partial}{\partial u^{1}}(D_\rho u^1)\\ \frac{\partial}{\partial u^{2}}(D_\rho u^2) & -\frac{\partial}{\partial u^{1}}(D_\rho u^2)\end{vmatrix}=$$

$$\frac{\partial}{\partial u^{1}}(D_\rho u^1)\frac{\partial}{\partial u^{2}}(D_\rho u^2)-\frac{\partial}{\partial u^{1}}(D_\rho u^2)\frac{\partial}{\partial u^{2}}(D_\rho u^1). \tag{30}$$

则由(29)式,我们可以用 Ψ_α 来表示 Φ_α:

$$\Phi_1=\frac{1}{D_2}\left(\left[\frac{\partial}{\partial u^{1}}(D_\rho u^2)\frac{\partial}{\partial u^{2}}(D_1\varepsilon_{\alpha}^{\cdot 1}u^{\alpha})-\frac{\partial}{\partial u^{1}}(D_\rho u^1)\frac{\partial}{\partial u^{2}}(D_1\varepsilon_{\alpha}^{\cdot 2}u^{\alpha})\right]\Psi_1+\right.$$

$$\left.\left[\frac{\partial}{\partial u^{1}}(D_\rho u^1)\frac{\partial}{\partial u^{1}}(D_1\varepsilon_{\alpha}^{\cdot 2}u^{\alpha})-\frac{\partial}{\partial u^{1}}(D_\rho u^2)\frac{\partial}{\partial u^{1}}(D_1\varepsilon_{\alpha}^{\cdot 1}u^{\alpha})\right]\Psi_2\right),$$

$$\Phi_2=\frac{1}{D_2}\left(\left[\frac{\partial}{\partial u^{2}}(D_\rho u^2)\frac{\partial}{\partial u^{2}}(D_1\varepsilon_{\alpha}^{\cdot 1}u^{\alpha})-\frac{\partial}{\partial u^{2}}(D_\rho u^1)\frac{\partial}{\partial u^{2}}(D_1\varepsilon_{\alpha}^{\cdot 2}u^{\alpha})\right]\Psi_1+\right.$$

$$\left.\left[\frac{\partial}{\partial u^{2}}(D_\rho u^1)\frac{\partial}{\partial u^{1}}(D_1\varepsilon_{\alpha}^{\cdot 2}u^{\alpha})-\frac{\partial}{\partial u^{2}}(D_\rho u^2)\frac{\partial}{\partial u^{1}}(D_1\varepsilon_{\alpha}^{\cdot 1}u^{\alpha})\right]\Psi_2\right).$$

从而根据 $\Phi_{12}=\Phi_{21}$, 有

$$\frac{\partial}{\partial u^{\alpha}}\left(m^{\alpha\beta}\frac{\partial}{\partial u^{\beta}}\Psi\right)=0,$$

其中

$$m^{\alpha\beta}=\frac{\sqrt{a}}{D_2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\varepsilon_{\nu\mu}\frac{\partial}{\partial u^{\lambda}}(D_\rho u^{\mu})\frac{\partial}{\partial u^{\sigma}}(D_1\varepsilon_{\gamma}^{\cdot\nu}u^{\gamma}).\qquad\square \tag{31}$$

定理 3 的证明 由速度图变换公式,(30)式可改写为

$$D_2=\frac{\partial(D_\rho u^1,D_\rho u^2)}{\partial(u^1,u^2)}=J\frac{\partial(D_\rho u^1,D_\rho u^2)}{\partial(V,\Theta)}=JQ,$$

其中

$$J=\frac{\partial(V,\Theta)}{\partial(u^1,u^2)},\ Q=\frac{\partial(D_\rho u^1,D_\rho u^2)}{\partial(V,\Theta)}.$$

从而,(31)式转化为

$$m^{\alpha\beta}=\frac{a}{J}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\left(A_{VV}\frac{\partial V}{\partial u^\lambda}\frac{\partial V}{\partial u^\sigma}+A_{V\Theta}\frac{\partial V}{\partial u^\lambda}\frac{\partial\Theta}{\partial u^\sigma}+A_{\Theta V}\frac{\partial\Theta}{\partial u^\lambda}\frac{\partial V}{\partial u^\sigma}+A_{\Theta\Theta}\frac{\partial\Theta}{\partial u^\lambda}\frac{\partial\Theta}{\partial u^\sigma}\right),\tag{32}$$

其中

$$A_{\xi\eta}=\frac{1}{Q\sqrt{a}}\varepsilon_{\alpha\beta}\frac{\partial}{\partial\xi}(D_\rho u^\beta)\frac{\partial}{\partial\eta}(D_1\varepsilon_{\gamma}^{\cdot\alpha}u^\gamma),\qquad \xi,\eta=V\text{ 或 }\Theta.\tag{33}$$

而(28)式变为

$$A\frac{\partial^2\Psi}{\partial V^2}+B\frac{\partial^2\Psi}{\partial V\partial\Theta}+C\frac{\partial^2\Psi}{\partial\Theta^2}+E\frac{\partial\Psi}{\partial V}+F\frac{\partial\Psi}{\partial\Theta}=0,\tag{34}$$

其中

$$\begin{cases}A=m^{\alpha\beta}\dfrac{\partial V}{\partial u^\alpha}\dfrac{\partial V}{\partial u^\beta},\ B=m^{\alpha\beta}\left(\dfrac{\partial V}{\partial u^\alpha}\dfrac{\partial\Theta}{\partial u^\beta}+\dfrac{\partial\Theta}{\partial u^\alpha}\dfrac{\partial V}{\partial u^\beta}\right),\ C=m^{\alpha\beta}\dfrac{\partial\Theta}{\partial u^\alpha}\dfrac{\partial\Theta}{\partial u^\beta},\\ E=\dfrac{\partial}{\partial V}\left(m^{\alpha\beta}\dfrac{\partial V}{\partial u^\beta}\right)\dfrac{\partial V}{\partial u^\alpha}+\dfrac{\partial}{\partial\Theta}\left(m^{\alpha\beta}\dfrac{\partial V}{\partial u^\beta}\right)\dfrac{\partial\Theta}{\partial u^\alpha},\\ F=\dfrac{\partial}{\partial V}\left(m^{\alpha\beta}\dfrac{\partial\Theta}{\partial u^\beta}\right)\dfrac{\partial V}{\partial u^\alpha}+\dfrac{\partial}{\partial\Theta}\left(m^{\alpha\beta}\dfrac{\partial\Theta}{\partial u^\beta}\right)\dfrac{\partial\Theta}{\partial u^\alpha}.\end{cases}\tag{35}$$

易证

$$\begin{cases}\sqrt{a}\varepsilon^{\alpha\beta}\dfrac{\partial V}{\partial u^\alpha}\dfrac{\partial V}{\partial u^\beta}=0,\ \sqrt{a}\varepsilon^{\alpha\beta}\dfrac{\partial V}{\partial u^\alpha}\dfrac{\partial\Theta}{\partial u^\beta}=J,\ \sqrt{a}\varepsilon^{\alpha\beta}\dfrac{\partial\Theta}{\partial u^\alpha}\dfrac{\partial V}{\partial u^\beta}=-J,\\ \sqrt{a}\varepsilon^{\alpha\beta}\dfrac{\partial\Theta}{\partial u^\alpha}\dfrac{\partial\Theta}{\partial u^\beta}=0,\ \dfrac{\partial}{\partial u^\alpha}(\sqrt{a}\varepsilon^{\alpha\lambda}\dfrac{\partial V}{\partial u^\lambda})=0,\ \dfrac{\partial}{\partial u^\alpha}(\sqrt{a}\varepsilon^{\alpha\lambda}\dfrac{\partial\Theta}{\partial u^\lambda})=0.\end{cases}\tag{36}$$

通过计算可知

$$\begin{cases}m^{\alpha\beta}\dfrac{\partial V}{\partial u^\beta}=\sqrt{a}\varepsilon^{\alpha\lambda}\left(A_{V\Theta}\dfrac{\partial V}{\partial u^\lambda}+A_{\Theta\Theta}\dfrac{\partial\Theta}{\partial u^\lambda}\right),\\ m^{\alpha\beta}\dfrac{\partial\Theta}{\partial u^\beta}=-\sqrt{a}\varepsilon^{\alpha\lambda}\left(A_{VV}\dfrac{\partial V}{\partial u^\lambda}+A_{\Theta V}\dfrac{\partial\Theta}{\partial u^\lambda}\right),\end{cases}\tag{37}$$

从而

$$\begin{cases}A=JA_{\Theta\Theta},\ B=-J(A_{\Theta V}+A_{V\Theta}),\ C=JA_{VV},\\ E=J\left(\dfrac{\partial A_{\Theta\Theta}}{\partial V}-\dfrac{\partial A_{V\Theta}}{\partial\Theta}\right),\ F=J\left(\dfrac{\partial A_{VV}}{\partial\Theta}-\dfrac{\partial A_{\Theta V}}{\partial V}\right).\end{cases}\tag{38}$$

(34)式为

$$A_{\Theta\Theta}\frac{\partial^2\Psi}{\partial V^2}-(A_{V\Theta}+A_{\Theta V})\frac{\partial^2\Psi}{\partial V\partial\Theta}+A_{VV}\frac{\partial^2\Psi}{\partial\Theta^2}+\left(\frac{\partial A_{\Theta\Theta}}{\partial V}-\frac{\partial A_{V\Theta}}{\partial\Theta}\right)\frac{\partial\Psi}{\partial V}+\left(\frac{\partial A_{VV}}{\partial\Theta}-\frac{\partial A_{\Theta V}}{\partial V}\right)\frac{\partial\Psi}{\partial\Theta}=0.$$

证毕. □

以下的定理将对定理3中得到的二阶速度图方程进行分类.

定理4 速度图方程(23)对应的特征方程和特征根分别为

$$\lambda^2-(A_{VV}+A_{\Theta\Theta})\lambda+A_{VV}A_{\Theta\Theta}-\frac{(A_{V\Theta}+A_{\Theta V})^2}{4}=0$$

和

$$\lambda_{1,2}=\frac{A_{VV}+A_{\Theta\Theta}\pm\sqrt{(A_{VV}-A_{\Theta\Theta})^2+(A_{V\Theta}+A_{\Theta V})^2}}{2}.$$

证明 方程(23)的特征矩阵为

$$\begin{pmatrix} A_{VV} & -\dfrac{A_{V\Theta}+A_{\Theta V}}{2} \\ -\dfrac{A_{V\Theta}+A_{\Theta V}}{2} & A_{\Theta\Theta} \end{pmatrix},$$

从而特征方程为

$$\begin{vmatrix} A_{VV}-\lambda & -\dfrac{A_{V\Theta}+A_{\Theta V}}{2} \\ -\dfrac{A_{V\Theta}+A_{\Theta V}}{2} & A_{\Theta\Theta}-\lambda \end{vmatrix}=(A_{VV}-\lambda)(A_{\Theta\Theta}-\lambda)-\frac{(A_{V\Theta}+A_{\Theta V})^2}{4}=$$

$$\lambda^2-(A_{VV}+A_{\Theta\Theta})\lambda+A_{VV}A_{\Theta\Theta}-\frac{(A_{V\Theta}+A_{\Theta V})^2}{4}=0.$$

相应的判别式为

$$\Delta=(A_{VV}+A_{\Theta\Theta})^2-4\left(A_{VV}A_{\Theta\Theta}-\frac{(A_{V\Theta}+A_{\Theta V})^2}{4}\right)=$$

$$(A_{VV}-A_{\Theta\Theta})^2+(A_{V\Theta}+A_{\Theta V})^2\geqslant 0.$$

从而可以得到结论,所有的特征根都是实数,其值分别为

$$\lambda_{1,2}=\frac{A_{VV}+A_{\Theta\Theta}\pm\sqrt{(A_{VV}-A_{\Theta\Theta})^2+(A_{V\Theta}+A_{\Theta V})^2}}{2}.$$ □

另外,我们有

$$\lambda_1\lambda_2=A_{VV}A_{\Theta\Theta}-\frac{(A_{V\Theta}+A_{\Theta V})^2}{4}.$$

从二阶偏微分方程的经典理论可以得出,方程(23)当$\lambda_1\lambda_2>0$时是椭圆的,当$\lambda_1\lambda_2<0$时是双曲的. 而$\{\lambda_1\lambda_2=0\}$是两种类型的分离线.

注2 特别地,对于平面或柱面流动,我们知道$\lambda_1\lambda_2=V^2(1-Ma^2)$,从而$\lambda_1\lambda_2=0$当且仅当Mach数$Ma=1$.

一种特殊但最重要的情况,S上的Gauss坐标系选为正交曲线坐标系时,即度量张量有如下的形式:

$$a_{12}=a_{21}=0,\ a^{12}=a^{21}=0,\ a^{11}=\frac{1}{a_{11}},\ a^{22}=\frac{1}{a_{22}},\ a=a_{11}a_{22}.$$

设速度$\boldsymbol{u}$的物理分量为u,v, 我们有

$$u^1=\frac{u}{\sqrt{a_{11}}},\ u^2=\frac{v}{\sqrt{a_{22}}},\ u_1=\sqrt{a_{11}}u,\ u_2=\sqrt{a_{22}}v. \tag{39}$$

另一方面,由于

$$u=V\cos\Theta,\ v=V\sin\Theta \text{ 及 } V^2=u^2+v^2,$$

因此

$$u_1=\sqrt{a_{11}}V\cos\Theta,\ u_2=\sqrt{a_{22}}V\sin\Theta,\ u^1=\frac{V\cos\Theta}{\sqrt{a_{11}}},\ u^2=\frac{V\sin\Theta}{\sqrt{a_{22}}}. \tag{40}$$

注意到

$$\begin{cases}D_\rho=\dfrac{1}{V^2},\ D_\rho u^1=\dfrac{\cos\Theta}{V\sqrt{a_{11}}},\ D_\rho u^2=\dfrac{\sin\Theta}{V\sqrt{a_{22}}},\\ D_1\varepsilon_{\alpha}^{\cdot 1}u^\alpha=-\dfrac{u_2}{D}=-\dfrac{\sin\Theta}{b\rho V\sqrt{a_{11}}},\ D_1\varepsilon_{\alpha}^{\cdot 2}u^\alpha=\dfrac{u_1}{D}=\dfrac{\cos\Theta}{b\rho V\sqrt{a_{22}}},\end{cases}$$

因此(33)式为

$$A_{\xi\eta}=-\frac{1}{Q}\left[\frac{\partial}{\partial\xi}\left(\frac{\cos\Theta}{V\sqrt{a_{11}}}\right)\frac{\partial}{\partial\eta}\left(\frac{\cos\Theta}{b\rho V\sqrt{a_{22}}}\right)+\frac{\partial}{\partial\xi}\left(\frac{\sin\Theta}{V\sqrt{a_{22}}}\right)\frac{\partial}{\partial\eta}\left(\frac{\sin\Theta}{b\rho V\sqrt{a_{11}}}\right)\right],\qquad \xi,\eta=V \text{ 或 } \Theta, \tag{41}$$

其中

$$Q=\frac{\partial(D_\rho u^1,D_\rho u^2)}{\partial(V,\Theta)}=\partial\left(\frac{\cos\Theta}{V\sqrt{a_{11}}},\frac{\sin\Theta}{V\sqrt{a_{22}}}\right)\Big/\partial(V,\Theta)=$$
$$-\frac{1}{V^3\sqrt{a}}+\frac{\sin(2\Theta)}{2V^2\sqrt{a}}\left(\frac{\partial}{\partial V}\ln\sqrt{a_{11}}\frac{\partial}{\partial\Theta}\ln\sqrt{a_{22}}-\frac{\partial}{\partial V}\ln\sqrt{a_{22}}\frac{\partial}{\partial\Theta}\ln\sqrt{a_{11}}\right)+$$
$$\frac{\sin(2\Theta)}{2V^3\sqrt{a}}\frac{\partial}{\partial\Theta}\ln\sqrt{\frac{a_{22}}{a_{11}}}-\frac{\cos^2\Theta}{V^2\sqrt{a}}\frac{\partial}{\partial V}\ln\sqrt{a_{11}}-\frac{\sin^2\Theta}{V^2\sqrt{a}}\frac{\partial}{\partial V}\ln\sqrt{a_{22}}. \tag{42}$$

这说明速度图方程(34)仅依赖流面 S 的度量张量($a_{\alpha\beta}$).

3 一些特殊流动实例

我们考虑可压缩位势流动问题.假设流体是理想气体,我们知道有关系式

$$\left(\frac{\rho_0}{\rho}\right)^2=1+\frac{V^2}{c_0^2} \text{ 及 } \rho^2c^2=\rho_0^2c_0^2,$$

其中 c_0 和 ρ_0 分别为滞止音速和密度,c 为当地音速.从而有

$$\frac{\partial\rho}{\partial V}=-\frac{\rho V}{c^2},\ \frac{\partial\rho}{\partial\Theta}=0. \tag{43}$$

例 1 平面流动情形,在直角坐标系下我们有

$$a_{11}=a^{11}=a_{22}=a^{22}=1,\ a=1,\ b=1.$$

从而由(41)和(42)式可以得出

$$\begin{cases}Q=-\dfrac{1}{V^3},\ A_{VV}=V^3\left(\dfrac{1}{\rho V^4}-\dfrac{1}{V^3}\dfrac{\partial}{\partial V}\dfrac{1}{\rho}\right),\\ A_{\Theta\Theta}=\dfrac{V}{\rho},\ A_{V\Theta}=-\dfrac{\partial}{\partial\Theta}\dfrac{1}{\rho},\ A_{\Theta V}=0.\end{cases} \tag{44}$$

另外,由(43)式

$$A_{VV}=\frac{1-Ma^2}{\rho V},\ A_{V\Theta}=0, \tag{45}$$

其中 $Ma=V/c$ 为 Mach 数. 由(43)至(45)式,有

$$\frac{\partial A_{\Theta\Theta}}{\partial V}=\frac{1}{\rho}-\frac{V}{\rho^2}\frac{\partial\rho}{\partial V}=\frac{1+Ma^2}{\rho},\ \frac{\partial A_{VV}}{\partial\Theta}=0. \tag{46}$$

从而速度图方程(23)为

$$V^2\frac{\partial^2\psi}{\partial V^2}+(1-Ma^2)\frac{\partial^2\psi}{\partial\Theta^2}+V(1+Ma^2)\frac{\partial\psi}{\partial V}=0, \tag{47}$$

这和经典结果是一致的.

例 2 当流面 S 是半径为 $r=R_0$ 的圆柱面时,取 $\boldsymbol{y}=(r,\phi,z)$ 为 R^3 中的圆柱坐标系,度量张量为

$$(\tilde g_{ij})=\begin{pmatrix}1&0&0\\0&r^2&0\\0&0&1\end{pmatrix}.$$

在圆柱面上,取 $x^1=\phi, x^2=z$, 并根据公式

$$a_{\alpha\beta}=\tilde g_{ij}\frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\beta}, \tag{48}$$

我们知道在 S 上

$$a_{11}=R_0^2,\ a_{22}=1,\ a_{12}=a_{21}=0,\ a=R_0^2. \tag{49}$$

因此由(41)和(42)式有

$$Q=-\frac{1}{V^3R_0},\ b=1,$$

$$A_{VV}=V^3R_0\left[\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{VR_0}\right)\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{b\rho V}\right)+\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{V}\right)\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{b\rho VR_0}\right)\right]=\frac{1-Ma^2}{\rho V},$$

$$A_{\Theta\Theta}=V^3R_0\left[\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{VR_0}\right)\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{b\rho V}\right)+\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{V}\right)\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{b\rho VR_0}\right)\right]=\frac{V}{\rho},$$

$$A_{V\Theta}=A_{\Theta V}=0.$$

从而速度图方程和(46)式相同.

例 3 流面是半径为 $r=R_0$ 的球面,取 $\boldsymbol{y}=(r,\theta,\phi)$ 为 R^3 中的球极坐标系,其度量张量为

$$(\tilde g_{ij})=\begin{pmatrix}1&0&0\\0&r^2&0\\0&0&r^2\cos^2\theta\end{pmatrix}.$$

在球面 S 上,取球面坐标 $x^1=\theta, x^2=\phi$ ($\theta\in(-\pi/2,\pi/2], \phi\in(-\pi,\pi]$), 由关系式(48)有

$$a_{11}=R_0^2,\ a_{22}=R_0^2\cos^2\theta,\ a_{12}=a_{21}=0,\ a=R_0^4\cos^2\theta. \tag{50}$$

从而由(41)和(42)式可以得到

$$Q=-\frac{1}{V^3R_0^2\cos\theta}\left(1-\frac{1}{2}\sin(2\Theta)\frac{\partial}{\partial\Theta}\ln(\cos\theta)+V\sin^2\Theta\frac{\partial}{\partial V}\ln(R_0\cos\theta)\right),\ b=1,$$

$$A_{VV}=-\frac{1}{Q}\left[\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{VR_0}\right)\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{b\rho VR_0\cos\theta}\right)+\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{VR_0\cos\theta}\right)\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{b\rho VR_0}\right)\right]=$$

$$- \frac{1}{Q\rho V^4 R_0^2 \cos\theta}\left(1 - Ma^2 + V\frac{\partial}{\partial V}\ln(\cos\theta) - VMa^2\sin^2\Theta\frac{\partial}{\partial V}\ln(\cos\theta)\right),$$

$$A_{\Theta\Theta} = -\frac{1}{Q}\left[\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{VR_0}\right)\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{b\rho VR_0\cos\theta}\right) + \frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{VR_0\cos\theta}\right)\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{b\rho VR_0}\right)\right] =$$

$$- \frac{1}{Q\rho V^2 R_0^2\cos\theta},$$

$$A_{V\Theta} = -\frac{1}{Q}\left[\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{VR_0}\right)\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{b\rho VR_0\cos\theta}\right) + \frac{\partial}{\partial V}\left(\frac{\sin\Theta}{VR_0\cos\theta}\right)\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{b\rho VR_0}\right)\right] =$$

$$- \frac{1}{Q\rho V^3 R_0^2\cos\theta}\left(\cos^2\Theta\frac{\partial}{\partial\Theta}\ln(\cos\theta) - \frac{1}{2}V\sin(2\Theta)\frac{\partial}{\partial V}\ln(\cos\theta)\right),$$

$$A_{\Theta V} = -\frac{1}{Q}\left[\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{VR_0}\right)\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{b\rho VR_0\cos\theta}\right) + \frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{VR_0\cos\theta}\right)\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{b\rho VR_0}\right)\right] =$$

$$- \frac{1}{Q\rho V^3 R_0^2\cos\theta}\left(\frac{1}{2}V\sin(2\Theta)\frac{\partial}{\partial V}\ln(\cos\theta) + (1 - Ma^2)\sin^2\Theta\frac{\partial}{\partial\Theta}\ln(\cos\phi)\right).$$

例 4 流面是回转面的情形,它在文献[9]中被 Wu(吴仲华)称为 S_1 流面. 设回转面 S 的旋转轴是 z 轴,生成曲线为 C, 其参数表示为

$$r = R(s),\ z = Z(s) = s,$$

其中 s 为 C 的弧长参数,C 的弧长为 L,$0 \leqslant s \leqslant L$,$R(s) > 0$. 令 $x^1 = s$,$x^2 = \phi$,x^2 是 S 上一点的经度,$\phi \in (0,2\pi]$,由(48)式,回转面 S 的度量张量为

$$a_{11} = R'(s)^2 + Z'(s)^2 = R'(s)^2 + 1,\ a_{12} = a_{21} = 0,$$

$$a_{22} = R(s)^2,\ a = (R'(s)^2 + 1)R(s)^2.$$

从而(41)和(42)式给出

$$Q = -\frac{1}{V^3R(s)\sqrt{R'(s)^2+1}}\left[1 - \frac{1}{2}V\sin(2\Theta)\left(\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1}\,\frac{\partial}{\partial\Theta}\ln R(s) - \right.\right.$$

$$\left.\frac{\partial}{\partial V}\ln R(s)\frac{\partial}{\partial\Theta}\ln\sqrt{R'(s)^2+1}\right) - \frac{1}{2}\sin(2\Theta)\frac{\partial}{\partial\Theta}\ln\frac{R(s)}{\sqrt{R'(s)^2+1}} +$$

$$\left.V\cos^2\Theta\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1} + V\sin^2\Theta\frac{\partial}{\partial V}\ln R(s)\right],$$

$$b = 1,$$

$$A_{VV} = -\frac{1}{Q}\left[\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{V\sqrt{R'(s)^2+1}}\right)\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{b\rho VR(s)}\right) + \frac{\partial}{\partial V}\left(\frac{\sin\Theta}{VR(s)}\right)\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{b\rho V\sqrt{R'(s)^2+1}}\right)\right] =$$

$$- \frac{1}{Q\rho V^4R(s)\sqrt{R'(s)^2+1}}\left(1 - Ma^2 + V\frac{\partial}{\partial V}\ln R(s) + V\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1} - \right.$$

$$VMa^2\cos^2\Theta\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1} - VMa^2\sin^2\Theta\frac{\partial}{\partial V}\ln R(s) +$$

$$\left.V^2\frac{\partial}{\partial V}\ln R(s)\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1}\right),$$

$$A_{\Theta\Theta} = -\frac{1}{Q}\left[\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{V\sqrt{R'(s)^2+1}}\right)\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{b\rho VR(s)}\right) + \frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{VR(s)}\right)\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{b\rho V\sqrt{R'(s)^2+1}}\right)\right] =$$

$$\frac{1}{Q\rho V^2R(s)\sqrt{R'(s)^2+1}}\left(1 + \frac{\partial}{\partial V}\ln R(s)\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1}\right),$$

$$A_{V\Theta}=-\frac{1}{Q}\left[\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{V\sqrt{R'(s)^2+1}}\right)\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{b\rho VR(s)}\right)+\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{VR(s)}\right)\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{b\rho V\sqrt{R'(s)^2+1}}\right)\right]=$$

$$\frac{1}{Q\rho V^3R(s)\sqrt{R'(s)^2+1}}\Big(\cos^2\Theta\frac{\partial}{\partial V}\ln R(s)+\sin^2\Theta\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1}+$$

$$\frac{1}{2}V\sin(2\Theta)\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1}-\frac{1}{2}V\sin(2\Theta)\frac{\partial}{\partial V}\ln R(s)+$$

$$V\cos^2\Theta\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1}\frac{\partial}{\partial\Theta}\ln R(s)+V\sin^2\Theta\frac{\partial}{\partial V}\ln R(s)\frac{\partial}{\partial\Theta}\ln\sqrt{R'(s)^2+1}\Big),$$

$$A_{\Theta V}=-\frac{1}{Q}\left[\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{V\sqrt{R'(s)^2+1}}\right)\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{b\rho VR(s)}\right)+\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{VR(s)}\right)\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{b\rho V\sqrt{R'(s)^2+1}}\right)\right]=$$

$$\frac{1}{Q\rho V^3R(s)\sqrt{R'(s)^2+1}}\Big(\frac{1}{2}V\sin(2\Theta)\frac{\partial}{\partial V}\ln R(s)-\frac{1}{2}V\sin(2\Theta)\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1}+$$

$$(1-Ma^2)\cos^2\Theta\frac{\partial}{\partial\Theta}\ln\sqrt{R'(s)^2+1}+(1-Ma^2)\sin^2\Theta\frac{\partial}{\partial\Theta}\ln R(s)+$$

$$V\cos^2\Theta\frac{\partial}{\partial\Theta}\ln\sqrt{R'(s)^2+1}\frac{\partial}{\partial V}\ln R(s)+V\sin^2\Theta\frac{\partial}{\partial\Theta}\ln R(s)\frac{\partial}{\partial V}\ln\sqrt{R'(s)^2+1}\Big).$$

例 5 螺旋面情形,在直角坐标系下其参数表示是

$$y^1=r\cos(\omega_1 t),\ y^2=r\sin(\omega_1 t),\ y^3=\omega_2 t,$$

其中, $0\leqslant r\leqslant R_0, 0<t\leqslant 2\pi, \omega_1, \omega_2>0$. 令 $x^1=r, x^2=t$, 由(48)式有

$$a_{11}=1,\ a_{22}=\omega_1^2r^2+t^2, a_{12}=a_{21}=0,\ a=\omega_1^2r^2+t^2.$$

从而根据(41)和(42)式可以得到

$$Q=-\frac{1}{V^3\sqrt{\omega_1^2r^2+t^2}}\left(1-\frac{1}{2}\sin(2\Theta)\frac{\partial}{\partial\Theta}\ln\sqrt{\omega_1^2r^2+t^2}+V\sin^2\Theta\frac{\partial}{\partial V}\ln\sqrt{\omega_1^2r^2+t^2}\right),$$

$$b=1,$$

$$A_{VV}=-\frac{1}{Q}\left[\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{V}\right)\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{b\rho V\sqrt{\omega_1^2r^2+t^2}}\right)+\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{V\sqrt{\omega_1^2r^2+t^2}}\right)\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{b\rho V}\right)\right]=$$

$$-\frac{1}{Q\rho V^4\sqrt{\omega_1^2r^2+t^2}}\left(1-Ma^2+V\frac{\partial}{\partial V}\ln\sqrt{\omega_1^2r^2+t^2}-VMa^2\sin^2\Theta\frac{\partial}{\partial V}\ln\sqrt{\omega_1^2r^2+t^2}\right),$$

$$A_{\Theta\Theta}=-\frac{1}{Q}\left[\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{V}\right)\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{b\rho V\sqrt{\omega_1^2r^2+t^2}}\right)+\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{V\sqrt{\omega_1^2r^2+t^2}}\right)\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{b\rho V}\right)\right]=$$

$$-\frac{1}{Q\rho V^2\sqrt{\omega_1^2r^2+t^2}},$$

$$A_{V\Theta}=-\frac{1}{Q}\left[\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{V}\right)\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{b\rho V\sqrt{\omega_1^2r^2+t^2}}\right)+\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{V\sqrt{\omega_1^2r^2+t^2}}\right)\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{b\rho V}\right)\right]=$$

$$-\frac{1}{Q\rho V^3\sqrt{\omega_1^2r^2+t^2}}\left(\cos^2\Theta\frac{\partial}{\partial\Theta}\ln\sqrt{\omega_1^2r^2+t^2}-\frac{V\sin(2\Theta)}{2}\frac{\partial}{\partial V}\ln\sqrt{\omega_1^2r^2+t^2}\right),$$

$$A_{\Theta V}=-\frac{1}{Q}\left[\frac{\partial}{\partial\Theta}\left(\frac{\cos\Theta}{V}\right)\frac{\partial}{\partial V}\left(\frac{\cos\Theta}{b\rho V\sqrt{\omega_1^2r^2+t^2}}\right)+\frac{\partial}{\partial\Theta}\left(\frac{\sin\Theta}{V\sqrt{\omega_1^2r^2+t^2}}\right)\frac{\partial}{\partial V}\left(\frac{\sin\Theta}{b\rho V}\right)\right]=$$

$$-\frac{1}{Q\rho V^3\sqrt{\omega_1^2r^2+t^2}}\Big(\frac{1}{2}V\sin(2\Theta)\frac{\partial}{\partial V}\ln\sqrt{\omega_1^2r^2+t^2}+$$

$$(1 - Ma^2)\sin^2\Theta \frac{\partial}{\partial\Theta}\ln\sqrt{\omega_1^2 r^2 + t^2}\bigg).$$

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Hodograph Method of Flow on Two-Dimensional Manifold

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Abstract: For some special flow, especially the potential flow in the plane, there are obvious advantages using the tool of hodograph method. For the realistic flow, there exists stream surface, namely, two-dimensional manifold, on which the velocity vector of the flow lies its tangent space. By introducing the stream function and potential function, the hodograph method for potential flow on a surface was established with the help of tensor analysis, which provided a kind of analysis method. For the derived hodograph equation, the characteristic equation and its characteristic roots were also derived, from which the type of the hodograph equation of the second order can be classified. Moreover, some examples for special surfaces were given.

Key words: hodograph method; potential flow; stream surface; stream function; potential function