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一类拟线性抛物型方程组的爆破率和爆破模式*

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摘要: 研究了一类非局部退化拟线性抛物型方程组, 在齐次 Dirichlet 边界条件下的正解, 在参数和初始数据满足一定的条件下,得到了解的爆破率和爆破模式.

关 键 词: 退化抛物型方程组; 非局部源; 爆破率; 爆破模式

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引 言

设 Ω 是 R^N 中的有界区域具有光滑边界 $\partial\Omega$, 下述带有非局部源的退化抛物型方程组

$$\begin{cases} u_{1\tau} = \Delta u_1^m + u_1^{p_1} \| v_1 \|_\alpha^{p_2}, & x \in \Omega, \ \tau > 0, \\ v_{1\tau} = \Delta v_1^n + v_1^{q_1} \| u_1 \|_\beta^{q_2}, & x \in \Omega, \ \tau > 0, \\ u_1(x, \tau) = v_1(x, \tau) = 0, & x \in \partial\Omega, \ \tau > 0, \\ u_1(x, 0) = u_{10}(x), \ v_1(x, 0) = v_{10}(x), & x \in \Omega \end{cases} \tag{1}$$

解的整体存在性和有限时刻爆破已经被广泛研究过[1-5], 其中 $m, n > 1$, $\alpha, \beta \geqslant 1$, $p_1, q_1 \geqslant 0$, $p_2, q_2 > 0$. 与文献[5] 中的方法类似, 我们可以得到下面 3 个定理, 其详细证明将在我们今后文章中给出:

定理 A 设 $u_{10}, v_{10} \geqslant 0$, $u_{10}, v_{10} \in L^\infty(\Omega)$, 则存在某个 $T^* = T^*(u_{10}, v_{10}) > 0$, 使得对每个 $T < T^*$, 问题(1) 存在一个非负弱解 $(u_1(x, t), v_1(x, t))$. 进一步, 或者 $T^* = \infty$, 或者

$$\limsup_{t \to T^*} (\| u_1(\cdot, t) \|_\infty + \| v_1(\cdot, t) \|_\infty) = \infty.$$

定理 B 假设下列条件之一成立, 方程(1) 的任一非负解都是整体存在的.

(i) $m > p_1, n > q_1, p_2 q_2 < (m - p_1)(n - q_1)$;

(ii) $m > p_1, n > q_1, p_2 q_2 = (m - p_1)(n - q_1)$, 且区域 Ω 充分小;

(iii) $m \leqslant p_1$, 或 $n \leqslant q_1$, 抑或 $m > p_1, n > q_1, p_2 q_2 > (m - p_1)(n - q_1)$, 且初值 u_{10}, v_{10}

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充分小.

定理 C　假设下列条件之一成立, 方程(1)的任一非负解在有限时刻爆破.

(ⅰ) $m > p_1, n > q_1, p_2q_2 > (m-p_1)(n-q_1)$, 且初值 u_{10}, v_{10} 充分大;

(ⅱ) $m > p_1, n > q_1, p_2q_2 = (m-p_1)(n-q_1)$, 且区域充分大, 即它包含了一个充分大的球, 初值 u_{10}, v_{10} 为 Ω 内的正连续函数;

(ⅲ) $m \leqslant p_1$, 或 $n \leqslant q_1$, 且初值 u_{10}, v_{10} 充分大.

带有非局部源或者局部化源的非线性抛物型方程组的爆破解的爆破率和爆破模式已经有许多结果[3-4,7-9]. 这方面最早的工作是文献[8]. 关于问题(1)的爆破解的爆破率和爆破模式, 文献[3-4]对于 $\alpha = p_2, \beta = q_2$ 的特殊情况得到了若干结果. 受文献[3-4]的启发, 本文讨论问题(1)的爆破解的爆破率和爆破模式, 其中参数满足

$$q_2 > p_1 - 1,\ p_2 > q_1 - 1,\ p_2q_2 > (p_1-1)(q_1-1). \tag{2}$$

我们将推广和改进文献[3-4]中的结果, 并且给出临界情形时的爆破模式. 首先, 对问题(1)的变量作一些变换, 设

$$u_1^m(x,\tau) = u(x,t),\ v_1^n(x,\tau) = v(x,t)(m/n)^{\alpha/(n-1)}, \qquad \tau = t/m,$$

则问题(1)变为

$$\begin{cases} u_t = u^{r_1}(\Delta u + au^{\rho_1}\|v\|_\mu^{\rho_2}), & x\in\Omega,\ t>0,\\ v_t = v^{r_2}(\Delta v + bv^{\sigma_1}\|u\|_\nu^{\sigma_2}), & x\in\Omega,\ t>0,\\ u(x,t) = v(x,t) = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{3}$$

这里

$$r_1 = (m-1)/m,\ \rho_1 = p_1/m,\ \rho_2 = p_2/n,\ \mu = \alpha/n,\ a = (m/n)^{p_2/(n-1)},$$
$$u_0(x) = u_{10}^m,\ r_2 = (n-1)/n,\ \sigma_1 = q_1/n,\ \sigma_2 = q_2/m,\ \nu = \beta/m,$$
$$b = (m/n)^{(q_1-n)/(n-1)},\ v_0(x) = v_{10}^n(n/m)^{n/(n-1)}.$$

这样, 条件(2)等价于

$$\sigma_2 + 1 - r_1 - \rho_1 > 0,\ \rho_2 + 1 - r_2 - \sigma_1 > 0,\ \sigma_2\rho_2 > (1-r_1-\rho_1)(1-r_2-\sigma_1). \tag{4}$$

下面, 我们仅讨论问题(3). 方便起见, 记

$$\Lambda_1 = 1 - r_1 - \rho_1,\ \beta_1 = \sigma_2 + \Lambda_1,\ \Lambda_2 = 1 - r_2 - \sigma_1,\ \beta_2 = \rho_2 + \Lambda_2,\ d = \sigma_2\rho_2 - \Lambda_1\Lambda_2.$$

由条件(4)知 $\beta_1, \beta_2, d > 0$.

本文中, 我们总假设

(H1) 存在某个 $\alpha \in (0,1)$, $u_0, v_0 \in C^{2+\alpha}(\Omega) \cap C^1(\Omega)$, 在 Ω 内, $u_0, v_0 > 0$, 在 $\partial\Omega$ 上 $u_0 = v_0 = 0$, $\partial u_0/\partial\eta < 0$, $\partial v_0/\partial\eta < 0$, 这里 η 是 $\partial\Omega$ 上的外法线方向;

(H2) 存在常数 $\delta \geqslant \max\left\{\delta_0, ak_2^{-1}|\Omega|^{\rho_2/\mu}, bk_1^{-1}|\Omega|^{\sigma_2/\nu}\right\}$, 使得 $\Delta u_0 + au_0^{\rho_1}\|v_0\|_\mu^{\rho_2} - \delta u_0^{k_1+1-r_1} \geqslant 0$, $\Delta v_0 + bv_0^{\sigma_1}\|u_0\|_\nu^{\sigma_2} - \delta v_0^{k_2+1-r_2} \geqslant 0$, 这里 $k_1 = d/\beta_2$, $k_2 = d/\beta_1$,

$$\delta_0 = \max\left\{\frac{ak_2}{r_1}\left(\frac{k_1+1-\rho_1}{k_2+\rho_2}\right)^{\rho_2/k_2+1}|\Omega|^{\rho_2/\mu},\ \frac{bk_1}{r_2}\left(\frac{k_2+1-\sigma_1}{k_1+\sigma_2}\right)^{\sigma_2/k_1+1}|\Omega|^{\sigma_2/\nu}\right\}.$$

注 1　由条件(4)容易推出 $k_1 + 1 - \rho_1 > r_1 > 0$, $k_2 + 1 - \sigma_1 > r_2 > 0$. 因此, $\delta_0 > 0$.

本文的主要结果为如下的两个定理.

定理 1 假设问题(3)的解 (u, v) 在有限时刻 T_* 爆破. 则 u 和 v 同时爆破, 且下列估计成立:

$$\begin{cases} c_1^{1/\beta_1} \leqslant \max_{x\in\Omega} u(x,t)(T_* - t)^{1/k_1} \leqslant (k_1\delta)^{-1/k_1}, \\ c_2^{1/\beta_2} \leqslant \max_{x\in\Omega} v(x,t)(T_* - t)^{1/k_2} \leqslant (k_2\delta)^{-1/k_2}, \end{cases} \tag{5}$$

这里 c_1, c_2 如(16)式定义.

定理 2 在定理 1 的条件下, 假设 $\Lambda_1 \geqslant 0$, $\Lambda_2 \geqslant 0$, $(1-\rho_1)\Lambda_2 < \rho_2(\sigma_2 - r_1)$, $(1-\sigma_1)\Lambda_1 < \sigma_2(\rho_2 - r_2)$, 且在 Ω 上有 $\Delta u_0 \leqslant 0$, $\Delta v_0 \leqslant 0$. 则在 Ω 的任意紧子集上一致地有

$$\lim_{t\to T_*}(T_* - t)^{1/k_1}u(x,t) = \lim_{t\to T_*}(T_* - t)^{1/k_1}\|u(\cdot,t)\|_\infty =$$
$$|\Omega|^{\theta_1}d^{-1/k_1}\left(\frac{\beta_2}{a}\right)^{\Lambda_2/d}\left(\frac{\beta_1}{b}\right)^{\rho_2/d}, \qquad 若\ \Lambda_1 > 0,$$
$$\lim_{t\to T_*}(T_* - t)^{1/k_2}v(x,t) = \lim_{t\to T_*}(T_* - t)^{1/k_2}\|v(\cdot,t)\|_\infty =$$
$$|\Omega|^{\theta_2}d^{-1/k_2}\left(\frac{\beta_1}{b}\right)^{\Lambda_1/d}\left(\frac{\beta_2}{a}\right)^{\sigma_2/d}, \qquad 若\ \Lambda_2 > 0,$$
$$\lim_{t\to T_*}\frac{\ln u(x,t)}{|\ln(T_* - t)|} = \lim_{t\to T_*}\frac{\|\ln u(\cdot,t)\|_\infty}{|\ln(T_* - t)|} = \frac{\beta_2}{\sigma_2\rho_2}, \qquad 若\ \Lambda_1 = 0,$$
$$\lim_{t\to T_*}\frac{\ln v(x,t)}{|\ln(T_* - t)|} = \lim_{t\to T_*}\frac{\|\ln v(\cdot,t)\|_\infty}{|\ln(T_* - t)|} = \frac{\beta_1}{\sigma_2\rho_2}, \qquad 若\ \Lambda_2 = 0,$$

其中

$$\theta_1 = -\left[\frac{\rho_2}{\mu} + \frac{\rho_2}{\beta_2}\left(\frac{\sigma_2}{\nu} - \frac{\rho_2}{\mu}\right)\right]\frac{1}{k_1}, \quad \theta_2 = -\left[\frac{\sigma_2}{\nu} + \frac{\sigma_2}{\beta_1}\left(\frac{\rho_2}{\mu} - \frac{\sigma_2}{\nu}\right)\right]\frac{1}{k_2}.$$

注 2 我们把定理 1 和定理 2 中的参数变回到原问题(1)中的参数, 可以发现本文要求参数满足的条件弱于文献[3-4]中的要求.

1 定理 1 的证明

下面两节, 我们假设问题(3)的解 (u, v) 在有限时刻 T_* 爆破, 且用 C 或 c 表示一般常数, 它们仅依赖于问题(3)的固有参数, 与其解无关, 在同一个式子中, C 或 c 有可能不一样.

记

$$U(t) = \max_{x\in\Omega} u(x,t), \quad V(t) = \max_{x\in\Omega} v(x,t), \tag{6}$$

则 $U(t)$, $V(t)$ Lipschitz 连续[1].

引理 1 对于由下面的(9)式确定的常数 $c_0 > 0$, 成立

$$U^{\beta_1} + V^{\beta_2} \geqslant c_0(T_* - t)^{-\beta_1\beta_2/d}. \tag{7}$$

证明 由问题(3)可知, $U(t)$, $V(t)$ 满足

$$U_t \leqslant a|\Omega|^{\rho_2/\mu}U^{r_1+\rho_1}V^{\rho_2}, \quad V_t \leqslant b|\Omega|^{\sigma_2/\nu}V^{r_2+\sigma_1}U^{\sigma_2}, \qquad \text{a.e. } t \in (0, T_*). \tag{8}$$

由于 $\beta_1, \beta_2, d > 0$, $\sigma_2/\beta_1 + \rho_2/\beta_2 > 1$. 由 Young 不等式, 我们有

$$(U^{\beta_1} + V^{\beta_2})_t \leqslant (a\beta_1|\Omega|^{\rho_2/\mu} + b\beta_2|\Omega|^{\sigma_2/\nu})(U^{\beta_1})^{\sigma_2/\beta_1}(V^{\beta_2})^{\rho_2/\beta_2} \leqslant$$
$$K(U^{\beta_1} + V^{\beta_2})^{\sigma_2/\beta_1 + \rho_2/\beta_2},$$

这里 $K = (a\beta_1|\Omega|^{\rho_2/\mu} + b\beta_2|\Omega|^{\sigma_2/\nu})(\sigma_2/\beta_1 + \rho_2/\beta_2)^{-1}\max\{\sigma_2/\beta_1, \rho_2/\beta_2\}$. 将上面不等式在 (t, T_*) 积分可得(7)式, 这里

$$c_0=\left(\frac{dK}{\beta_1\beta_2}\right)^{-\beta_1\beta_2/d}. \tag{9}$$ □

引理 2　假设 k_1, k_2, δ 由条件(H2)定义，则

$$u_t-\delta u^{k_1+1}\geqslant 0,\quad v_t-\delta v^{k_2+1}\geqslant 0,\qquad (x,t)\in\Omega\times(0,T_*). \tag{10}$$

证明　设 $J_1(x,t)=u_t-\delta u^{k_1+1}$，$J_2(x,t)=v_t-\delta v^{k_2+1}$。通过一系列计算可得

$$\begin{aligned}
u_{tt}={}& r_1u^{-1}(J_1^2+2\delta u^{k_1+1}J_1+\delta^2u^{2k_1+2})+u^{r_1}\Delta J_1+\delta(k_1+1)k_1u^{k_1-1+r_1}|\nabla u|^2+\\
&\delta(k_1+1)u^{k_1+r_1}\Delta u+a\rho_1u^{\rho_1-1+r_1}J_1\|v\|_\mu^{\rho_2}+a\rho_1\delta u^{\rho_1+k_1+r_1}\|v\|_\mu^{\rho_2}+\\
&a\rho_2u^{\rho_1+r_1}\|v\|_\mu^{\rho_2-\mu}\int_\Omega v^{\mu-1}(J_2+\delta v^{k_2+1})\,\mathrm{d}x=\\
&u^{r_1}\Delta J_1+(2r_1\delta u^{k_1}+a\rho_1u^{\rho_1-1+r_1}\|v\|_\mu^{\rho_2}+\delta(k_1+1)u^{k_1})J_1+\\
&r_1u^{-1}J_1^2+\delta k_1(k_1+1)u^{k_1-1+r_1}|\nabla u|^2+(r_1\delta^2+(k_1+1)\delta^2)u^{2k_1+1}+\\
&(\rho_1-k_1-1)a\delta u^{\rho_1+k_1+r_1}\|v\|_\mu^{\rho_2}+a\rho_2u^{\rho_1+r_1}\|v\|_\mu^{\rho_2-\mu}\int_\Omega v^{\mu-1}J_2\,\mathrm{d}x+\\
&a\rho_2\delta u^{\rho_1+r_1}\|v\|_\mu^{\rho_2-\mu}\|v\|_{\mu+k_2}^{\mu+k_2},
\end{aligned}$$

$$J_{1t}=u_{tt}-\delta(k_1+1)u^{k_1}u_t.$$

由于 $k_1-\rho_1+1>r_1>0$，因此

$$\begin{aligned}
&J_{1t}-u^{r_1}\Delta J_1-(2r_1\delta u^{k_1}+a_1\rho_1u^{\rho_1-1+r_1}\|v\|_\mu^{\rho_2})J_1-\\
&\quad a\rho_2u^{\rho_1+r_1}\|v\|_\mu^{\rho_2-\mu}\int_\Omega v^{\mu-1}J_2\,\mathrm{d}x\geqslant\\
&\quad r_1\delta^2u^{2k_1+1}+a\delta\rho_2u^{\rho_1+r_1}\|v\|_\mu^{\rho_2-\mu}\|v\|_{\mu+k_2}^{\mu+k_2}-\\
&\quad a\delta(k_1-\rho_1+1)u^{\rho_1+r_1+k_1}\|v\|_\mu^{\rho_2}=\\
&\quad a\delta(k_1-\rho_1+1)u^{\rho_1+r_1}\left[\frac{r_1\delta}{a(k_1-\rho_1+1)}u^{2k_1+\Lambda_1}+\right.\\
&\quad\left.\frac{\rho_2}{k_1-\rho_1+1}\|v\|_\mu^{\rho_2-\mu}\|v\|_{\mu+k_2}^{\mu+k_2}-u^{k_1}\|v\|_\mu^{\rho_2}\right].
\end{aligned} \tag{11}$$

由 Hölder 不等式，对任何 $0<\theta<1$，有

$$\begin{aligned}
\|v\|_\mu^{\rho_2}={}&\|v\|_\mu^{(\rho_2-\mu)\theta}\|v\|_\mu^{\rho_2-(\rho_2-\mu)\theta}\leqslant\\
&\|v\|_\mu^{(\rho_2-\mu)\theta}\|v\|_{\mu+k_2}^{\rho_2-(\rho_2-\mu)\theta}|\Omega|^{(k_2(\rho_2-(\rho_2-\mu)\theta))/(\mu(\mu+k_2))}.
\end{aligned}$$

进一步，由 Young 不等式，对任何 $\varepsilon>0$ 以及满足 $1/l_1+1/l_2=1$ 的 $l_1, l_2>1$，我们有

$$\begin{aligned}
u^{k_1}\|v\|_\mu^{\rho_2}\leqslant{}&|\Omega|^{(k_2(\rho_2-(\rho_2-\mu)\theta))/(\mu(\mu+k_2))}u^{k_1}\|v\|_\mu^{(\rho_2-\mu)\theta}\|v\|_{\mu+k_2}^{\rho_2-(\rho_2-\mu)\theta}\leqslant\\
&|\Omega|^{(k_2(\rho_2-(\rho_2-\mu)\theta))/(\mu(\mu+k_2))}\left(\frac{(\varepsilon u^{k_1})^{l_1}}{l_1}+\right.\\
&\left.\frac{1}{l_2}\left(\frac{1}{\varepsilon}\|v\|_\mu^{(\rho_2-\mu)\theta}\|v\|_{\mu+k_2}^{\rho_2-(\rho_2-\mu)\theta}\right)^{l_2}\right).
\end{aligned} \tag{12}$$

取

$$l_1=\frac{k_2+\rho_2}{k_2},\quad l_2=\frac{k_2+\rho_2}{\rho_2},\quad \theta=\frac{\rho_2}{k_2+\rho_2},$$

$$\varepsilon=\left(\frac{k_1-\rho_1+1}{k_2+\rho_2}|\Omega|^{k_2\rho_2/(\mu(\rho_2+k_2))}\right)^{\rho_2/(k_2+\rho_2)},$$

由于

$$\delta > \frac{ak_2}{r_1}\left(\frac{k_1+1-\rho_1}{k_2+\rho_2}\right)^{\rho_2/k_2+1}|\Omega|^{\rho_2/\mu},$$

我们也有

$$\frac{r_1\delta}{a(k_1-\rho_1+1)} \geqslant \frac{\varepsilon_1^{l_1}}{l_1}|\Omega|^{(k_2(\rho_2-(\rho_2-\mu)\theta))/(\mu(\mu+k_2))}.$$

因此由(11)式和(12)式可得

$$J_{1t}-u^{r_1}\Delta J_1-(2r_1\delta u^{k_1}+a\rho_1 u^{\rho_1-1+r_1}\|v\|_{\mu}^{\rho_2})J_1- a\rho_2 u^{\rho_1+r_1}\|v\|_{\mu}^{\rho_2-\mu}\int_\Omega v^{\mu-1}J_2\mathrm{d}x \geqslant r_1\delta(\delta-\delta_1)u^{2k_1+1}\geqslant 0,$$

其中 $\delta_1=(ak_2/r_1)\left((k_1+1-\rho_1)/(k_2+\rho_2)\right)^{\rho_2/k_2+1}|\Omega|^{\rho_2/\mu}$. 与上面相似的方法可得

$$J_{2t}-v^{r_2}\Delta J_2-(2r_2\delta v^{k_2}+b\sigma_1 v^{\sigma_1-1+r_2}\|u\|_{\nu}^{\sigma_2})J_2- b\sigma_2 v^{\sigma_1+r_2}\|u\|_{\nu}^{\sigma_2-\nu}\int_\Omega u^{\nu-1}J_1\mathrm{d}x \geqslant r_2\delta(\delta-\delta_2)v^{2k_2+1}\geqslant 0,$$

其中 $\delta_2=(bk_1/r_2)\left((k_2+1-\sigma_1)/(k_1+\sigma_2)\right)^{\sigma_2/k_1+1}|\Omega|^{\sigma_2/\nu}$. 进一步, 由(H2)可得

$$\lim_{x\to\partial\Omega}J_1(x,t)=\lim_{x\to\partial\Omega}J_2(x,t)\geqslant 0,\qquad J_1(x,0),\ J_2(x,0)\geqslant 0,\ x\in\Omega.$$

由比较原理(见文献[3]的引理1)可得 $J_1, J_2\geqslant 0$. □

定理1的证明 由(10)式可得

$$U_t\geqslant \delta U^{k_1+1},\quad V_t\geqslant \delta V^{k_2+1},\qquad t\in(0,T_*).\tag{13}$$

根据(8)式和(13)式有

$$\begin{cases}\delta U^{k_1+1}\leqslant U_t\leqslant a|\Omega|^{\rho_2/\mu}U^{r_1+\rho_1}V^{\rho_2},\\ \delta V^{k_2+1}\leqslant V_t\leqslant b|\Omega|^{\sigma_2/\nu}V^{r_2+\sigma_1}U^{\sigma_2},\end{cases}\qquad \text{a.e. } t\in(0,T_*),$$

此即表明

$$U^{k_1+\Lambda_1}\leqslant\frac{a}{\delta}|\Omega|^{\rho_2/\mu}V^{\rho_2},\quad V^{k_2+\Lambda_2}\leqslant\frac{b}{\delta}|\Omega|^{\sigma_2/\mu}U^{\sigma_2},\qquad \text{a.e. } t\in(0,T_*).\tag{14}$$

通过直接计算可得 $k_1+\Lambda_1=\rho_2\beta_1/\beta_2$, $k_2+\Lambda_2=\sigma_2\beta_2/\beta_1$. 故由(14)式得

$$U^{\beta_1}(t)\leqslant\left(\frac{a}{\delta}\right)^{\beta_2/\rho_2}|\Omega|^{\beta_2/\mu}V^{\beta_2}(t),\quad V^{\beta_2}(t)\leqslant\left(\frac{b}{\delta}\right)^{\beta_1/\sigma_2}|\Omega|^{\beta_1/\mu}U^{\beta_1}(t).\tag{15}$$

上式表示 u 和 v 同时爆破. 结合(15)式和(7)式得

$$U(t)\geqslant c_1^{1/\beta_1}(T_*-t)^{-1/k_1},\quad V(t)\geqslant c_2^{1/\beta_2}(T_*-t)^{-1/k_2},$$

这里

$$c_1=\left[1+\left(\frac{b}{\delta}\right)^{\beta_1/\sigma_2}|\Omega|^{\beta_1/\mu}\right]^{-1}c_0,\quad c_2=\left[1+\left(\frac{a}{\delta}\right)^{\beta_2/\rho_2}|\Omega|^{\beta_2/\mu}\right]^{-1}c_0.\tag{16}$$

再将(13)式在 $(0,T_*)$ 积分就完成了本定理的证明. □

2 定理2的证明

为了证明定理2, 首先证明几个引理.

引理3 假设在 Ω 上 $\Delta u_0\leqslant 0$, $\Delta v_0\leqslant 0$, 则在 Ω 的任意紧子集上有 $\Delta u\leqslant 0$, $\Delta v\leqslant 0$.

证明 本引理的证明类似于文献[4]的引理5.1的证明, 略去.

记

$$g_1(t) = a\|v\|_{\mu^2}^{\rho_2},\ G_1(t) = \int_0^t g_1(s)\mathrm{d}s,$$

$$g_2(t) = b\|u\|_{\nu^2}^{\sigma},\ G_2(t) = \int_0^t g_2(s)\mathrm{d}s.$$

引理 4 若 $\Lambda_1, \Lambda_2 \geqslant 0$, 则

$$\lim_{t\to T_*} g_i(t) = \infty,\ \lim_{t\to T_*} G_i(t) = \infty \qquad (i = 1, 2).$$

证明 记 $U(t), V(t)$ 如(6)式定义, 则

$$U'(t) \leqslant U^{\rho_1+r_1}(t)g_1(t), \qquad \text{a.e. } t\in[0, T^*). \tag{17}$$

对任何 $t_0\in[0, T_*)$, 将(17) 式在(t_0, t) 积分可得

$$U^{\Lambda_1}(t) \leqslant \Lambda_1 G_1(t) + U^{\Lambda_1}(t_0), \qquad 若\ \rho_1 + r_1 < 1, \tag{18}$$

$$\ln U(t) \leqslant G_1(t) + \ln U(t_0), \qquad 若\ \rho_1 + r_1 = 1. \tag{19}$$

注意到 $\lim_{t\to T_*} U(t) = \infty$, 因此, 如果 $\Lambda_1 \geqslant 0$, 则 $\lim_{t\to T_*} G_1(t) = \lim_{t\to T_*}\sup g_1(t) = \infty$. 又由于 $v_t \geqslant 0$, 因此 $g_1(t)$ 单调不减, 故 $\lim_{t\to T_*} g_1(t) = \infty$. 类似地, $\lim_{t\to T_*} G_2(t) = \lim_{t\to T_*} g_2(t) = \infty$. 本引理得证. □

注 3 回到问题(1), $\Lambda_1 \geqslant 0$, $\Lambda_2 \geqslant 0$ 等价于 $p_1 \leqslant 1$, $q_1 \leqslant 1$.

引理 5 如果 $(1-\rho_1)\Lambda_2 < \rho_2(\sigma_2 - r_1)$, $(1-\sigma_1)\Lambda_1 < \sigma_2(\rho_2 - r_2)$, 且在 Ω 上 $\Delta u_0 \leqslant 0$, $\Delta v_0 \leqslant 0$. 则在 Ω 的任意紧子集上, 一致地有

$$\begin{cases} \lim\limits_{t\to T_*} \dfrac{u^{\Lambda_1}(x,t)}{\Lambda_1 G_1(t)} = \lim\limits_{t\to T_*} \dfrac{\|u(\cdot,t)\|_\infty^{\Lambda_1}}{\Lambda_1 G_1(t)} = 1, & 若\ \Lambda_1 > 0, \\ \lim\limits_{t\to T_*} \dfrac{\ln u(x,t)}{G_1(t)} = \lim\limits_{t\to T_*} \dfrac{\|\ln u(\cdot,t)\|_\infty}{G_1(t)} = 1, & 若\ \Lambda_1 = 0, \\ \lim\limits_{t\to T_*} \dfrac{v^{\Lambda_2}(x,t)}{\Lambda_2 G_2(t)} = \lim\limits_{t\to T_*} \dfrac{\|v(\cdot,t)\|_\infty^{\Lambda_2}}{\Lambda_2 G_2(t)} = 1, & 若\ \Lambda_2 > 0, \\ \lim\limits_{t\to T_*} \dfrac{\ln v(x,t)}{G_2(t)} = \lim\limits_{t\to T_*} \dfrac{\|\ln v(\cdot,t)\|_\infty}{G_2(t)} = 1, & 若\ \Lambda_2 = 0. \end{cases} \tag{20}$$

证明

情形 1 $\Lambda_1, \Lambda_2 > 0$. 证明方法类似于文献[3]的引理 15, 证明细节略去.

情形 2 $\Lambda_1 = 0$, $\Lambda_2 > 0$. 记 λ_1, $\varphi(x)$ 为特征值问题

$$-\Delta\varphi(x) = \lambda\varphi(x), \qquad x\in\Omega;\ \varphi(x) = 0, \qquad x\in\partial\Omega$$

的第一特征值和相应的特征函数, 则 $\lambda_1 > 0$. 将 $\varphi(x)$ 标准化, 使得 $\varphi|_\Omega > 0$, $\int_\Omega \varphi(x)\mathrm{d}x = 1$. 定义

$$z(x,t) = G_1(t) - \ln u(x,t),\ \lambda(t) = \int_\Omega z(y,t)\varphi(y)\mathrm{d}y.$$

直接计算可得

$$\lambda'(t) = \int_\Omega (g_1(t) - u^{-1}(y,t)u_t(y,t))\varphi(y)\mathrm{d}y =$$

$$-\int_\Omega (u^{r_1-1}(y,t)\Delta u(y,t)\varphi(y))\mathrm{d}y \leqslant$$

$$-\frac{1}{r_1}\int_\Omega \varphi(y)\Delta u^{r_1}(y,t)\mathrm{d}y=\frac{\lambda_1}{r_1}\int_\Omega u^{r_1}(y,t)\varphi(y)\mathrm{d}y=$$
$$C\int_\Omega \exp\left\{r_1[G_1(t)-z(y,t)]\right\}\varphi(y)\mathrm{d}y.$$

利用(19)式得

$$z(x,t)\geqslant -M,\qquad (x,t)\in\Omega\times[0,T_*).\tag{21}$$

因此

$$\lambda'(t)\leqslant C\int_\Omega \exp\left\{r_1G_1(t)\right\}\varphi(y)\mathrm{d}y=C\exp\left\{r_1G_1(t)\right\}.$$

上式从0到 t 积分可得

$$\lambda(t)\leqslant\lambda(0)+C\int_0^t\exp\left\{r_1G_1(s)\right\}\mathrm{d}s\leqslant C\left(1+\int_0^t\exp\left\{r_1G_1(s)\right\}\mathrm{d}s\right).$$

同于情形1的证明,可得

$$\int_\Omega |z(y,t)|\varphi(y)\mathrm{d}y\leqslant C\left(1+\int_0^t\exp\left\{r_1G_1(s)\right\}\mathrm{d}s\right).\tag{22}$$

任给 $\zeta>0$,定义 $\Omega_\zeta=\left\{y\in\Omega:\mathrm{dist}(y,\partial\Omega)\geqslant\zeta\right\}$. 由引理3得 $-\Delta z\leqslant 0$. 注意到(22)式,再次利用文献[8]的引理4.5可得

$$\max_{\Omega_\zeta} z(x,t)\leqslant\frac{C}{\zeta^{N+1}}\left(1+\int_0^t\exp\left\{r_1G_1(s)\right\}\mathrm{d}s\right).\tag{23}$$

由(21)式和(23)式,当 $x\in\Omega_\zeta$, $t\in(0,T_*)$ 时,

$$-\frac{M}{G_1(t)}\leqslant\frac{z(x,t)}{G_1(t)}=1-\frac{\ln u(x,t)}{G_1(t)}\leqslant\frac{C}{\zeta^{N+1}G_1(t)}\left(1+\int_0^t\exp\left\{r_1G_1(s)\right\}\mathrm{d}s\right).\tag{24}$$

不失一般性,可以假设 $T_*>1$. 由定理1可得

$$G_1(t)=\int_0^t g_1(s)\mathrm{d}s\leqslant a\int_0^t\|V(s)\|_\mu^{\rho_2}\mathrm{d}s\leqslant\frac{a}{k_2\delta}|\Omega|^{\rho_2/\mu}\int_0^t(T_*-s)^{-1}\mathrm{d}s\leqslant$$
$$\ln(T_*-t)^{-1}+\ln T_*.\tag{25}$$

根据(19)式以及(5)式中的第1个不等式,当 t 趋近于 T_* 时,存在某个常数 $0<\varepsilon_1<1$,使得 $G_1(t)\geqslant\varepsilon_1|\ln(T_*-t)|$. 再结合(25)式和(24)式可知,当 $x\in\Omega_\zeta$, $t\in(0,T_*)$ 时,

$$-\frac{M}{G_1(t)}\leqslant 1-\frac{\ln u(x,t)}{G_1(t)}\leqslant\frac{C}{\zeta^{N+1}|\ln(T_*-t)|}\left(1+T_*^{r_1}\int_0^t(T_*-s)^{-r_1}\mathrm{d}s\right).\tag{26}$$

由于 $1-r_1>0$, 容易得到

$$\lim_{t\to T_*}\frac{1}{|\ln(T_*-t)|}\int_0^t(T_*-s)^{-r_1}\mathrm{d}s=0.$$

注意到 $t\to T_*$ 时, $G_1(t)\to\infty$, 由(26)式得

$$\lim_{t\to T_*}\frac{\ln u(x,t)}{G_1(t)}=1$$

在 Ω_ζ 上一致成立. 与情形1类似地可证

$$\lim_{t\to T_*}\frac{\|\ln u(\cdot,t)\|_\infty}{G_1(t)}=1.$$

同理可以证明,在 Ω_ζ 上一致地成立

$$\lim_{t\to T_*}\frac{v^{\Lambda_2}(x,t)}{\Lambda_2G_2(t)}=\lim_{t\to T_*}\frac{\|v(\cdot,t)\|_\infty^{\Lambda_2}}{\Lambda_2G_2(t)}=1.$$

用上面类似的方法可证情形 3: $\Lambda_1 > 0$, $\Lambda_2 = 0$ 和情形 4: $\Lambda_1 = 0$, $\Lambda_2 = 0$ 时的结论. □

定理 2 的证明 (i) 若 $\Lambda_1, \Lambda_2 > 0$, 由引理 5 可得, 当 $t \to T_*$ 时,

$$G_1'(t) = a\|v\|_{\mu}^{\rho_2} \sim a|\Omega|^{\rho_2/\mu}(\Lambda_2 G_2(t))^{\rho_2/\Lambda_2},$$

$$G_2'(t) = b\|u\|_{\nu}^{\sigma_2} \sim b|\Omega|^{\sigma_2/\nu}(\Lambda_1 G_1(t))^{\sigma_2/\Lambda_1},$$

这里 $u \sim v$ 表示 $\lim_{t\to T_*} u(t)/v(t) = 1$. 因此

$$\frac{\mathrm{d}G_1}{\mathrm{d}G_2} \sim \frac{a}{b}|\Omega|^{\rho_2/\mu-\sigma_2/\nu}\Lambda_2^{\rho_2/\Lambda_2}\Lambda_1^{-\sigma_2/\Lambda_1}G_1^{-\sigma_2/\Lambda_1}G_2^{\rho_2/\Lambda_2}.$$

通过一系列计算可得

$$G_1(t) \sim \Lambda_1^{-1}|\Omega|^{\theta_1\Lambda_1}d^{-\Lambda_1\beta_2/d}\left(\frac{\beta_2}{a}\right)^{\Lambda_1\Lambda_2/d}\left(\frac{\beta_1}{b}\right)^{\Lambda_1\rho_2/d}(T_*-t)^{-\Lambda_1\beta_2/d},$$

$$G_2(t) \sim \Lambda_2^{-1}|\Omega|^{\theta_1\Lambda_1}d^{-\Lambda_2\beta_1/d}\left(\frac{\beta_1}{b}\right)^{\Lambda_1\Lambda_2/d}\left(\frac{\beta_2}{a}\right)^{\Lambda_2\sigma_2/d}(T_*-t)^{-\Lambda_2\beta_1/d}.$$

结合引理 5 可得结论.

(ii) 若 $\Lambda_1 = 0$, $\Lambda_2 > 0$. 则 $\beta_1 = \sigma_2$, $d = \sigma_2\rho_2$, $k_2 = \rho_2$, $\theta_2 = -1/\mu$. 任给 ε: $0 < \varepsilon \ll \rho_2/(\beta_2 + \Lambda_2)$, 选取 $\Omega_\varepsilon \subset\subset \Omega$, 使得 $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$. 由引理 5, 存在 $0 < t_0 < T_*$ 使得

$$\begin{cases} \ln u(x,t) \geqslant (1-\varepsilon)G_1(t), \\ v^{\Lambda_2}(x,t) \geqslant (1-\varepsilon)\Lambda_2 G_2(t), \end{cases} \quad x \in \Omega_\varepsilon,\ t \in [t_0, T_*),$$

$$\begin{cases} \ln u(x,t) \leqslant \|\ln u(\cdot,t)\|_\infty \leqslant (1+\varepsilon)G_1(t), & x \in \Omega,\ t \in [t_0, T_*), \\ v^{\Lambda_2}(x,t) \leqslant \|v^{\Lambda_2}(\cdot,t)\|_\infty \leqslant (1+\varepsilon)\Lambda_2 G_2(t), & x \in \Omega,\ t \in [t_0, T_*). \end{cases}$$

因此

$$\begin{aligned} & b|\Omega_\varepsilon|^{\sigma_2/\nu}\exp\{\sigma_2(1-\varepsilon)G_1(t)\} \leqslant G_2'(t) \leqslant \\ & \quad b|\Omega|^{\sigma_2/\nu}\exp\{\sigma_2(1+\varepsilon)G_1(t)\}, \\ & a|\Omega_\varepsilon|^{\rho_2/\mu}[(1-\varepsilon)\Lambda_2 G_2(t)]^{\rho_2/\Lambda_2} \leqslant G_1'(t) \leqslant \\ & \quad a|\Omega|^{\rho_2/\mu}[(1+\varepsilon)\Lambda_2 G_2(t)]^{\rho_2/\Lambda_2}, \qquad t \in [t_0, T_*). \end{aligned} \tag{27}$$

从而

$$\begin{aligned} & \frac{a|\Omega_\varepsilon|^{\rho_2/\mu}}{b|\Omega|^{\sigma_2/\nu}}\frac{[(1-\varepsilon)\Lambda_2 G_2(t)]^{\rho_2/\Lambda_2}}{\exp\{\sigma_2(1+\varepsilon)G_1(t)\}} \leqslant \frac{\mathrm{d}G_1(t)}{\mathrm{d}G_2(t)} \leqslant \\ & \quad \frac{a|\Omega|^{\rho_2/\mu}}{b|\Omega_\varepsilon|^{\sigma_2/\nu}}\frac{[(1+\varepsilon)\Lambda_2 G_2(t)]^{\rho_2/\Lambda_2}}{\exp\{\sigma_2(1-\varepsilon)G_1(t)\}}, \qquad t \in [t_0, T_*). \end{aligned} \tag{28}$$

根据(28)式右端可得

$$\begin{aligned} & \exp\{\sigma_2(1-\varepsilon)G_1(t)\}\mathrm{d}G_1(t) \leqslant \\ & \quad \frac{a|\Omega|^{\rho_2/\mu}}{b|\Omega_\varepsilon|^{\sigma_2/\nu}}[(1+\varepsilon)\Lambda_2 G_2(t)]^{\rho_2/\Lambda_2}\mathrm{d}G_2(t), \qquad t \in [t_0, T_*). \end{aligned}$$

将上式从 t_0 到 t 积分得

$$\begin{aligned} & \frac{1}{\sigma_2(1-\varepsilon)}\exp\{\sigma_2(1-\varepsilon)G_1(t)\}\Big|_{t_0}^{t} \leqslant \\ & \quad \frac{a|\Omega|^{\rho_2/\mu}}{b|\Omega_\varepsilon|^{\sigma_2/\nu}}(1+\varepsilon)^{\rho_2/\Lambda_2}\frac{1}{\beta_2}(\Lambda_2 G_2(t))^{\beta_2/\Lambda_2}\Big|_{t_0}^{t} \leqslant \\ & \quad \frac{a|\Omega|^{\rho_2/\mu}}{b|\Omega_\varepsilon|^{\sigma_2/\nu}}(1+\varepsilon)^{\rho_2/\Lambda_2}\frac{1}{\beta_2}(\Lambda_2 G_2(t))^{\beta_2/\Lambda_2}. \end{aligned} \tag{29}$$

根据 $\lim_{t\to T_*} G_1(t)=\infty$, 存在 $t_1: t_0\leqslant t_1<T_*$ 使得

$$\frac{1}{\sigma_2(1-\varepsilon)}\exp\left\{\sigma_2(1-\varepsilon)G_1(t_0)\right\}\leqslant \frac{\varepsilon}{\sigma_2(1-\varepsilon)}\exp\left\{\sigma_2(1-\varepsilon)G_1(t)\right\},\qquad t\in[t_1,T_*).$$

因此,由(29)式得

$$\exp\left\{\sigma_2(1-\varepsilon)G_1(t)\right\}\leqslant \frac{a|\Omega|^{\rho_2/\mu}}{b|\Omega_\varepsilon|^{\sigma_2/\nu}}(1+\varepsilon)^{\rho_2/\Lambda_2}\frac{\sigma_2}{\beta_2}[\Lambda_2G_2(t)]^{\beta_2/\Lambda_2},\qquad t\in[t_1,T_*).\tag{30}$$

同理,存在 $t_2: t_0\leqslant t_2<T_*$ 使得

$$\exp\left\{\sigma_2(1+\varepsilon)G_1(t)\right\}\geqslant \frac{a|\Omega_\varepsilon|^{\rho_2/\mu}}{b|\Omega|^{\sigma_2/\nu}}(1+\varepsilon)(1-\varepsilon)^{\rho_2/\Lambda_2}\frac{\sigma_2}{\beta_2}[\Lambda_2G_2(t)]^{\beta_2/\Lambda_2},\qquad t\in[t_2,T_*).\tag{31}$$

设 $t=\max\left\{t_1,t_2\right\}$, (30)、(31)式和(27)式相结合,得

$$m(\varepsilon)[\Lambda_2G_2(t)]^{\vartheta_1(\varepsilon)}\leqslant G_2'(t)\leqslant M(\varepsilon)[\Lambda_2G_2(t)]^{\vartheta_2(\varepsilon)},\qquad t\in[t,T_*),\tag{32}$$

其中

$$m(\varepsilon)=b|\Omega_\varepsilon|^{\sigma_2/\nu}\left(\frac{a|\Omega_\varepsilon|^{\rho_2/\mu}}{b|\Omega|^{\sigma_2/\nu}}(1+\varepsilon)(1-\varepsilon)^{\rho_2/\Lambda_2}\frac{\sigma_2}{\beta_2}\right)^{(1-\varepsilon)/(1+\varepsilon)},$$

$$\vartheta_1(\varepsilon)=\frac{\beta_2}{\Lambda_2}\frac{1-\varepsilon}{1+\varepsilon},$$

$$M(\varepsilon)=b|\Omega|^{\sigma_2/\nu}\left(\frac{a|\Omega|^{\rho_2/\mu}}{b|\Omega_\varepsilon|^{\sigma_2/\nu}}(1+\varepsilon)^{\rho_2/\Lambda_2}\frac{\sigma_2}{\beta_2}\right)^{(1+\varepsilon)/(1-\varepsilon)},$$

$$\vartheta_2(\varepsilon)=\frac{\beta_2}{\Lambda_2}\frac{1+\varepsilon}{1-\varepsilon}.$$

对 $t\in[t,T_*)$, 从 t 到 T_* 分别积分(32)式的两个不等式并令 $\varepsilon\to0$, 我们有

$$\Lambda_2\lim_{t\to T_*}G_2(t)(T_*-t)^{\Lambda_2/\rho_2}=|\Omega|^{-\Lambda_2/\mu}\left(\frac{\beta_2}{a\sigma_2\rho_2}\right)^{\Lambda_2/\rho_2}.\tag{33}$$

根据(20)式的第3个等式可得,在 Ω 的任意紧子集上一致成立

$$\lim_{t\to T_*}v(x,t)(T_*-t)^{1/\rho_2}=\lim_{t\to T_*}\|v(x,t)\|_\infty(T_*-t)^{1/\rho_2}=|\Omega|^{-1/\mu}\left(\frac{\beta_2}{a\sigma_2\rho_2}\right)^{1/\rho_2}.$$

由(32)式和(27)式可得,存在 $\hat t\in(t,T_*)$ 使得对任意 $t\in(\hat t,T_*)$,

$$\sigma_2(1-\varepsilon)G_1(t)\leqslant C_1(\varepsilon)+\vartheta_2(\varepsilon)\ln G_2(t)\leqslant C_2(\varepsilon)+\vartheta_2(\varepsilon)\frac{\Lambda_2}{\rho_2}|\ln(T_*-t)|,$$

$$\sigma_2(1+\varepsilon)G_1(t)\geqslant C_3(\varepsilon)+\vartheta_1(\varepsilon)\ln G_2(t)\geqslant C_4(\varepsilon)+\vartheta_1(\varepsilon)\frac{\Lambda_2}{\rho_2}|\ln(T_*-t)|,$$

这里 $C_i(\varepsilon)$ 是关于 $0<\varepsilon\ll1$ 的有界函数. 由(33)式可得

$$\frac{\vartheta_1(\varepsilon)\Lambda_2}{(1-\varepsilon)\sigma_2\rho_2}\leqslant\lim_{t\to T_*}\frac{G_1(t)}{|\ln(T_*-t)|}\leqslant\frac{\vartheta_2(\varepsilon)\Lambda_2}{(1+\varepsilon)\sigma_2\rho_2}.$$

令 $\varepsilon\to0^+$ 有

$$\lim_{t\to T_*}\frac{G_1(t)}{|\ln(T_*-t)|}=\frac{\beta_2}{\sigma_2\rho_2}.$$

此式结合(20)式的第2个等式可得,在 Ω 的任意紧子集上一致地成立

$$\lim_{t \to T_*} \frac{\ln u(x,t)}{|\ln(T_* - t)|} = \lim_{t \to T_*} \frac{\|\ln u(\cdot,t)\|_\infty}{|\ln(T_* - t)|} = \frac{\beta_2}{\sigma_2 \rho_2}.$$

同样的方法可以证明结论(iii)和结论(iv).

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Blow-up Rate and Profile for a Class of Quasilinear Parabolic System

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Abstract: The positive solutions to a class of nonlocal and degenerate quasilinear parabolic system with null Dirichlet boundary conditions are dealt with. The blow-up rate and blow-up profile were gained if the parameters and the initial data satisfy some conditions.

Key words: degenerate parabolic system; nonlocal sources; blow-up rate; blow-up profile